Meromorphic fcns. on $\mathbb{C}P^1$ are rationals.

Notation. Let $f: R \to S$ be proper & hol, and non-constant.

For $q \in S$ write $\deg_q(f)$ for

$$\sum_{f(p)=q} \nu_p(f).$$

Then if S is connected we have $\deg(f) = \deg_q(f)$ which is independent of q. "Local degree formula".

Prop.

$f: R \to S$

A non-constant map between compact connected Riemann surfaces is (where R is compact) a branched covering. In particular it has a degree $\deg(f) = \deg_q(f)$ for any $q \in S$.

Proof. If R is compact then f is automatically proper.

Prop. Let $f: R \to \mathbb{C}$ be a holomorphic function where R is compact then f is constant.

Proof. Clearly $f(R)$ is compact. On the other hand, since R is compact f is proper and so $\deg f = \deg_q f$ is constant. Assume $\deg f > 0$. We can see this by picking a $q \in f(R)$

$\deg f = \deg_q f = \sum_{f(p)=q} \nu_p(f)$ where $\nu_p(f) \geq 1$ at every $p \in f^{-1}(q)$. Since $\deg_q(f)$ is independent of q, f is surjective which contradicts the assertion that $\mathrm{im}(f)$ is compact

This result tells us that holomorphic functions are not a useful tool in studying the compact Riemann surfaces.

On the other hand meromorphic functions on compact Riemann surfaces are very useful.

Recall that a meromorphic functions can be viewed as holomorphic functions to $\mathbb{C} \cup \{\infty\}$, which, if they are constant, do not take the value ∞. The points where $f(p) = \infty$ are called poles and are isolated.

Since $\mathbb{CP}^1$ is compact there are only finitely many poles. If p is a pole then we can choose a a local coordinate $\varphi : U \xrightarrow{\ni p} \mathbb{C}$ with

$f \circ \varphi^{-1}(z) = \sum_{j \in \mathbb{Z}} a_j z^j$ and $a_j = 0$ for $j \leq N$.

$\mathrm{ind}_p(f) = n$ where $a_n \neq 0$ but $a_j = 0$ for $j < n$.

$= \infty$ if all a_n vanish.

Say we have a holomorphic function defined in the complement of some discrete set of points. When is it meromorphic?

If f is a holomorphic function defined in $U-p$ then we can still write a Laurent ~~power~~ series for

$$f\circ\varphi^{-1}(z) = \sum_{j\in\mathbb{Z}} a_j z^j$$

Here we also have the possibility that $a_j \neq 0$ for infinitely many negative j's. In this case we write $\mathrm{ind}_p(f) = -\infty$. and observe that it is independent of φ.

Observation. A holomorphic function $f: R-\Sigma \to \mathbb{C}$ (with Σ discrete in R) has a holomorphic extension to $\bar{f}: R \to \mathbb{C}\cup\{\infty\}$ iff $\mathrm{ind}_p(f) > -\infty$ at each $p\in\Sigma$.

Also observe that if f, g are holomorphic in the complement of p then

$\mathrm{ind}_p(f\cdot g) = \mathrm{ind}_p(f) + \mathrm{ind}_p(g)$. in the sense that if any 2 of these are finite then the third is as well.

$\mathrm{ind}(f+g) \geq \min(\mathrm{ind}(f), \mathrm{ind}(g))$.

$f\circ\varphi^{-1}(z) = z^{\mathrm{ind}_p(f)} \cdot h$ where h is holomorphic and non zero at 0.

$g\circ\varphi^{-1} = z^{\mathrm{ind}_p(g)} \cdot h'$

$f\circ\varphi^{-1} \cdot g\circ\varphi^{-1} = z^{\mathrm{ind}(f)+\mathrm{ind}(g)} \cdot h \cdot h'$.

p is a zero of f if $\mathrm{ind}_p(f) > 0$
pole $\mathrm{ind}_p(f) < 0$
finite value $(\neq 0)$ $\mathrm{ind}_p(f) = 0$.

If p is a zero then $\mathrm{ind}_p(f)$ is $\nu_p(f)$

the local degree of f.
(mapping p to 0)

If p is a pole then $\mathrm{ind}_p(f)$ is $-\nu_p(f)$
(mapping p to $\infty \in \mathbb{CP}^1$).

Theorem. If f is a meromorphic function on a compact connected Riemann surface R then the number of zeros of f is equal to the number of poles counted with multiplic
(where the mult. of a zero is its index and the ~~pole~~ mult. of a pole is the negative of the index.)

Proof. We have $\deg(f) =$ $\deg_0(f) = \deg_\infty(f)$.

$\deg_0$

$$\deg_0 f = \sum_{f(p)=0} \nu_p(f) = \sum \mathrm{ind}_p(f).$$

$$\deg_\infty f = \sum_{f(p)=\infty} \nu_p(f) = \sum -\mathrm{ind}_p(f).$$

To see the second statement recall that we get a chart φ around 0 by considering $\frac{1}{\varphi(z)} = \frac{1}{f(z)}$. & $\nu_p(f) = \mathrm{ind}_p\left(\frac{1}{f}\right) = -\mathrm{ind}_p(f)$

since $f \cdot \frac{1}{f} = 1$ and $\mathrm{ind}_p(f) + \mathrm{ind}_p(f^{-1}) = \mathrm{ind}_p(1) = 0$.

(~~cons~~ we consider $n=1\ \&\ 2$)

$R = P/Q$

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Def. A rational function on $\mathbb{CP}^n$ is the quotient of two functions P, Q on $\mathbb{C}^{n+1}$ where P, Q are homogeneous polynomi[als] of the same degree.

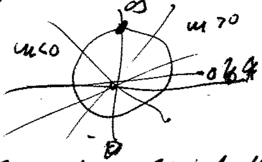
Note that R is constant on lines ~~are~~ through the origin on $\mathbb{C}^{n+1} - \{0\}$ since $R(\lambda x_0 \dots \lambda x_n)$

$$= \frac{P(\lambda x_0 \dots \lambda x_n)}{Q(\lambda x_0 \dots \lambda x_n)} = \frac{\lambda^d}{\lambda^d} \frac{P(x_0 \dots x_n)}{Q(x_0 \dots x_n)} = R(x_0 \dots x_n).$$

Thus we can think of R as being well defined on $\mathbb{CP}^n$ where P and Q are not both 0.
When $Q = 0$ and $P \neq 0$ we say $Q = \infty$.
If both P and Q vanish then we have a point of indetermi[nacy]

If $n=1$ then we can devide P and Q by common factors and assume that R is well defined at every point.

Recall $P(X,Y)$ is homo. then $P(X,Y) = \prod(\cancel{Y} - r_j \dots$



$\cdot X^{n_\infty}$

zero at ∞ zero at [r]

If $n>1$ this is not always possible.

If $n=1$ a rational function is a meromorphic function.

Theorem. Every meromorphic function on $\mathbb{CP}^1$ is rational.

Remark: To be meromorphic is an analytic property ~~Conditions~~ [illegible]
(Related to convergent power series. as many coefs.)

Rationality is an algebraic property.
(Related to polynomials. Finitely many coefficients.) Result is very strong.

Analyticity + compactness is a powerful assumption. We will see this principal applied again

$\mathbb{C} \cup \infty \quad \mathbb{C} \cup \infty \leftarrow$ this notation indicates a preferred coord. chart.

Proof. Let $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ be meromorphic.
f is a branched covering map from a compact space so it has a finite collection of zeros $z_1 \ldots z_n$ and poles $p_1 \ldots p_m$. Let us construct a rational

function with the same collection of finite zeros and poles.

Let $P(X,Y) = \prod (X - \cancel{z_j}\, Y)^{u_j}$ $\quad z_j$ a zero of f

$Q(X,Y) = \prod (X - \cancel{p_j}\, Y)^{z_j}$ $\quad u_j$ a pole of f.

P and Q need not have the same degree but we can adjust the degrees by including a power of Y^n, say $P(X,Y) = Y^n \prod (X - z_j Y)^{u_j}$

$Q(X,Y) = \prod (X - p_j Y)^{z_j}$

This may change the # of roots at ∞.

Now $R(X,Y) = \dfrac{P(X,Y)}{Q(X,Y)}$ and f have the same finite zeros and poles with same multiplicity but we know nothing about how the values or ~~value a~~ behavior at ∞.

Say f has a pole ~~at ∞ then~~

more finite zeros than poles then

since $\quad$ the sum of multiplicity

of zeros is equal to the sum of ~~mo~~ the multiplicity of the poles f must have a pole at ∞ and its multiplicity is the difference of the sum of multiplicity of the finite zeros and the multiplicities of the finite poles.

Same argument applies to R, its value at ∞ is determined by the difference ~~of~~ of the multiplicities of the zeros and poles.

$\mathrm{ind}_\infty(f) = \mathrm{ind}_\infty(R)$.

Now consider the quotient f/R.

At ~~an~~ any $p \in \mathbb{CP}^1$ $\mathrm{ind}_p(f/R) = \mathrm{ind}_p(f) - \mathrm{ind}_p(R) = 0$

so f/R has a holomorphic extension to a function $f/R : \mathbb{CP}^1 \to \mathbb{C} \cup \infty$ which never takes the value ∞. In particular $\frac{f}{R}$ is a holomorphic function from a compact Riemann surface to $\mathbb{C}$ so it must be constant. So $f = c \cdot R$ and $c \cdot R$ is again a rational function.

Claim: $\operatorname{ind}_\infty(f) = -\sum_{w\in\mathbb{C}} \operatorname{ind}_w(f)$.

Proof. $\sum_{z_j\in\mathbb{C}\cup\infty} \nu_{z_j}(f) = \sum_{p_j\in\mathbb{C}\cup\infty} \nu_{p_j}(f)$

$$\sum_{z_j\in\mathbb{C}\cup\infty} \operatorname{ind}_{z_j}(f) = -\sum_{p_j\in\mathbb{C}\cup\infty} \operatorname{ind}_{p_j}(f)$$

So $\sum_{z_j\in\mathbb{C}\cup\infty} \operatorname{ind}_{z_j}(f) + \sum_{p_j\in\mathbb{C}\cup\infty} \operatorname{ind}_{p_j}(f) = 0$

$$\sum_{w\in\mathbb{C}\cup\infty} \operatorname{ind}_w(f) = 0$$

$$\operatorname{ind}_\infty(f) + \sum_{w\in\mathbb{C}} \operatorname{ind}_w(f) = 0,$$

also $\operatorname{ind}_\infty(R) = -\sum_{w\in\mathbb{C}} \operatorname{ind}_w(R) = -\sum_{w\in\mathbb{C}} \operatorname{ind}_w(f)$.

So $\operatorname{ind}_\infty(R) = \operatorname{ind}_\infty(f)$.

Remark. The algebraic degree of a rational ~~meromorphic~~ function on $\mathbb{CP}^1$ is the same as its topological degree.

$R(X:Y) = \dfrac{P(X,Y)}{Q(X,Y)}$ defines a map from

lines in $\mathbb{C}^2$

$\mathbb{CP}^1$ to $\mathbb{C}\cup\infty\ (=\mathbb{CP}^1)$. (Notation distinguishes between 0 and ∞)
$(X:Y)$ $\quad z\ (= Y/X)$ (slope coordinates)

If P and Q have no ~~factors~~ linear factors in common then the algebraic degree is $\deg(P) = \deg(Q)$.

The topological degree is $\displaystyle\sum_{(X:Y)\in\mathbb{CP}^1} \nu_{(X:Y)} R$ = # zeros = # poles

(counted with multiplicity). Since $P = X^{d_0}\prod(Y - \rho_j X)^{d_j}$

# of zeros = deg P.

To formalize the last argument in the proof from Friday
Useful to rewrite the local expression for degree in terms of $\operatorname{ind}(f)$ instead of $\nu(f)$.

$f: \mathbb{C}\cup\infty \to \mathbb{C}\cup\infty$.

Let z_j be the zeros of f, p_k the poles

We have $\displaystyle\sum_{z_j\in\mathbb{C}\cup\infty} \nu_{z_j}(f) = \sum_{p_k\in\mathbb{C}\cup\infty} \nu_{p_k}(f)$

$$\sum_{z_j\in\mathbb{C}\cup\infty} \operatorname{ind}_{z_j}(f) = -\sum_{p_k\in\mathbb{C}\cup\infty} \operatorname{ind}_{p_j}(f)$$

so $\displaystyle\sum_{w \in \mathbb{C} \cup \infty} \mathrm{Ind}_w(f) = 0$ since $\mathrm{Ind}_w(f) = 0$ if w is neither a zero or pole.

Thus $\displaystyle \mathrm{Ind}_\infty(f) + \sum_{w \in \mathbb{C}} \mathrm{Ind}_w(f) = 0$ or

$$\mathrm{Ind}_\infty(f) = -\sum_{w \in \mathbb{C}} \mathrm{Ind}_w(f).$$

Conclusion of argument from Friday:

Consider f, a meromorphic function and R a rational function. These have been constructed in order to have the same ~~finite~~ zeros and poles. Since

$$\mathrm{Ind}_\infty(f) = -\sum_{w \in \mathbb{C}} \mathrm{Ind}_w(f) = -\sum_{w \in \mathbb{C}} \mathrm{Ind}_w(R) = \mathrm{Ind}_\infty(R)$$

they also have the same index at ∞.

Now consider $\frac{f}{R}$. A priori this is only defined where f and R are finite and non-zero since $\frac{0}{0}$ and $\frac{\infty}{\infty}$ do not have clear values. Nevertheless we can calculate the indices of f and R at any pt. using the def of the index in the deleted nbd. (Recall that $\nu_p(f)$ only makes sense when f has an actual value at p.)

Outside of $\{z_1 .. z_n, p_m, \infty$

We calculate that $\mathrm{ind}_w\left(\frac{f}{R}\right) = \mathrm{ind}_w(f) - \mathrm{ind}_w(R)$ $= 0$ at every point in $\mathbb{C} \cup \infty$. This means that $\frac{f}{R}$ has a holomorphic (non-zero) extension to each of these deleted points. It follows that $\frac{f}{R}$ is constant so $\frac{f}{R} = c$ and $f = c \cdot R = \frac{cP}{Q}$ is rational.

Remark. To say that f is meromorphic is an analytic and local property in terms of Laurent series. To say that f is rational is an algebraic and global property.

Equality here indicates why analytic objects can be studied purely algebraically.

Prop. If $f : R \to \mathbb{C} \cup \infty$ is meromorphic and R is compact then

① If f has no poles then f is constant

② If f has 1 pole counted with multiplicity then R is conformally equivalent to $\mathbb{CP}^1$.

Remark. We have shown that in $\mathbb{CP}^1$ we can assign zeros and poles arbitrarily subject only to the condition that sums of indices are zero. This is the

Compact Riemann surface in which we can do this

Proof. If f has only one pole then the degree of f is 1. This means that for each $q \in \mathbb{C} \cup \infty$
$$\sum_{f(p)=q} \nu_p = 1.$$ Since $\nu_p \geq 1$ this means that $\# f^{-1}(q) = 1$. So f is a holomorphic bijection. A holomorphic bijection is a conformal automorphism.

Prop. Every conformal automorphism of $\mathbb{C} \cup \infty$ is a linear fractional transformation $z \mapsto \frac{az+b}{cz+d}$ with $ad-bc \neq 0$.

Proof. A holomorphic map f is a rational map. $f = \frac{P}{Q}$.
where P, Q have no common zeros. Since it is an automorphism
it has topological degree 1. Thus it has algebraic degree 1 $z \mapsto \frac{az+b}{cz+d}$ (using $\mathbb{C} \cup \infty$ notation). The condition $ad-bc \neq 0$ is equivalent to P, Q having no common zero.

We can also define rational functions on $\mathbb{CP}^2$.

Def: $R(X:Y:Z) = \frac{P(X,Y,Z)}{Q(X,Y,Z)}$ where P and Q are homogeneous polynomials of the same degree.

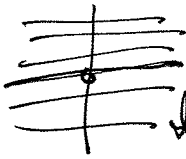
A rational "function" fails to be a well defined function where P and Q have common zeros. In the case of $\mathbb{CP}^1$ we could remove common linear factors from P and Q so that this did not occur. In the case of $\mathbb{CP}^2$ we ~~can~~ may not be able to do this.
For example consider

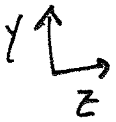
$R(X:Y:Z) = Y/X.$

Lets look at this function with respect to a coordinate chart.

$(Y,Z) \mapsto (1:Y:Z)$, ~~$R(X:Y,Z) = Y/X$~~

$R(1:Y:Z) = Y$

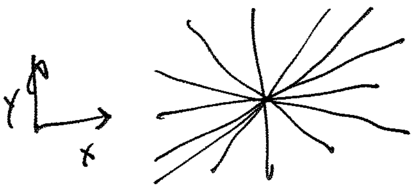




In this chart R is linear.

Consider the chart

$(X,Y) \mapsto (X:Y:1)$ $\quad (X,Y) \mapsto Y/X$



Here R is undefined at $X=Y=0$.
and R has no continuous extension to $X=Y=0$.

We call such a point a point of indeterminacy.

It is nevertheless true that a rational function on $\mathbb{CP}^2$ restricts to a rational function on a projective algebraic curve $R \subset \mathbb{CP}^2$.

Prop. If C is a non-singular projective algebraic curve then there exist non-constant meromorphic functions on C.

Proof. Consider $R(X:Y:Z) = \dfrac{Y-Y_0}{X-X_0}$

where $(X_0 ; Y_0 ; 1)$ is not a point in the curve

alternatively we can shift C so that it does not contain $(0:0;1)$.

In fact if C is not the line at ∞ $Z=0$ we can consider the restrictions of $\frac{X}{Z}$ and $\frac{Y}{Z}$ to C and we get two meromorphic functions.

If C is the curve corresponding to $P(X, Y, Z)=0$ where P is a homogeneous polynomial then

$$P\left(\frac{X}{Z}, \frac{Y}{Z}, 1\right) = 0$$

Say

$P(X, Y, Z)$ $\quad X^2+Y^2-Z^2=0$

Want to think about the ring of functions on C