As we have seen
non-constant proper holomorphic maps ⑪
~~are~~ have the structure of "branched covers".
~~They~~ These are closely related to covering spaces. Any $F: R \to S$ is a ~~noncomc~~ branched cover than $\hat{F}/(R - F^{-1}(B(F)) \to S - B(F)$ ~~is~~ has no critical points, is still proper so $\hat{F}$ is a covering space. ($F: R \to S$ is proper. Remove a set from S and its inverse image from R and it is still proper)

Let us recall some of what we know about covering spaces.

{ Throwing out finitely many points means we still have a proper map between surfaces of finite type. Now it is a local homeomorphism so, as we have seen, it ~~th~~ is a covering map of finite degree.

Let

$f: R \to S$ be a covering map.

$\boxed{\text{Review of covering space theory;}}$

~~A path~~

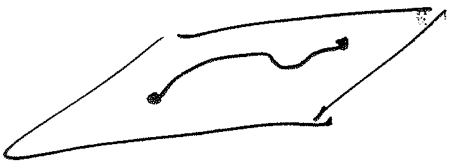
Consider a path $\gamma:[0,1] \to S$ and a point p in $f^{-1}(\gamma(0))$.

A lift of γ starting at p is a path $\bar{\gamma}:[0,1] \to R$ with $\bar{\gamma}(0) = p$ and $f \circ \bar{\gamma} = \gamma$.

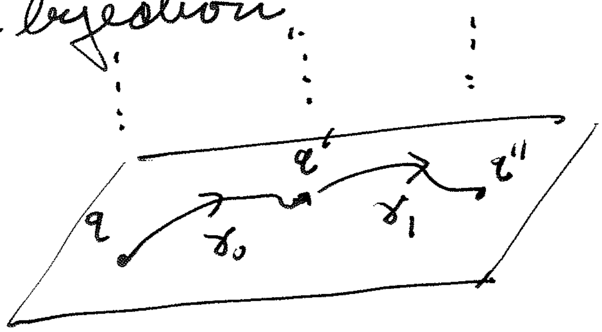
~~Lifts~~ Given any path γ and point $p \in f^{-1}(\gamma(0))$ there is a unique lift $\bar{\gamma}$ with $\bar{\gamma}(0) = p$.

Given points q and $q' \in S$ and a path γ with $\gamma(0) = q$ $\gamma(1) = q'$, the operation of path lifting

gives a map from ~~f~~ $f^{-1}(q)$ to $f^{-1}(q')$.



Uniqueness of the lift implies that this map is a bijection.



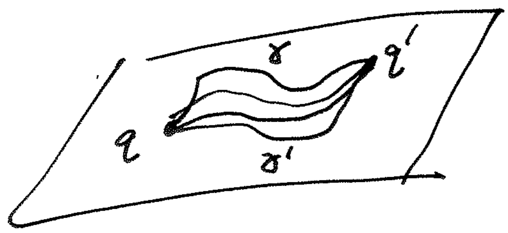
Notation for monodromy.
$h(\gamma_0): f^{-1}(q) \to f^{-1}(q')$

This map satisfies a composition property

$$\begin{tikzcd} f^{-1}(q) \arrow[r, "\gamma_0"] \arrow[rr, bend right, "h(\gamma_1)\circ h(\gamma_0)"'] & f^{-1}(q') \arrow[r, "\gamma_1"] & f^{-1}(q'') \end{tikzcd}.$$

Functor from category of points and paths to category of sets and maps.

This map satisfies a homotopy invariance property: If γ is homotopic to γ' through a homotopy that fixes the endpoints then the induced maps from $f^{-1}(q)$ to $f^{-1}(q')$ are equal.



~~Putting this together we get a~~

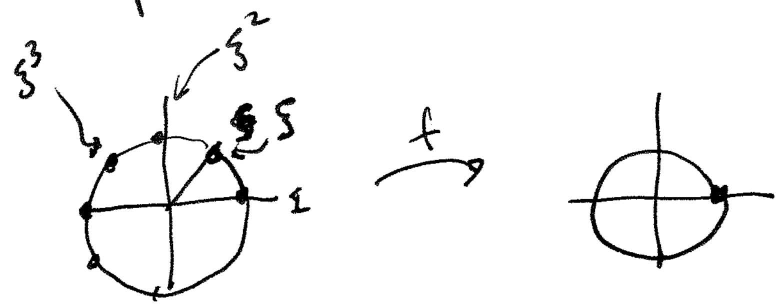
If we pick a basepoint q_0 for S ~~we get~~ then a loop based at q_0 gives an automorphism of the fiber $f^{-1}(q_0)$.

Homotopy invariance gives us an anti homomorphism from $\pi_1(S, q_0)$ to $\mathrm{Perm}(f^{-1}(q_0)) \cong$ symmetric group Σ_d.

R is connected iff $\pi_1(S, q_0)$ acts transitively on $f^{-1}(q_0)$.

$$h(\gamma_1) \circ h(\gamma_0) = h(\gamma_0 \gamma_1).$$

Example: $f(z)=z^n$. $n\ge 1$. $C(f)=\{0\}$ $B(f)=\{0\}$,



Write $\zeta = e^{2\pi i/n}$ then

$f^{-1}(1)=\{n\text{-th roots of unity}\} \to \{\zeta^0, \zeta^1 \dots \zeta^{n-1}\}$.

Lift of the loop $s \mapsto e^{2\pi i s}$ $\quad s\in[0,1]$

starting at ζ^k with $\zeta = e^{2\pi i/n}$ is the path

$\gamma(s) \mapsto e^{2\pi i s/n}\cdot \zeta^k$ $\Big|$ $\gamma(1)=e^{2\pi i/n}\cdot\zeta^k=\zeta^{k+1}$. $\quad f(\gamma(s))=e^{2\pi i s}$ $\quad (\zeta^{nk}=1)$.

This lift takes ζ^k to $\zeta^{k+1 \,(\mathrm{mod}\, n)}$.

So corresponding element of Perm(n) is the cyclic permutation of order n.

We can extend this analysis to small loops around punctures for the surface S. $\bar p\in \bar S$

~~then~~ We consider a point $\bar q$ mapping to $\bar p$.

We can choose a chart in $\bar R$ around $\bar q$ so that

F has the form $z\mapsto z^n$ $\quad n=V_f(\bar q)$.
$F: R\to S$

$\boxed{\begin{array}{ll} (1,2)(3)(4,5) & \\ 1\mapsto 2 & 3\mapsto 3 \\ 2\mapsto 1 & 4\mapsto 5 \quad 5\mapsto 4 \end{array}}$

Prop. The lift of a simple loop around a puncture is ~~an~~ ~~a~~ has cycles of length $V_f(\bar p_1)\dots V_f(\bar p_k)$ where $F^{-1}(\bar q)=\{\bar p_1\dots \bar p_k\}$.

Proof. Choose coords around $\bar p_1 \dots \bar p_k$ so $F(z) = z^{V_f(\bar p_i)}$

$\cancel{z^3-3z-2}=0$

Regular covering spaces.
(or normal or Galois)

Example of a non-regular cover.
(Covering space is not given by taking the quotient of a finite group action.)

Let $f:\mathbb{C}\to\mathbb{C}$ be given by $f(z)=z^3-3z$.

$C(f)=\{z: f'(z)=0\}$ $\quad 0=f'=3z^2-3.\quad C(f)=\{\pm 1\}$

$0=z^2-1.$

$B(f)=f(\{\pm1\})=\quad f(1)=-2\quad f(-1)=2$

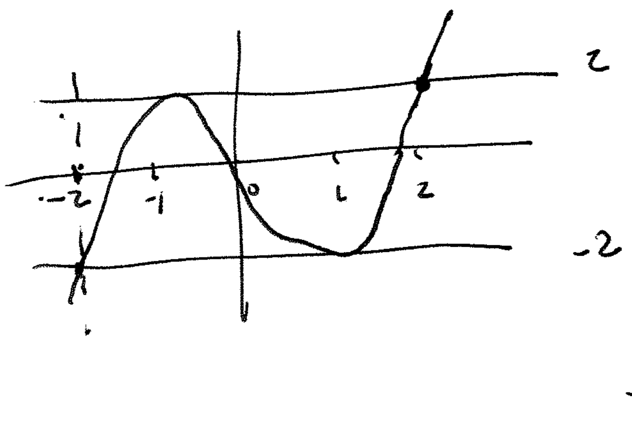
$f^{-1}(B(f))=$

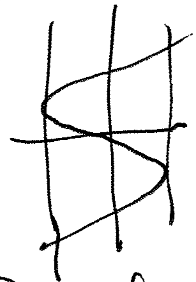
$f(-1)=-1+3=2$

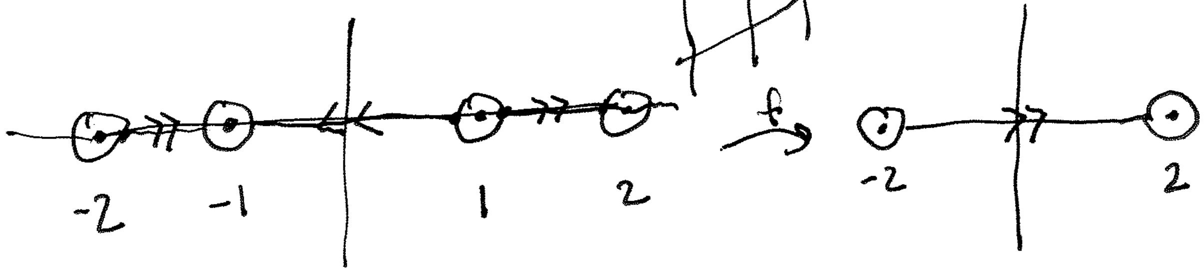
$f(1)=1-3=-2.$

$f(-2)=-8+6=-2$

$f(2)=8-6=2.$

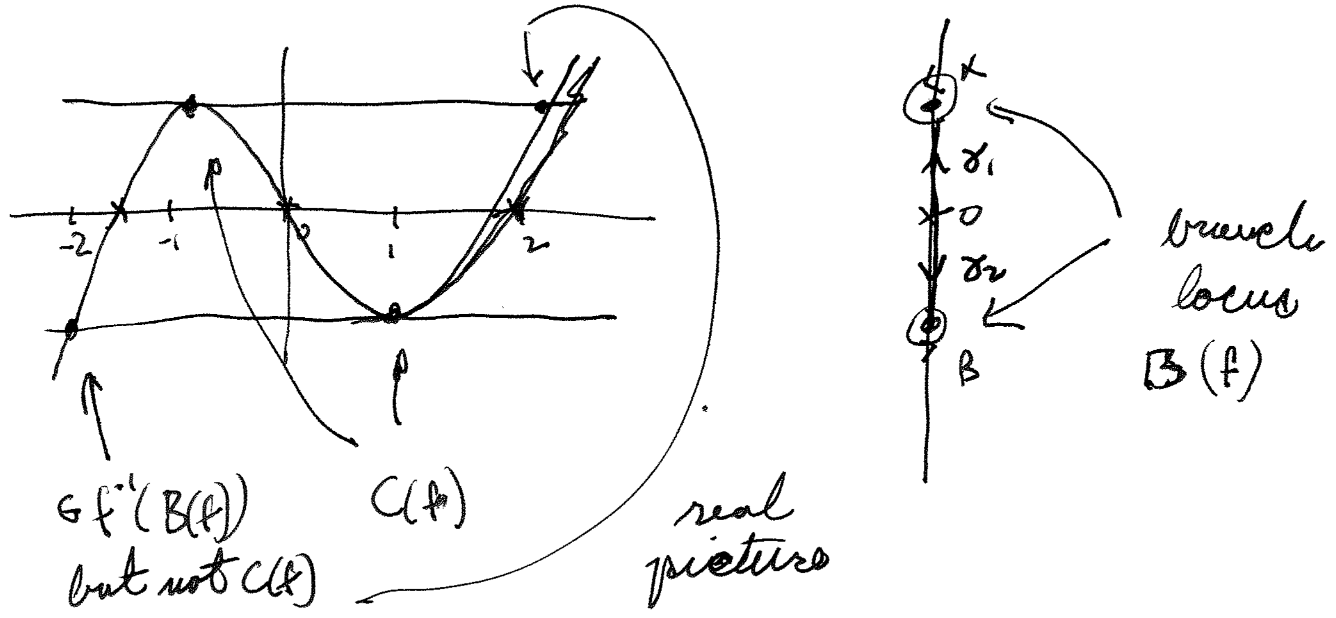






Let's review the example from last time with more details.

$f: \mathbb{C} \to \mathbb{C} \qquad f(z) = z^3 - 3z.$



Build a combinatorial model of the map.

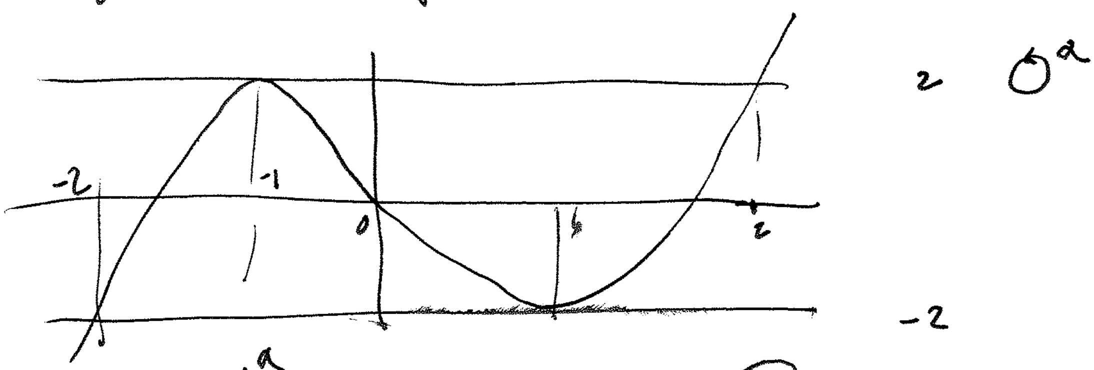
Range $= \mathbb{C} - \{\pm 1\}$. Insert a small circle around each puncture. Connect the two circles by a segment in the ~~axis~~.

~~What does~~ This complex is homotopy equivalent to $\mathbb{C} - \{\pm 1\}$. In particular it carries the fundamental group which is a free group on two generators α, β (anti-clockwise loops around 1 and -1).

What is the inverse image of G?

Inverse image of the segment consists of 3 segments. (Map ~~is a covering space of degree 3),~~ has degree $3 = \deg z^3 - z$.)

We can see from the graph that each of these segments is real.



Inverse image of ~~a near~~ 2 is -1, 2.

Inverse image of α near 2 is a loop mapping with degree 1, Inverse imag near -1.

since 1 is a regular value $V_f(1) = 1$.

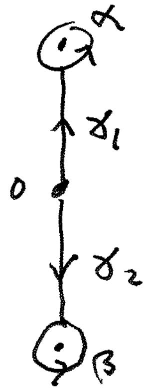
~~Inverse image near~~ is a loop mapping with degree 2 since $V_f(-1) = 2$.

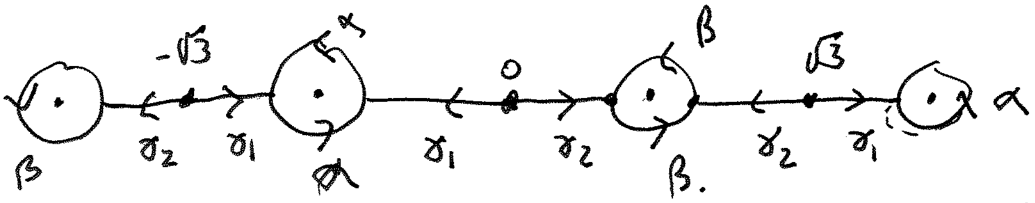
Get

Generators:

$\gamma_1 \alpha \gamma_1^{-1}$

$\gamma_2 \beta \gamma_2^{-1}$



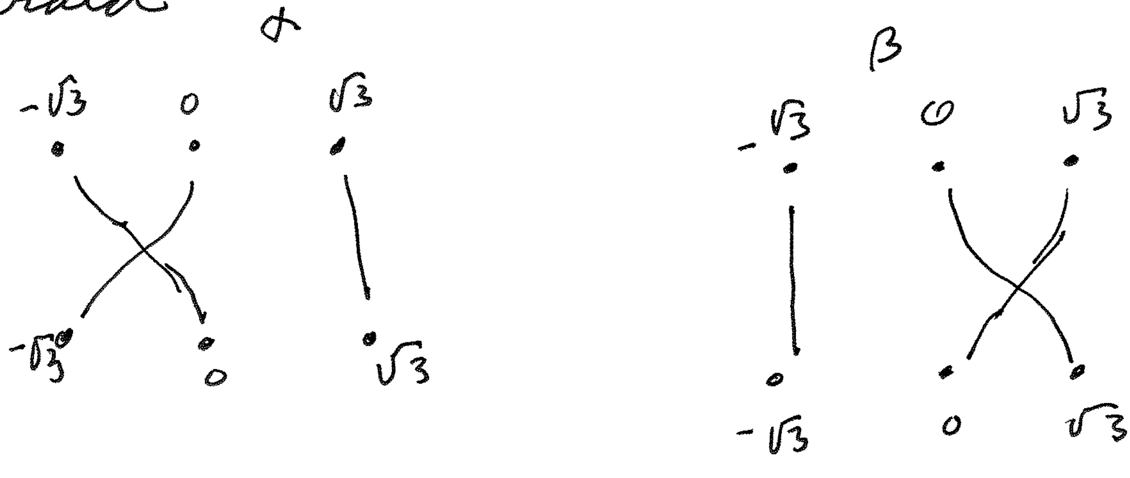


Paths and their lifts are denoted by the same letter.

We get a map from $\pi_1(\mathbb{C} - \{\pm 1\})$ to $\mathrm{Perm}(f^{-1}(0))$.
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $

What is this? Let me draw this as a $\mathrm{Perm}\{\sqrt{3}, 0, \sqrt{3}\}$

braid



What permutation corresponds to $\alpha\beta$?

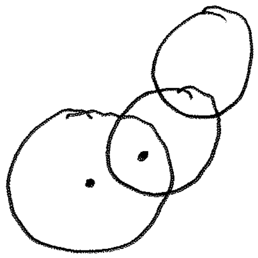
Cyclic perm of order 3.

Note that the path $\alpha\beta$ is also homotopic to a puncture, the puncture at ∞.

$V_f(\infty) = 3$ so we expect the permutation corresponding to $\alpha\beta$ to be a cycle of length 3 by our analysis of monodromy around punctures.

~~This description of covering spaces in terms of "monodromy" is~~

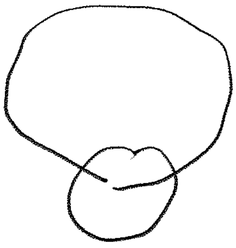
Classic idea in Riemann surface theory is to build surfaces by analytically continuing functions. (Original approach to Riemann surfaces.)



disk in which the power series converges

Analytic extension may not be possible, perhaps.

~~In general the surface~~



If you do that the surface that you build need not ~~map to $\mathbb{C}$ via a proper map,~~

~~If it~~ ~~need not be~~ Could be a cover but ~~might~~ be a covering space, but as sheeted

eg $\int \frac{dz}{z}$ builds the logarithmic cover.

We are analyzing the nice case when we get a ~~finite sheet~~ covering space of finite degree.

We are keeping track of the ~~way in which~~ ambiguity that arises when we analytically continue along loops. ⑥

A general term for this ambiguity is "monodromy".

~~We~~ From this viewpoint the action of ~~$\pi_1(\mathbb{C}-\ldots$~~ $\pi_1(U)$ $U \subset \mathbb{C}$ on sheets is a natural thing to consider. $\pi_1(U) \to \mathrm{Perm}(f^{-1}(p))$.

On the other hand this is not how covering spaces are usually presented in topology. What is the connection.

~~If~~ Say $f: R \to S$ is a covering space where R and S are connected. Let q be a point in S and p_0 an element of $f^{-1}(q)$ so $f(p_0) = q$. We get a homomorphism $f_*: \pi_1(R, p_0) \to \pi_1(S, q)$. This map is an injection (by homotopy lifting) and the covering space is determined by the image group $f_*(\pi_1(R, p_0))$ in $\pi_1(S, q)$.

Recall:

Thm. If we have two covering spaces $f:(R,p) \to (S,q)$ and $f':(R',p') \to (S,q)$ and if $f_*(\pi_1(R,p)) = f'_*(\pi_1(R',p'))$ then there is a homeomorphism $H:(R,p) \to (R',p')$ so that

$$\begin{tikzcd} (R,p) \arrow[rr, "H"] \arrow[dr, "f"'] & & (R',p') \arrow[dl, "f'"] \\ & (S,q) & \end{tikzcd}$$

$f = f' \circ H$ and
$f' = f \circ H^{-1}$.
$H(p) = p'$

In other words the covering spaces are equivalent.

Note that this is an equivalence of <u>pointed</u> covering spaces.

Prop. Any $f:(R,p_0) \to (S,q)$. Let $h: \pi_1(S,q) \to \mathrm{Perm}(f^{-1}(q))$ be the monodromy representation. Then $f_*(\pi_1(R,p_0))$ ~~Stab(p_0)~~ is the set of $\gamma \in \pi_1(S,q)$ such that $h(\gamma)(p_0) = p_0$. (Call this $\mathrm{Stab}(p_0)$.)

Proof. If $h(\gamma)(p_0) = p_0$ then the lift of γ starting at p_0 ends at p_0 so it is a loop in $\pi_1(R,p_0)$ mapping to γ. Conversely if α is a loop in $\pi_1(R,p_0)$ then $h(f_*(\alpha))(p_0) = p_0$.

Def. The deck translation group of a covering map $f: R \to S$ is the group of homeomorphisms $H: R \to R$ so that

$$\begin{tikzcd} R \arrow[r, "H"] \arrow[dr, "f"'] & R \arrow[d, "f"] \\ & S \end{tikzcd}$$

$fH = f$. Call it $\mathrm{Deck}(f)$.

Note that there is a homomorphism from $\mathrm{Deck}(f) \longrightarrow \mathrm{Perm}(f^{-1}(q))$.

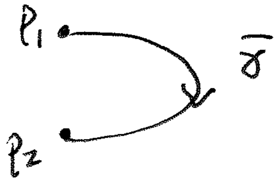
Prop. The following are equivalent

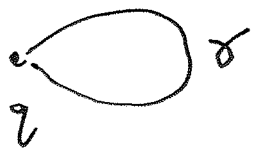
1. $\mathrm{Deck}(f) \to \mathrm{Perm}(f^{-1}(q))$ acts transitively
2. $f_*(\pi_1(R, p_0))$ is normal in $\pi_1(S, q)$.
3. The stabilizer of any point $p \in f^{-1}(q)$ under the monodromy map is the same.

Def. If any of these conditions hold we say that f is a ~~normal~~ regular cover or a Galois cover.

Say p_1 and p_2 are in $f^{-1}(q)$ and $\bar{\gamma}$ is a path from p_1 to p_2. Let $\gamma = f(\bar{\gamma})$.

~~The stabilizer of~~ $\mathrm{Stab}(p_1) = \tilde{\gamma}_1^{-1}\,\mathrm{Stab}(p_2)\,\tilde{\gamma}_1$.





Thus ② $\Rightarrow$ ③.

If ~~there is std~~ $\mathrm{Stab}(p_1) = \mathrm{Stab}(p_2)$ then there is a deck translation which takes p_1 to p_2 by the covering space equivalence thm.

Thus ③ $\Rightarrow$ ①.

If the deck group acts transitively then on $f^{-1}(q)$ then for any $\gamma \in \pi_1(S,q)$ with a lift $\bar{\gamma}$ taking any p_0 to p_1 we have an H taking p_0 to p_1 so

$$\begin{tikzcd} \pi_1(R,p_0) \arrow[rr, "H"] \arrow[dr, "f"'] & & \pi_1(R,p_1) \arrow[dl, "f"] \\ & \pi_1(S,q) & \end{tikzcd}$$

$$f_*(\pi_1(R,p_1)) = \gamma^{-1} f_*(\pi_1(R,p_0))\,\gamma$$

$$= H_* f_*(\pi_1(R,p_0)) = f_*(\pi_1(R,p_0))$$

so ① $\Rightarrow$ ②.

Cor. $f(z) = z^3 - 3z$ does not give a regular covering space $f: \mathbb{C} - f^{-1}(B(f)) \to \mathbb{C} - B(f)$.

Proof. $\alpha \in \mathrm{Stab}(\sqrt{3})$ but $\alpha \notin \mathrm{Stab}(0)$ so the stabilizer of a point in $f^{-1}(0)$ ~~depends on~~ is not independent of the point.