Next topic is surfaces in $\mathbb{C}^2 = \{(z,w) : z,w \in \mathbb{C}\}$.
These will be subsets of $\mathbb{C}^2$ defined by equations.

3 examples.

Riemann surfaces as objects on which multivalued functions become single valued.

Covering space associated to log.

"Graphs" in $\mathbb{C}^2$ of multivalued functions.

In the discussion I will make some assumptions which will be addressed later in the course.

Examples 1 and 2 could be handled more simply by other methods but I will introduce some techniques which will be useful in example 3.

Example 1: Let $R \subset \mathbb{C}^2$ be the graph of "$\log z$".

$$R = \{(z, w) : \text{"}w = \log z\text{"}\}.$$

Since $\log z$ is not an actual function we take this as suggestive.

More precisely $R = \{(z, w) : z = e^w\}$.

Viewing z as a function of w this is an actual graph. But we want to think of this as a "graph" in the other direction $w =$ "$\log z$".

In particular we want to view this in terms of the function $\pi_1 : \mathbb{C}^2 \to \mathbb{C}$ where $\pi_1(z, w) = z$.

Claim: $\pi_1 : R \to \mathbb{C} - \{0\}$ is a covering space.

Let $U_0 = \mathbb{C} - \{\text{neg real axis}\}$. Since U_0 is simply connected $\pi^{-1}(U_0)$ any covering space over U_0 is trivial so $\pi^{-1}(U_0)$ consists of disjoint sets each of which is mapped bijectively to U_0. Can decompose R into "copies of U_0".

$$L_{w_0}(z) = w_0 + \int_1^z \frac{d\zeta}{\zeta}$$

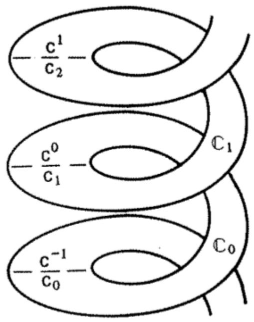


Figure 3.5.1.

$$L_w(z) = w + \int_{z_1}^{z} \frac{d\zeta}{\zeta}$$

This expression is well defined in V_0 since V_0 is simply connected.

How do you glue these sheets U_w together?

We would like to use path integration (of 1-forms) to construct an explicit formula for lifting paths.

We will make use of the formula:

$$\exp\left(\int_\gamma \frac{dz}{z}\right) = \frac{\gamma(1)}{\gamma(0)}.$$

There is an easy fake proof of this:

$$\exp\left(\int_\gamma \frac{dz}{z}\right) = \exp\left(\log z \Big|_{\gamma(0)}^{\gamma(1)}\right) = \exp\left(\log \gamma(1) - \log \gamma(0)\right)$$
$$= \frac{e^{\log \gamma(1)}}{e^{\log \gamma(0)}}$$
$$= \gamma(1)/\gamma(0).$$

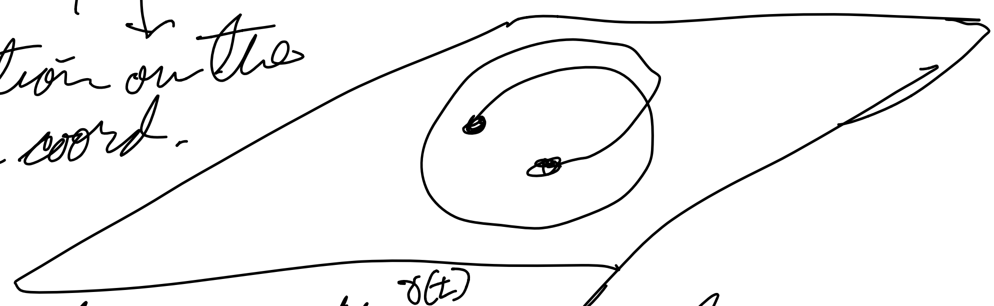
Since this proof uses the "function" $\log z$ it is not an actual proof. See Example sheet 2 for a better proof.

$\gamma : [0,1] \to \mathbb{C} - 0.$

Want to use the fact $\exp\left(\int_\gamma \frac{dz}{z}\right) = \frac{\gamma(1)}{\gamma(0)}$ to find an integral formula for lifting paths.

$R =$ 

π_1 projection on the first coord.



Start with a path $\gamma(t)$ in the base and a lift of the basepoint.

$\gamma : [0,1] \to \mathbb{C} - \{0\}$ ← path

lift of endpoint.

$(\gamma(0), w_0)$ → $\exp(w_0) = \gamma(0)$.

"lift" is in R.

Define:

$$\tilde{\gamma}(t) = w_0 + \int_0^t \frac{\gamma'(s)}{\gamma(s)}\,ds$$

Claim: $t \mapsto (\gamma(t), \tilde{\gamma}(t)) \in R$

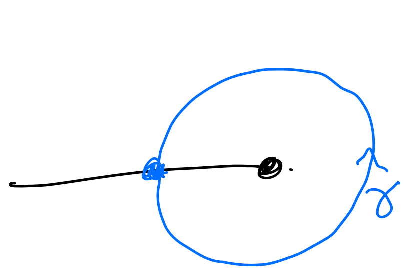
ie need to show $\exp(\tilde{\gamma}(t)) = \gamma(t)$

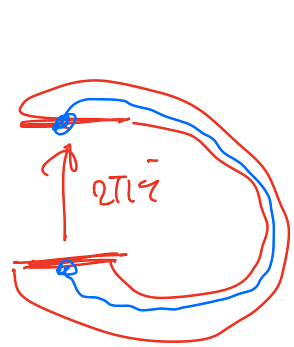
Now $\exp(\tilde{\gamma}(t))$

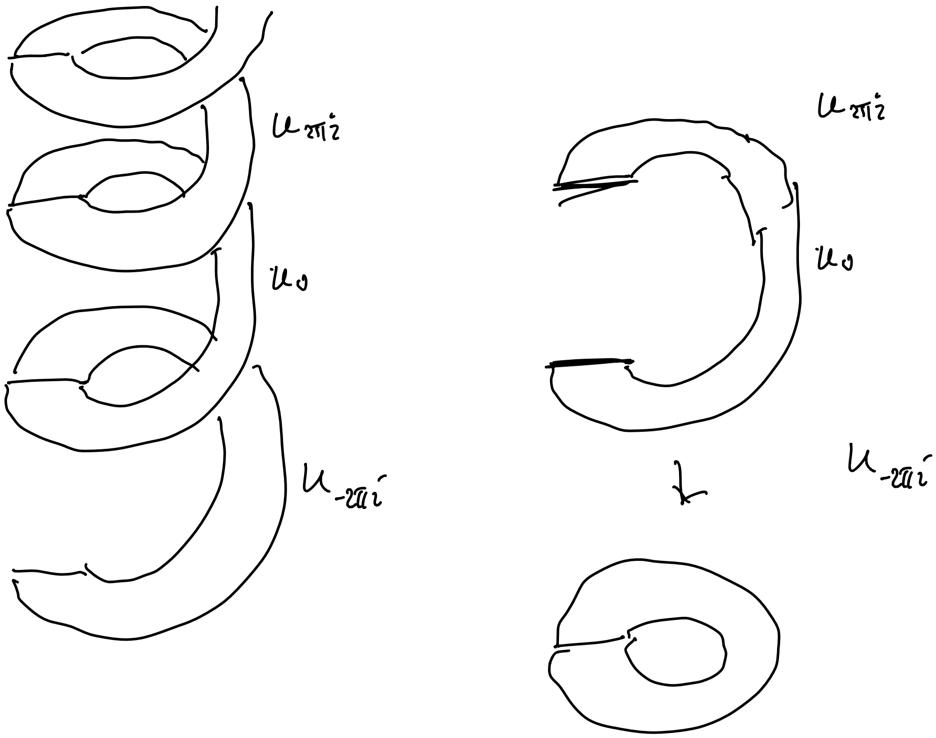
$$= e^{w_0} \cdot \exp \int_0^t \frac{\gamma'(s)}{\gamma(s)}\,ds$$

$$= e^{w_0} \cdot \frac{\gamma(t)}{\gamma(0)}$$

$$= \gamma(0) \cdot \frac{\gamma(t)}{\gamma(0)} = \gamma(t)$$







The "slit" downstairs lifts to two copies upstairs. We build the surface upstairs by gluing together copies of $U_{n\cdot 2\pi i}$ along slits.

Connection between surfaces and integrals.

Connection between topology & integrals.

$\pi^{-1}(z)$ consists of 2 pts. if $z \neq 0$

Next example: Covering space associated to the square root function.

$w = \sqrt{z}$

$R = \{(z, \sqrt{z}) : z \in \mathbb{C}\}$

$= \{(w^2, w) : w \in \mathbb{C}\}$

$= \{(z, w) : w = \sqrt{z}$ or $z = w^2\}$

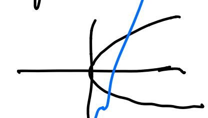
Use the fact that

$z^{1/2} = \exp \frac{1}{2} \log z.$

Use our path lifting construction for "log z".

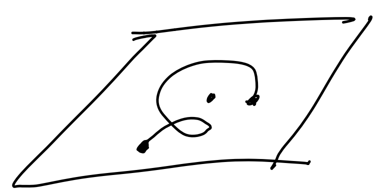
"$w = \exp\left(\frac{1}{2} \log z\right)$"

Real picture:



Projection onto the first coordinate gives a covering space away from $z=0$.

$\gamma(t) : [0,1] \longrightarrow \mathbb{C} - \{0\}$



Pick a lift of $\gamma(0)$ to R.

$(\gamma(0), w_0)$ where $w_0^2 = \gamma(0)$ $\quad (w_0 = \sqrt{\gamma(0)})$

Define $\tilde{\gamma}(t) = w_0 \cdot \exp\left(\frac{1}{2} \int_0^t \frac{\gamma'(s)}{\gamma(s)}\, ds\right)$

Lifted path should be $t \mapsto (\gamma(t), \tilde{\gamma}(t))$.

Check $(\gamma(t), \tilde{\gamma}(t)) \in R$.

In other words $\gamma(t) = \tilde{\gamma}^2(t)$. $\quad$ ($\tilde{\gamma}(t) = \sqrt{\gamma(t)}$ or $\tilde{\gamma}^2(t) = \gamma(t)$)

$$\tilde{\gamma}^2(t) = w_0^2 \cdot \exp^2\left(\frac{1}{2}\int_0^t \frac{\gamma'(s)}{\gamma(s)}\,ds\right)$$

$$= w_0^2 \cdot \exp \int_0^t \frac{\gamma'(s)}{\gamma(s)}\,ds$$

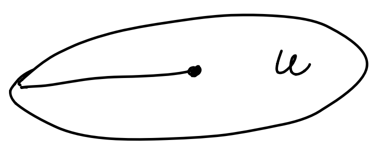
$$= w_0 \cdot \frac{\gamma(t)}{\gamma(0)} = \gamma(t).$$

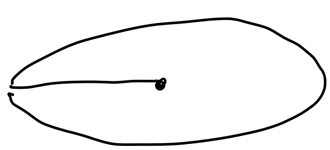
Use:

$$\exp \int_\gamma \frac{dz}{z} = \frac{\gamma(1)}{\gamma(0)}$$

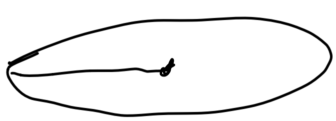
Consider $\mathbb{C}$ - neg. real axis. We have a covering space of degree 2.

Since U is simply connected U is evenly covered so $\pi^{-1}(U)$ consists of 2 copies of U. Call them U_0 and U_1.

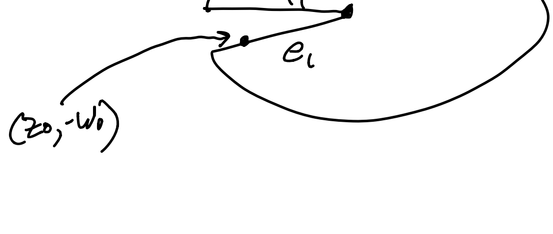




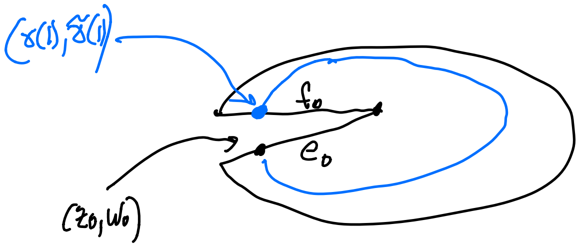
U_1



U_0



$U_1 \subset R$

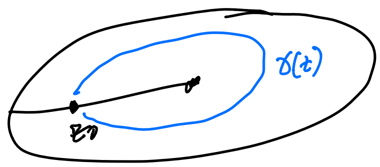


$u_0 \subset R$

$(\gamma(1), \tilde{\gamma}(1)) \in f_0$
Does f_0 connect to e_0 or e_1?

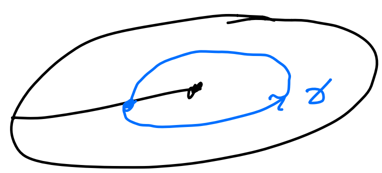
How do the "slits" pair up in R?

If (z_0, w_0)



$\tilde{\gamma}(1) = w_0$ | $\tilde{\gamma}(1) = -w_0$
$\boxed{\begin{array}{l} e_0 \sim f_0 \\ e_1 \sim f_1 \end{array}}$ or $\boxed{\begin{array}{l} e_0 \sim f_1 \\ e_1 \sim f_0 \end{array}}$
Case 1 | Case 2

We can see which pairing is compatible with our path lifting formula.



Say $\gamma(0) = \gamma(1)$ is in the slit.

Say $(\gamma(0), w_0) \in e_0$. Construct $\tilde{\gamma}(t)$ so that

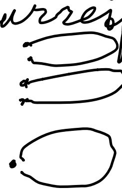
$(\gamma(t), \tilde{\gamma}(t))$ is a lift of γ.

Then $$\tilde{\gamma}(1) = w_0 \cdot \exp\left(\frac{1}{2}\int_\gamma \frac{dz}{z}\right) = w_0 \cdot \exp\left(\frac{1}{2} \cdot (2\pi i \cdot \text{wind}(\gamma, 0))\right)$$
$$= w_0 \cdot \exp(\pi i)$$
$$= -w_0.$$

If we restrict the map π to the inverse image of the circle we get a covering space of the circle of degree 2. There are exactly two possibilities:

We can distinguish these by their "monodromy", which is a map from π_1 of the base to the automorphisms of the fibre.

Case 1 corresponds to trivial monodromy
case 2 corresponds to non-trivial monodromy.



Monodromy corresponds to gluing of slits.