~~On the first~~

In the first lecture I discussed the curve given by $w^2 = f(z)$ where f had degree 4 and distinct zeros. Now let us ~~no~~ consider this example (and related examples) more carefully.

Say f has degree d. Let $P(z,w) = w^2 - f(z)$ and let C be the affine curve determined by P. $\{(z,w) \in \mathbb{C}^2 : P(z,w) = 0\}$. If f has distinct zeros then C is non-singular. Let us assume this.

This example differs from the first one in that we don't have an explicit parametrization of C, we have local parametrizations.

Consider the projection ~~onto~~ ~~on~~ to the z coordinate $\pi_z : C \to \mathbb{C}$. This projection has degree 2.

ion. Say that C is an affine curve defined by a polynomial $P(z,w) = w^n + Q_{n-1}(z)w^{n-1} + \dots Q_0$. Then $\pi_z : C \to \mathbb{C}$ is a proper map.

Let X be a compact subset of $\mathbb{C}$.

Proof. Since π_z is continuous the inverse image of a ~~compact set~~ X is closed. We need to show that it is bounded.

The value of $|Q_j|$ is bounded above by some b_j on X. Let $R = \sum b_j$. Claim that all the zeros of $P(z,w)$ for $z \in X$ satisfy $|w| < R$. So both the z and w coordinates of $\pi_z^{-1}(C)$ are bounded.

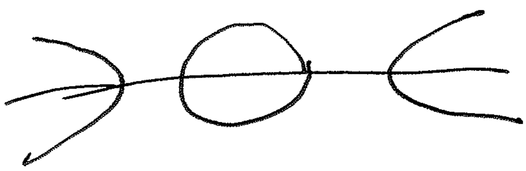
To see this assume we have (z,w) with $z \in X$ and $|w| > R$ then

$$
\begin{aligned}
|P(z,w)| &\geq |w^n| - \left|\sum Q_j w^j\right| \\
&\geq R^n - \sum b_j R^{n-1} \\
&\geq \left(R - \sum b_j\right) R^{n-1} > 0.
\end{aligned}
$$

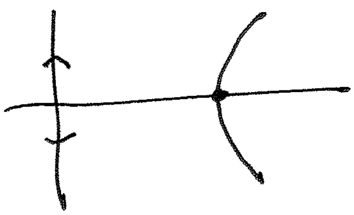
Remark. Can extend this by assuming we remove zeros of Q_n in the z plane.

Let's assume that f has distinct zeros. Recall that ~~$\frac{\partial P}{\partial w}$~~ this implies that C is a ~~Riess~~ Riemann surface. $\frac{\partial P}{\partial w} = 2w$, $\frac{\partial P}{\partial z} = f'(z)$

Claim. The critical points of the projection π_z are those points $\underset{(z,w)}{\text{on}}$ C where $\frac{\partial P}{\partial w} = 0$.



~~Recall the~~ If $\frac{\partial P}{\partial w} = 0$ then $\frac{\partial P}{\partial z} \neq 0$ and locally C can be written as $(\phi(w), w)$.



~~Now if we~~ We are looking at critical points of

$$w \mapsto (\phi(w), w) \xrightarrow{\pi_z} \phi(w).$$

Now differentiating implicitly we have

$$\cancel{\partial P} \quad \frac{\partial}{\partial w} P(\phi(w), w) = 0$$

so

$$\cancel{\frac{\partial P}{\partial w}} \quad \frac{\partial}{\partial w} P(\phi(w), w) = \frac{\partial P}{\partial z} \cdot \frac{\partial \phi}{\partial w} + \frac{\partial P}{\partial w} = 0.$$

and

$$\frac{\partial \phi}{\partial w} = - \frac{\partial P}{\partial w} \cdot \frac{\partial P}{\partial z}$$

non-zero.

In particular ϕ' vanishes iff $\frac{\partial P}{\partial w}$ vanishes.

Critical points of π_z ~ $\frac{\partial P}{\partial w} = 0$ ~ $w=0$ ~ $f(z)=0$.

since ~~...~~ $w^2 - f(z) = 0$. Write π_z for $\pi_z|C$.

~~Br~~ ~~$B(\pi_z|C)$~~ $C(\pi_z|C) = $ " $(z,0)$ where z is a root of f. $B(\pi_z|C) = \{$roots of $f\}$. $\pi_z^{-1}(B(\pi_z|C)) =$ the set of critical pts $= \{(z,0) : f(z)=0\}$.

So we get a covering space.

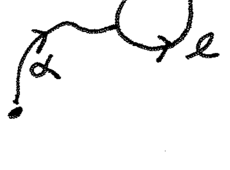
~~Results~~ Is it a regular covering?

Yes since $w \mapsto -w$ is a group of deck transformations acting transitively on the fibers. ~~We get a~~ Pick a q which is not a root.

We get a map $\pi_1(\mathbb{C} - \text{roots}, q) \to \text{Perm}\{\pm\sqrt{q}\} = \mathbb{Z}/2\mathbb{Z}$.

Note that something special happens when we have an abelian deck group.

We don't need to worry about the basepoint.



The image of a loop l does not depend on how we connect to the basepoint.

$$h(\alpha l \alpha^{-1}) = \cancel{h(\beta l \beta^{-1})}$$

$$h(\beta l \beta^{-1}) = h(\beta \alpha^{-1} \alpha l \alpha^{-1} \alpha \beta^{-1})$$

$$= h(\beta\alpha^{-1}) \cdot h(\alpha l \alpha^{-1}) \cdot h^{-1}(\beta\alpha^{-1})$$

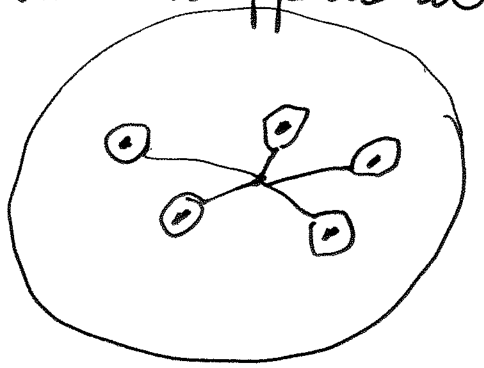
$$= h(\alpha l \alpha^{-1}).$$

(not true in previous examples)

~~Let's assume that C has finite type.~~

At each critical point we have $V_\pi((z_0,w_0)) = 2$ since we have checked that $V_\pi \neq 1$ and 2 is the maximum. ~~What about the~~

What happens at the puncture at ∞?



A large loop is equivalent to the sum of the loops around punctures.

So the monodromy around ∞ is trivial if d is even (~~sum~~ product of an even number of permutations).

Monodromy is ~~odd if~~ non-trivial if d is odd.

Let's assume C has finite type. What is $\bar{C}$, the result of filling in punctures? π_z extends to a map from $\bar{C}$ to S^2.

If d is even then

$$\chi(\bar{C}) = 2\cdot\chi(S^2) - d(V_\pi(r_i)-1)$$

$$= 2\cdot 2 - d = 4-d.$$

If d is odd then

$$\chi(\bar{C}) = 2\cdot\chi(S^2) - d(V_\pi(r_j)-1) - (V_\pi(\infty)-1)$$
$$= 4 - (d+1) = 3-d.$$

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d	$\chi(\bar{C})$	$g(\bar{C})$	
1	2	0	
2	2	0	
3	0	1	elliptic curves
4	0	1	
5	-2	2	hyper-elliptic curves
6	-2	2	
7	-4	3	
8	-4	3	
		$\vdots$	

In particular we have constructed Riemann surfaces of every genus.

Let's return to the question of the finite type of C. One approach is to ① form the projective curve: $P(z,w,u) = w^2u^{d-2} - z^d + a_{d-1}u + \cdots a_0u^d$
(Call this C^+.)

① Check that it is non-singular.

② See that the projective completion of C is the same as $\bar{C}$.

Proposition. Every finite ~~degree~~ sheeted covering space of a surface of finite type is a surface of finite type.

Proof. Say $f: R \to S$ is a ~~finite covering space of~~ finite sheeted covering space.

We can write S as a compact piece X together with open ~~disc~~ sets $U_1 \dots U_k$ each equivalent to a punctured disk. ~~Since f has finite is a finite cover~~ Can ~~we~~ describe X as a finite union of compact simply connected piece. Inverse image of each of these is homeo. ~~to~~ $\{1 \dots d\} \times Y_j$ so is compact.

Remains to show that each component of the inverse image of U_j is a punctured disk. Say V is such a component

$$\begin{tikzcd} V \arrow[dr, "f"'] & & \Delta - \{0\} \arrow[dl, "z \mapsto z^j"', "g"] \\ & \Delta - \{0\} & \end{tikzcd}$$

$f_*(\pi_1(V))$ is a subgroup of $\pi_1(\Delta - \{0\})$ of finite index say j. Let $g: \Delta - \{0\} \to \Delta - \{0\}$ be defined $g(z) = z^j$, $g_*(\pi_1(\Delta - \{0\})) = f_*(\pi_1(V))$. So covers are equiv.