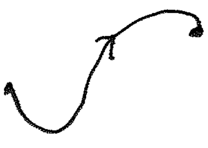
Our next objective is to prove that a plane curve with no singular points is a Riemann surface. The primary tool is the implicit function theorem. The proof is easier in the holomorphic case than in the real case. It makes use of some facts from the complex analysis course which we will now review (and will use later).

Recall that a fundamental tool in complex analysis is the path integral.

Let $\gamma : [a,b] \to \mathbb{C}$ be a smooth path and f a holomorphic function then

complex multiplication.

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t)) \cdot \frac{d\gamma}{dt}$$



complex # no.

complex number

This integral ~~only~~ doesn't change if we reparametrize γ. Depends only on the orientation.

Fundamental fact: If D is a disk and f is holomorphic in D then $\int_{\partial D} f\,dz = 0$.

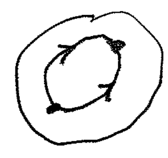
If f has a pole a z_0 and D is a disk centered at z_0 (then f is not holomorphic in D) we have:

$$\int_{\partial D} f\,dz = 2\pi i \cdot \mathrm{res}(f, z_0)$$

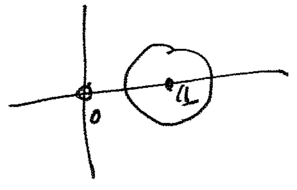
If $f(z) = \sum_{n \geq -k} a_n (z-z_0)^n$ then $\mathrm{res}(f, z_0) = a_{-1}$.

The path integral can be used to construct anti-derivatives. If f is holomorphic in D and we define $F(z) = \int_{z_0}^{z} f(z)\,dz$ then $\frac{dF}{dz} = f$.

This integral is defined by choosing a path from z_0 to z but the answer does not depend on the path. (Because the integral around a loop is 0.)

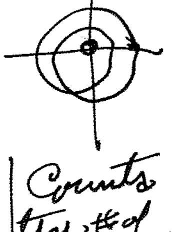


Classic case. To construct a branch of $\log(z)$ in a disk around 1 we define $\log(z) = \int_1^z \frac{du}{u}$.



(Because we know that if $\log$ exists its derivative is $\frac{1}{z}$. Thus we construct $\log$ by integrating $\frac{1}{z}$.)

This fails if our disk contains 0 because in that case the integral does depend on the path.



Def. $\frac{1}{2\pi i} \int_\gamma \frac{dz}{z}$ is the winding number of γ (because $\frac{1}{z}$ has residue 1 at 0) with respect to 0. This winding # is the obstruction to defining $\log z$ in a nbd. of 0.

Counts the # of times γ goes around

Also useful to consider the obstruction to constructing $\log f(z)$.

We don't know whether $\log f$ exists in a disk D but we know what its derivative should be and that is $\frac{d}{dz} \log f(z) = \frac{f'(z)}{f(z)}$.

We can try to define $\log f(z)$ by considering $\int \frac{f'}{f}\, dz$.

Write $u = f(z)$

$\int \frac{du}{u} = \int \frac{f' dz}{f}$

What we find is that the zeros of f give obstructions.

Prop. The residue of $\frac{f'}{f}$ at a zero of f is the order of the zero. ~~this is also the same~~

Proof. $f(z) = (z-z_0)^n \cdot g(z)$ where $g(z_0) \neq 0$

then
$$\frac{f'}{f} = \frac{n(z-z_0)^{n-1} \cdot g(z) + (z-z_0)^n g'(z)}{(z-z_0)^n g(z)}$$

$$= \frac{n}{(z-z_0)} + \frac{g'(z)}{g(z)}$$

holomorphic. (residue = 0)

Corollary. The number of zeros of a holomorphic function inside D is $\frac{1}{2\pi i}\int_{\partial D}\frac{f'}{f}\,dz$.

Note: $\frac{1}{2\pi i}\int_{\partial D}\frac{f'}{f}\,dz$ is an integer. (Note: This is the winding # of $f\circ\gamma$ around 0. Change of variables formula.) Example sheet

If we change f slightly without introducing any zeros on the boundary then $\frac{1}{2\pi i}\int\frac{f'}{f}\,dz$ varies continuously so it is constant.

We can do a similar analysis of the function

$$\frac{z\cdot f'(z)}{f(z)}.$$

Lemma. The residue of $\frac{z f'}{f}$ at a zero z_0 of f is $z_0\cdot$ order of the zero.

Proof. Say $f(z)=(z-z_0)^n\cdot g(z)$ then

$$\frac{z\cdot f'}{f}=\frac{z\cdot n\cdot(z-z_0)^{n-1}g(z)+z(z-z_0)^n\cdot g'(z)}{(z-z_0)^n\cdot g(z)}.$$

$$=\frac{z\cdot n}{(z-z_0)}+\frac{z\cdot g'}{g}=\frac{((z-z_0)+z_0)n}{(z-z_0)}+\frac{z\cdot g'}{g}$$

$$=n+\frac{n z_0}{(z-z_0)}+\frac{z g'}{g},$$

hol. hol.

Corollary. $\frac{1}{2\pi i}\int_{\partial D} \frac{z\cdot f'(z)}{f(z)}\,dz = \sum_{z_0 : f(z_0)=0} z_0\cdot(\text{ord of zero}).$

If there is just 1 simple zero in D then it is given by the integral.

We are now ready to prove the implicit function theorem.

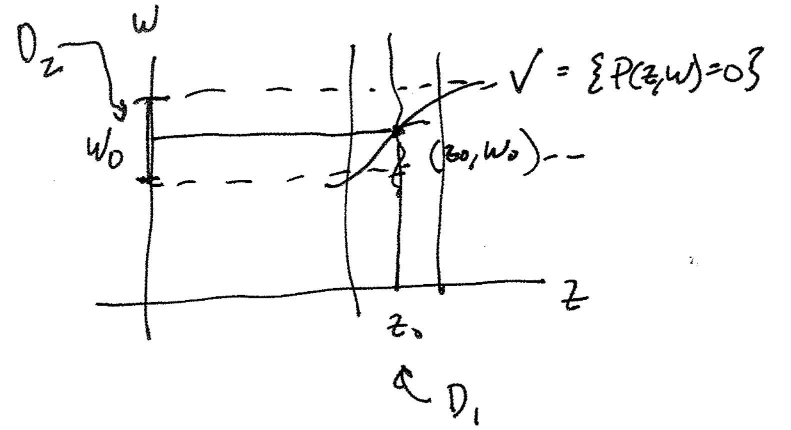
Let $P(z,w)$ be a polynomial in 2 complex variables. Let $V=\{(z,w)\in\mathbb{C}^2 : P(z,w)=0\}$.

Thm. Let $(z_0,w_0)\in V$ and assume that $\frac{\partial P}{\partial w}\neq 0$ at (z_0,w_0). Then there is a disk D_1 in $\mathbb{C}$ centered at z_0 and a disk D_2 centered at w_0 and a holomorphic function $\phi: D_1\to D_2$ with $\phi(z_0)=w_0$ such that

$$V\cap(D_1\times D_2)=\{(z,\phi(z)) : z\in D_1\}.$$

That is to say that V is locally a graph of a 1-variable function.

Implicit function theorem.



assumption.
$\frac{\partial P}{\partial w}(z_0,w_0) \neq 0$

To show: V is locally the graph of a function

Consider the function $\psi(w) \mapsto P(z_0,w)$.

ψ has a zero at $w=w_0$. $\frac{d}{dw}\psi\Big|_{w=w_0} = \frac{\partial P}{\partial w}(z_0,w_0) \neq 0$

so $\psi(w) = a_1(w-w_0) + a_2(w-w_0)^2 + \dots$ with $a_1 = \frac{d}{dw}\psi\Big|_{w=w_0} \neq 0$

Since zeros of holomorphic functions are isolated there is a disk D_2 around w_0 so that ψ has only one zero in D_2.

According to our lemma $\frac{1}{2\pi i}\int_{\partial D_2} \frac{\psi'}{\psi}\, dw = 1$.

Now let $\psi_z(w) = P(z,w)$. (not a derivative)

The integral counts solutions with multiplicity.

For each $w \in \partial D_2$ there is an ε_w so that $\psi_z(w) \neq 0$ for $|z-z_0| < \varepsilon_w$. Using compactness of the circle there is a single ε so that $\psi_z(w) \neq 0$ for all $w \in \partial D_2$ and $|z-z_0| < \varepsilon$, Let $D_1 = \{z : |z-z_0| < \varepsilon\}$

Thus the integrand of $\frac{1}{2\pi i}\int_{\partial D_2} \frac{\psi_z'(w)}{\psi_z(w)}\,dw$ varies continuously. It follows that the integral varies continuously and that the integral is constant, since it takes integral values.

Let $D_1 = \{z : |z-z_0| < \varepsilon\}$. For each $z \in D_1$ the equation $\psi_z(w)=0$ has a unique solution in D_2 of mult. 1.

Now recall that we have a formula for that solution: Define

(when it is unique) $$\phi(z) = \frac{1}{2\pi i}\int_{\partial D_2} w\,\frac{\psi_z'(w)}{\psi_z(w)}\,dw.$$

(phi) (Psi)
$w = \phi_\psi(z)$ is the unique soln of $\psi_z(w)=0$.

Since the integrand varies continuously in z, $\phi(z)$ is continuous. More is true though. For each w the integrand is a holomorphic function in z. In fact this implies that (Leibniz thm.)

ϕ is holomorphic. This follows from approximating the Riemann integral by a Riemann sum, checking that Riemann sums vary holomorphically in order to see that ϕ is a uniform limit of holomorphic functions. A uniform limit of holomorphic functions is holomorphic.

Theorem. Let $V = \{P(X,Y) = 0\}$ be an affine plane curve in $\mathbb{C}^2$. Assume that ~~V~~ V has no singular points then V has a Riemann surface atlas for which coordinate projections are holomorphic.

Proof. The assumption that V has no singular points means that there are no common zeros of $P(X,Y)$, $\frac{\partial P}{\partial X}(X,Y)$ and $\frac{\partial P}{\partial Y}(X,Y)$

Let $(X_0,Y_0) \in V$. Either $\frac{\partial P}{\partial X}(X_0,Y_0)$ or $\frac{\partial P}{\partial Y}(X_0,Y_0)$ is non-zero.

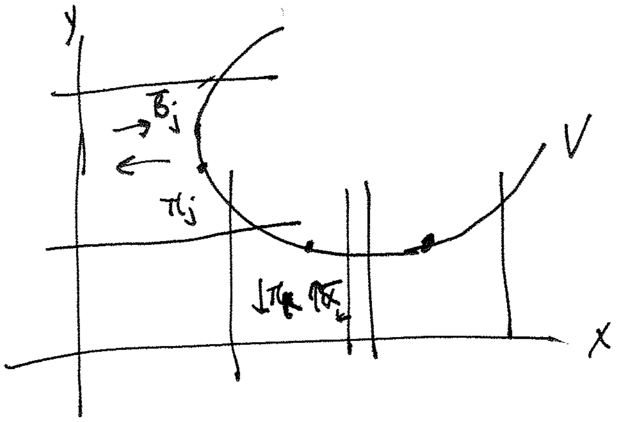
If $\frac{\partial P}{\partial Y} \neq 0$ we have D_1, D_2 so that $V \cap D_1 \times D_2 \ni (X_0,Y_0)$

$= \{(X, \alpha_j(X)) \cdot X \in D_1\}$. Define the chart

$\pi_X : V \cap \mathring{D}_1 \times \mathring{D}_2 \to \mathring{D}_1$ to be the restriction of the coordinate projection. $\bar{\alpha}(X) : D_1 \to D_1 \times D_2$ given by $\bar{\alpha}_j(X) = (X, \alpha(X))$ is an inverse to π_X showing that π_X is a homeomorphism.

If $\frac{\partial P}{\partial X} \neq 0$ then again we have D_1, D_2 so that $V \cap D_1 \times D_2 = \{(\beta(Y), Y) : Y \in D_2\}$ by interchanging the coordinates we get charts $\bar{\beta}_k(Y) = (\beta(Y), Y)$.

MS.05



This gives a collection of charts taking values in the x or y axes. Overlaps between two charts in the x-axis have the form:

$$x \xrightarrow{\bar{\alpha}} (x, \alpha(x)) \xmapsto{\pi_x} x$$

and are holomorphic.

Overlaps between an x-valued chart and a y-valued chart have the form:

$$x \xrightarrow{\bar{\alpha}} (x, \alpha(x)) \xrightarrow{\pi_y} \alpha(x).$$

also holomorphic. The collection of charts $\{\pi_x, \pi_y\}$ give an atlas.

$V = \{y^2 = f(x)\} = \{(x,y) : P(x,y) = 0\}$ where $P(x,y) = -y^2 + f$

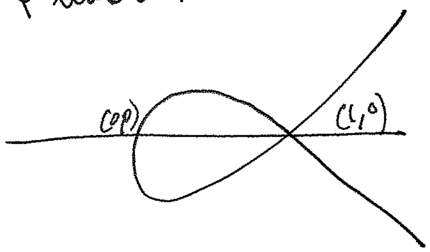
Example: $f(x) - y^2 = 0$ where $f(x) = ax^3 + \dots d$.

$$y^2 = ax^3 + bx^2 + cx + d$$

$\frac{\partial P}{\partial y} = -2y$ $\quad \frac{\partial P}{\partial x} = f'(x)$. Simultaneous zero of $\frac{\partial P}{\partial y}, \frac{\partial P}{\partial x}, P$ satisfies $y = 0$, $f'(x) = 0$, $f(x) = 0$. Thus it is a ~~higher~~ repeated root of f. of the form $(0, x)$ where x is a repeated root of f.

If f has distinct roots this is a Riemann surface.

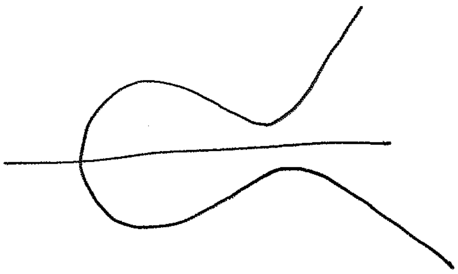
cubic node



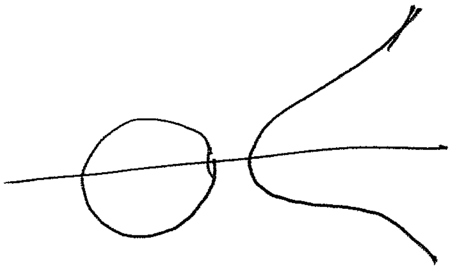
$$Y^2 = X(X-1)^2$$

↑ repeated root.

Non-singular:



$$Y^2 = X(X-1)^2 + 1/10,$$



$$Y^2 = X(X-1)^2 - 1/10$$

Def. This is an ~~example~~ of an elliptic curve if $\deg f = 3$ or 4. ~~If~~ It is a hyperelliptic curve if $\deg f \geq 5$.

We could consider the case of projective plane curves by checking each chart but there is a snappier way.

Thm. The projective curve $V \subset \mathbb{CP}^2$ given by the homogeneous polynomial $P(X,Y,Z)$ $\left(V = \{(X:Y:Z) : V(X,Y,Z)=0\} \right)$ is a Riemann surface if the only simultaneous solution of $\frac{\partial P}{\partial X} = 0$, $\frac{\partial P}{\partial Y} = 0$, $\frac{\partial P}{\partial Z} = 0$ and $P = 0$ is $(0,0,0)$.

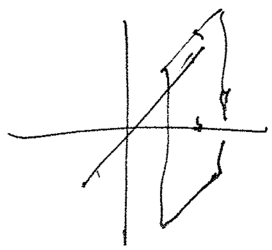
Lemma. $x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y} + z\frac{\partial P}{\partial z} = d \cdot P$ ← degree of P.

Proof of Lemma. We have $P(\lambda x, \lambda y, \lambda z) = \lambda^d P(x,y,z)$.
Differentiate both sides with respect to λ.

$$\frac{\partial}{\partial \lambda} P(\lambda x, \lambda y, \lambda z) = x\frac{\partial P}{\partial x} + y \cdot \frac{\partial P}{\partial y} + z \cdot \frac{\partial P}{\partial z}$$ by the chain rule,

$$\frac{\partial}{\partial \lambda} \lambda^d P(x,y,z) = d\lambda^{d-1} P(x,y,z).$$ Now set $\lambda = 1$.

Proof of Theorem. Pick a point $(x_0 : y_0 : z_0) \in V$.
Want to find a chart of $\mathbb{CP}^2$ in which this point is regular (in the affine sense).



Say $x_0 \neq 0$. Consider $\left(1, \frac{y_0}{x_0}, \frac{z_0}{x_0}\right)$.

$(y,z) \xrightarrow{\phi_x} (1,y,z)$. Let $P_x(y,z) = P(1,y,z)$.

$V_x = \{(y,z) : P_x(y,z) = 0\}$.

Want to know that $\frac{\partial P_x}{\partial y} \neq 0$ or $\frac{\partial P_x}{\partial z} \neq 0$

Assume they both vanish then

$$1 \cdot \frac{\partial P}{\partial x} + y \cdot \frac{\partial P}{\partial y} + z \cdot \frac{\partial P}{\partial z} = \frac{\partial P}{\partial x} \quad \text{at } (1,y,z)$$

$$= 0$$ by Lemma

So all first partials vanish contradicting our assumption.

Example

$$Y^2 = X^3 + aX^2 + bX + c = f(x) \qquad a \neq 0$$

$$P(X,Y) = -Y^2 + X^3 + aX^2 + bX + c$$

$$Q(X,Y,Z) = -Y^2Z + X^3 + aX^2Z + bXZ^2 + cZ^3.$$

$$\frac{\partial Q}{\partial X} = 3X^2 + 2aXZ + bZ^2$$

$$\frac{\partial Q}{\partial Y} = -3YZ$$

$$\frac{\partial Q}{\partial Z} = -Y^2 + aX^2 + 2bZX + 3cZ^2$$

$Y^2 = F(X)$

Hyperelliptic surface.
Every cubic and quartic can be put in this form.

elliptic if deg $= 3$ or 4

If $Z \neq 0$ we have checked before that this is non-singular, if $f(x) = x^3 + ax^2 + bx + c$ has simple zeros.

When $Z=0$ we get

$$0 = 3X^2$$
$$0 = 0$$
$$0 = -Y^2 + aX^2$$
$$0 = X^3$$

If these vanish then $X=0$ and $Y=0$ so the only solution of $Q = \frac{\partial Q}{\partial X} = \frac{\partial Q}{\partial Y} = \frac{\partial Q}{\partial Z} = 0$ is $X=Y=Z=0$.

We conclude that
Our projective curve is non-singular.