Last time we constructed Λ invariant functions $\mathcal{E}_\ell(z) = \sum_{\lambda \in \Lambda} (z-\lambda)^{-\ell}$ with poles of order ℓ exactly at lattice points for $\ell \geq 3$. We can alternatively think of $\mathcal{E}_\ell$ as a merom. fcn. on the compact Riem. surface $\mathbb{C}/\Lambda$. As such it has a degree and that degree is ℓ by the degree formula.

What about constructing a meromorphic function of degree 2?

One approach would be to integrate the 1-form $E_3 dz$. (Note that we are being careful to distinguish between functions f and 1-forms f dz here. We could get away with being sloppy here since we are working on $\mathbb{C}$. The underlying fact that is important is that the 1-form dz is Λ invariant.)

Presumably would produce something of the form $\sum_{\lambda \in \Lambda} \frac{1}{(z-\lambda)^2} + c_\lambda$ which was convergent.

constant of integration.

(Integral of inv. fcn need not be invariant.)

Instead we will simply pick constants c_λ that work. Consider

$$\frac{1}{z^2} + \sum_{\lambda \in \Lambda - \{0\}} \frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}$$

c_λ

for $\lambda = 0$ $c_\lambda = 0$

Then
$$\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} = \frac{\lambda^2 - (z-\lambda)^2}{\lambda^2 (z-\lambda)^2} = \frac{\lambda^2 - z^2 + 2\lambda z - \lambda^2}{\lambda^2 (z-\lambda)^2}$$

$$= \frac{z(2\lambda - z)}{\lambda^2 (z-\lambda)^2}$$

(say R say).

where $|z|$ is bounded
$$\left| \frac{z(2\lambda - z)}{\lambda^2 (z-\lambda)^2} \right| \leq C_0 \frac{|\lambda|}{|\lambda^4|} = \frac{C_0}{|\lambda^3|}.$$

Following the analysis we did before, recalling that the # of terms grows linearly we get $M_k \leq \frac{C_1}{k^2}$. So we get convergence.

Def $\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda} \frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}$.

Is $\wp(z)$ invariant?

Note $\wp'(z) = -2 \sum_{\lambda \in \Lambda} \frac{1}{(z-\lambda)^3} = -2\mathcal{E}_3(z)$.

$d\wp' = \mathcal{E}_3 dz$ in invariant language.

So $\wp'(z+\lambda) = \wp'(z)$ and $\wp(z+\lambda) = \wp(z) + c_\lambda$

In particular $\frac{d}{dz}\left(\wp(z+\lambda) - \wp(z)\right) = 0$ so

$\wp(z+\lambda) = \wp(z) + c$.

Note also that $\wp(z) = \wp(-z)$ since

$$\wp(z) = \frac{1}{z^2} + \sum_{k} \sum_{\lambda \in \Lambda_k} (z-\lambda)^{-2} - \lambda^{-2}$$

$$\wp(-z) = \frac{1}{z^2} + \sum_{k} \underbrace{\sum_{\lambda \in \Lambda_k} (-z-\lambda)^{-2} - \lambda^{-2}}$$

$$= \sum_{-\lambda \in \Lambda_k} (-z+\lambda)^{-2} - \lambda^{-2} = \sum_{-\lambda \in \Lambda_k} (z-\lambda)^{-2} - \lambda^{-2}.$$

Prop. $\wp(z) = \wp(z+\lambda)$.

How does Γ act on $\mathcal{E}_3 dz$? $\mathcal{E}_3 dz$ is invariant.

Let Γ be the group generated by $z \mapsto -z$ and $z \mapsto z+\lambda$ for $\lambda \in \Lambda$.

Elements of Γ linear part -1 act with fixed points:

Consider $\gamma(z) = -z+\lambda$. Solve $\gamma(z) = z$.
Get $z = -z+\lambda$ or $2z = \lambda$ or $z = \frac{\lambda}{2}$.

So the set of fixed points is the set $\Lambda/2$.

If $z_0 \in \Lambda/2 - \Lambda$ then

If $\lambda_0 \in \Lambda - 2\Lambda$ and $z_0 = \lambda_0/2$ then $z_0 \in \Lambda/2 - \Lambda$ is fixed by $z \mapsto -z+\lambda_0$ but $z_0 \notin \Lambda$ so z_0 is not a pole of P. We have $P(z_0) = P(-z_0)$
$= P(-z_0+\lambda_0) + C_{\lambda_0} = P(z_0) + C_{\lambda_0}$ so $C_{\lambda_0} = 0$.

Similarly $C_{\lambda_1} = 0$. $P(z_0+\lambda+\lambda') = P(z_0+\lambda) + C_{\lambda'}$
$= P(z_0) + C_\lambda + C_{\lambda'}$ so $\lambda \mapsto C_\lambda$ is a homomorphism.
Since it vanishes on generators of Λ it vanishes on all of Λ.

Thus P is periodic and induces a meromorphic function on $\mathbb{C}/\Lambda$ of degree 2.
P is called the Weierstrass P-function.

The Riemann-Hurwitz formula applied to $\wp : T^2 \to S^2$ tells us that

$$\chi(T^2) = d\,\chi(S^2) - \sum_{z \in T^2} \nu_\wp(z) - 1$$

so $$0 = 2 \cdot 2 - \sum_{z \in T^2} \nu_\wp(z) - 1 .$$

Since $\wp$ has deg. 2 $\nu_\wp(z) \leq 2$ so there are exactly 4 pts in T^2 where $\wp$ has valence 2.

Prop. $\wp$ has valence 2 exactly at the half-lattice points.

Proof.

$\wp' = -2\mathcal{E}_3$ and $\mathcal{E}_3 = \sum_{\lambda \in \Lambda} \frac{1}{(z-\lambda)^3}$ is an odd function.

If $\lambda_0 \in \Lambda/2$ then

$$\wp'(\lambda_0) = -\wp'(-\lambda_0) = -\wp'(-\lambda_0 + 2\lambda_0) = -\wp'(\lambda_0).$$

So $\wp'(\lambda_0) = 0$. This implies that for $\lambda_0/2$, $\lambda_1/2$ and $\lambda_0+\lambda_1/2$ $\wp$ has valence at least 2. But $\wp$ has degree 2 so the valence must be exactly 2.

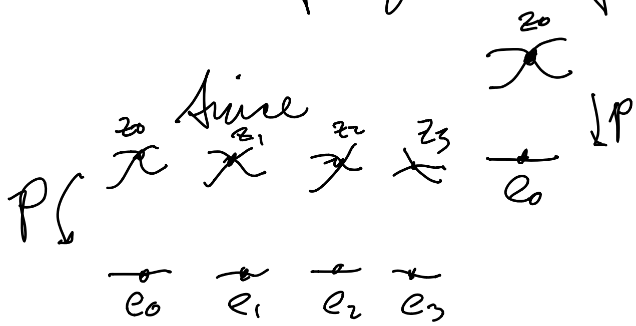
Since $\wp$ has a pole of order 2 at 0, $\wp$ also has valence 2 at 0.

Prop. The values of $\wp$ at $0, \frac{z_0}{2}, \frac{z_1}{2}, \frac{z_0+z_1}{2}$ are distinct.

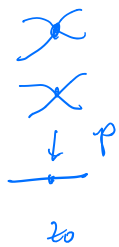
Proof. $\wp: \mathbb{C}/\Lambda \to \mathbb{C}_\infty$ has degree 2 and at each half-lattice point it has valence 2.

The degree formula gives $2 = \sum_{p : P(p) = q} V_p(p) = 2 + \sum_{p' \neq p :} V_p(p')$

Since



Any additional inverse images would give additional positive contributions to the degree.



This situation can't happen. It would give $\delta(z_0) = 4$ instead of $\delta(z_0) = 2$.