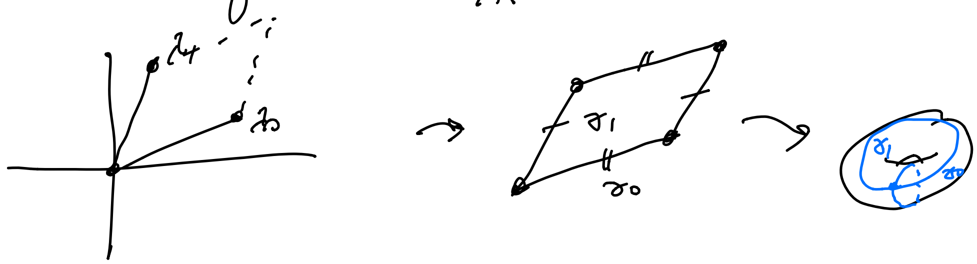
Space of marked surfaces

$$\mathcal{U} = \{(\lambda_0, \lambda_1) \in \mathbb{C}^2 : \det[\lambda_0\ \lambda_1] > 0\}.$$

(λ_0, λ_1) gives rise to a lattice $\Lambda = \{m_1\lambda_0 + m_2\lambda_1 : m_1, m_2 \in \mathbb{Z}\}$

and a surface $T^2 = \mathbb{C}/\Lambda$



Two marked surfaces (λ_0, λ_1) (λ_0', λ_1') are conformally equivalent (as marked surfaces) iff

$$(\lambda_0', \lambda_1') = (c\lambda_0, c\lambda_1) \text{ with } c \in \mathbb{C}^*.$$

Recall the construction of $\mathbb{CP}^1\ (=\mathbb{C}_\infty) = \mathbb{C}^* \backslash \mathbb{C}^2 - \{(0,0)\}$.

We get coordinate charts by sending

(λ_0, λ_1) to $\underset{\phi_1}{\frac{\lambda_1}{\lambda_0}}$ or $\underset{\phi_0}{\frac{\lambda_0}{\lambda_1}}$ to $\mathbb{C}$.

In our case we look at

$$\phi_1(\mathcal{U}) = \mathbb{H} \subset \mathbb{C}.$$

Let $SL(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{Z}^4 : \det = 1 \right\}$.

There is a natural left action of $SL(2,\mathbb{Z})$ on $\mathbb{H}$ given by $z \mapsto \frac{az+b}{cz+d}$. Note that $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ acts trivially. Let $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z}) / \{\pm I\}$.

The space of unmarked tori corresponds to $\mathbb{H} / PSL(2,\mathbb{Z})$ (actually a right action given by a similar formula but not the same formula)

What does this quotient space looks like?

$$\frac{b\lambda + a}{d\lambda + c}$$

Find a fundamental domain $F \subset \mathbb{H}$.

This is a subset of $\mathbb{H}$ which intersects a typical orbit in 1 point.

Typically we can identify $\mathbb{H}/PSL(2,\mathbb{Z})$ with $F/\sim$ where we glue boundary edges together.

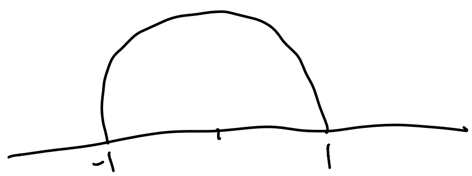
We can identify the orbit of a lattice Λ with the set of bases of Λ.

Finding a fundamental domain corresponds to the problem of given a lattice Λ selecting a particular basis. (up to complex scaling)

Given a lattice Λ give a recipe for finding a canonical oriented basis λ_0, λ_1. The set F will then correspond to the ratios $\frac{\lambda_1}{\lambda_0}$ for all canonical oriented bases.

Let λ_0 be the shortest non-zero element of Λ and let λ_1 be the shortest element of Λ which is not of the form $n\lambda_0$ so that (λ_0, λ_1) is oriented (replace λ_1 by $-\lambda_1$ if necessary).

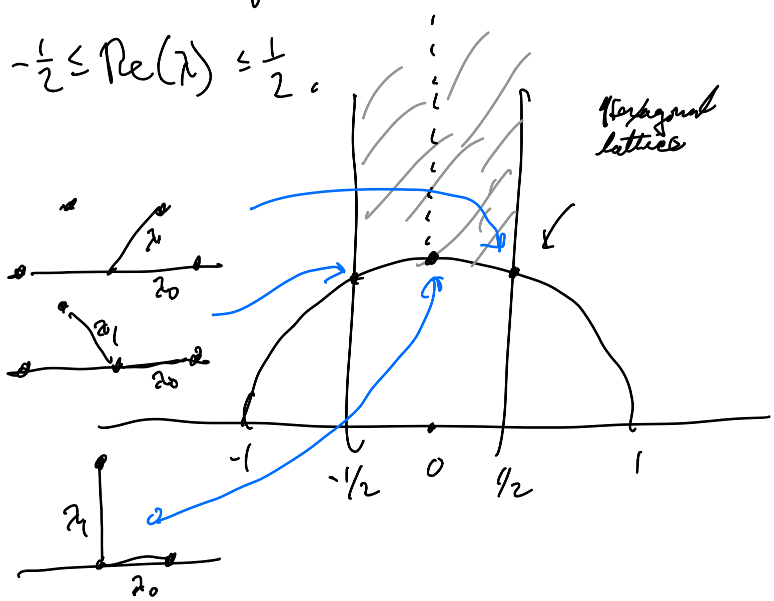
Plot $z = \frac{\lambda_1}{\lambda_0}$. Observe $|z| \geq 1$.



By scaling we can assume $z_0 = 1$ and $z_1 = \lambda$, $|\lambda| > 1$. If $\lambda \in \Lambda$ then so is $\Lambda + 1$ or $\Lambda + 2$.

Which of these is shortest? Must have

$-\frac{1}{2} \le \mathrm{Re}(\lambda) \le \frac{1}{2}$.



Def. Teichmüller space, moduli space.

For the torus both these spaces are again Riemann surfaces.

Metric of negative curvature on the moduli space.

In this geometric language we can identify a torus with a conformal structure with a point.

Recall that given a lattice Λ we defined a Weierstrass function P which satisfied

$$(P'(z))^2 = P^3(z) + g_2 P(z) + g_3.$$

g_2 and g_3 are functions on the space of lattices. What are they?

$$g_2 = 60 \sum_{\omega \in \Lambda - \{0\}} \omega^{-4} \qquad g_3 = 140 \sum_{\omega \in \Lambda - \{0\}} \omega^{-6}$$

$$\sum_{\lambda \in \Lambda - \{0\}} \frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}$$

$P(z) - z^{-2} = \lambda z^2 + \mu z^4.$
Differentiate both sides twice.
Plug in $z=0$.

Let $\mathcal{E}_d = \sum_{\omega \in \Lambda - \{0\}} \omega^{-d}$ for d even. These are examples modular forms and modular forms play an important role in number theory and many other areas of mathematics.

The rigidity phenomenon that we have seen for $\mathbb{C}_\infty$ and $\mathbb{C}/\Lambda$ works here to show establish polynomial relations between the E_k and show that every modular function can be written in terms of the E_k. This is an important tool in number theory. It leads to amazing explicit formulas for counting things.

To study Riemann surfaces of higher genus it is also convenient to look at the space of marked surfaces where we can think of a marking as a choice of a basis for π_1.

The collection of marked surfaces of genus g is a topological space which is called the Teichmüller space. Our discussion so far has constructed the Teichmüller space of surfaces of genus 1 which can be identified with $\mathbb{H}$.

For higher genera the Teich. space is again a manifold but of higher dimension. In fact it is a complex manifold with a holomorphic atlas.

This space has a natural metric on it.

We can interpret this metric as a measure of the minimal conformal distortion of maps between surfaces.

In analogy with the genus 1 case we can build a moduli space of surfaces of genus g by taking the quotient of the Teichmüller space by a discrete group action where this action preserves the complex structure and the metric. Teichmüller space is connected to interesting developments in topology. In particular the classification of 3-manifolds. Very popular at Warwick.