Prop. (1) The only surfaces with self-maps of positive degree are S^2 and T^2.

(2) If $f: R \to S$ hb is non-constant then

$g(S) \le g(R)$.

each term is positive.

Proof. $\chi(R) - d \cdot \chi(S) = -\left(\sum_{c(f)} V_f(p) - 1\right)$

so $\chi(R) - d \cdot \chi(S) \le 0$

$\chi(R) \le d \cdot \chi(S)$

$\boxed{\text{Assume If } \chi(S) \le 0}$ $\chi(R) \le d \cdot \chi(S) \le \chi(S)$ Equality $\Rightarrow d = 1$

This implies $g(R) \ge g(S)$ since $g = \frac{2-\chi}{2}$

If $\chi(S) > 0$ then $\chi(S) = 2$. If $\chi(S) = 2$ then S is the 2 sphere and S has minimal genus so $g(S) \le g(R)$.

$\chi = 2 - 2g$

$\chi - 2 = -2g$

$2g = 2 - \chi$

$g = \frac{2-\chi}{2}$

Examples. Have seen self maps of S^2 of positive degree.

~~Here is a self~~

Here is a self map of T^2.

Self-map of T^2.

$\Lambda = \mathbb{Z} \oplus i\mathbb{Z}$

$T^2 = \mathbb{C}/\mathbb{Z}\oplus i\mathbb{Z}$

$F(z) = 2z$

$$\begin{tikzcd} \mathbb{C} \arrow[r, "F"] \arrow[d, "\pi"] & \mathbb{C} \arrow[d, "\pi"] \\ \mathbb{C}/\Lambda \arrow[r, "f"'] & \mathbb{C}/\Lambda \end{tikzcd}$$

Want to check that $\pi \circ F \circ \pi^{-1}$ is well defined.

Any z', z'' map to p under π

$z' = z'' + \lambda \qquad \lambda \in \Lambda$

$2z' = 2z'' + 2\lambda$

so $F(z') = F(z'') + 2\lambda = F(z'') + \Lambda$.

Any $\gamma(z) = -z + m + ni$.

If $\gamma(z) = z + m + ni$ then there are no non-trivial stabilizers.

$$z = -z + m + ni$$

$$2z = m + ni$$

$$z \in 2\Lambda.$$

~~There are Then~~

For any point in 2Λ

4 critical points: $0, \frac{1}{2}, \frac{1}{2}i, \frac{1}{2}+\frac{1}{2}i$ (mod Λ). Order of each of these points is 2.

R-H $$\chi(R) = 2\cdot\chi(S) - \sum_{p\in C(f)} (\nu_f(p) - 1)$$

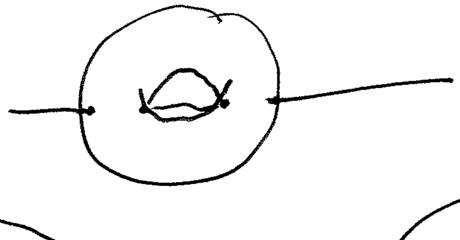
$$0 = 2\cdot\chi(S) - \sum (2-1)$$

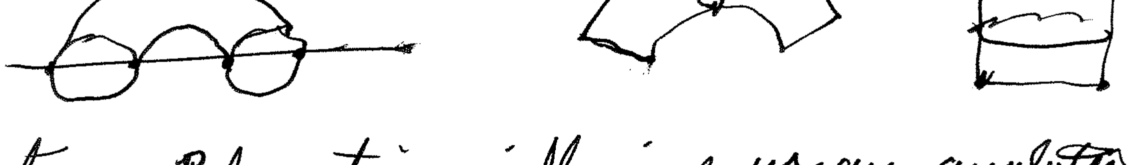
$$0 = 2\cdot\chi(S) - 4$$

$$4 = 2\cdot\chi(S)$$

$\chi(S) = 2.$ S is the 2-sphere.

Picture:





Weierstrass $\wp$ function will give us an analytic formula for this map.

Genus reducing maps 11.5

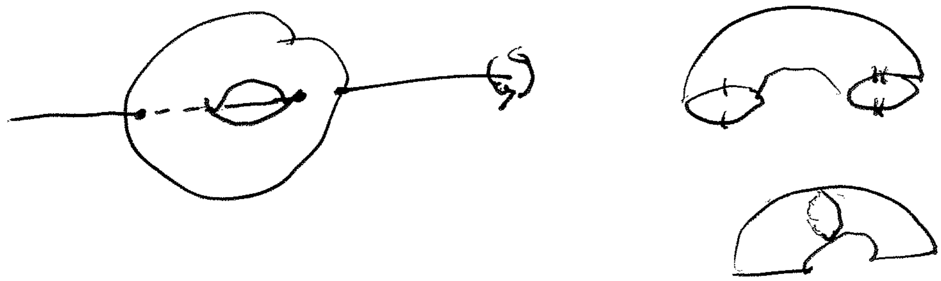
d = Degree = 2 (generically 2 pts. get identified)

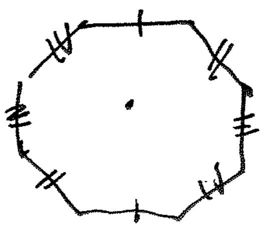
4 critical pts. $V_F(p) = 2$ at each half-integral point.

~~RH~~ $\chi(T^2) = 2 \cdot \chi(S^2) - \sum_{p \in C(F)} 2 - 1$

$0 = 4 - 4$.

Picture.





$z \mapsto -z$. Involution, order 2. Degree of all f.p. is 2.

0 is fixed and midpoints of 4 sides. Vertex is fixed.
6 fixed pts.
Give critical pts. of order 2

$\chi(R) = 2 \cdot \chi(S) - \sum_{p \in C(F)} V_F(p) - 1$

$g(R) = 2$

$\chi(R) = 2 - 2g = 2 - 4 = -2$

~~RH~~.

$-2 = 2 \cdot \chi(S) - 6$

$4 = 2\chi(S)$

$\chi(S) = 2$. Quotient is a sphere.