$\wp''(z) =$

$\wp'(z) =$

Fermat curve $\overset{R}{X^4+Y^4+Z^4=0}$

$f: \mathbb{CP}^2 \longrightarrow S^2 \quad f: (X:Y:Z) \mapsto \frac{X}{Y}.$ (Could also write $f(X:Y:Z) = (X:Y)$ $f: \mathbb{CP}^2 \to \mathbb{CP}^1$

Projection map has 4 critical points

at $\underset{q_1}{(1:\xi:0)}, \underset{q_2}{(1:\xi^3:0)}, \underset{q_3}{(1:\xi^5:0)}, \underset{q_4}{(1:\xi^7:0)} \qquad \xi^8=1$

Each critical pt. has multiplicity 4.

a $f: R \to S^2$ has degree 4.

~~Before we can~~

$$\chi(R) - 4\cdot\chi(S^2) = -\sum_{j=1\ldots4} v(q_j)-1$$

$$\chi(R) - 8 = -12$$

$$\chi(R) = -4.$$

Would like to say that $g(R)=3$. $(\chi = 2-2g)$

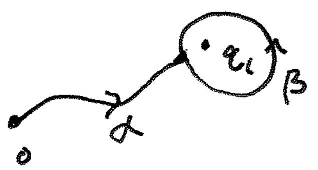
In order to do that we need ~~one more technique~~ to check one thing. Is $R$ connected?

How do we check?

Proposition: Let $\mu: \pi_1(S^2 - \{q_i\}) \to \mathrm{Perm}(f^{-1}(0))$ be the "monodromy map". ~~R is conn~~ $R - f^{-1}(\{q_i\})$ is connected iff $\pi_1(S^2 - \{q_i\})$ acts transitively on the fibre.

We can check that here since a small loop around $q_1$ say is a cycle of ~~degree~~ length 4.

($q_1$ has 1 inverse image $p_1$ and $V(p_1) = 4$ so small loops around $q_1$ lifts to a loop conjugate to a cycle of length 4.)




This element $\alpha\beta\alpha^{-1}$ acts transitively.
(only identified up to conjugacy)
so the whole group acts transitively.

So far in the course we have introduced basic notions, some techniques, ~~but we don't have a comprehensive~~ We understand all meromorphic functions on $S^2$, some meromorphic functions on other surfaces but we don't have a comprehensive theory of Riemann surfaces.

Our next topic is to develop a ~~pretty~~ pretty good theory of tori.

We have constructed tori in

~~Tori arise in~~ two different ways: ~~for ex~~ (4)

as quotients $\mathbb{C}/\Lambda$ (which can also be thought

of as parallelograms with ~~glued sides~~ identified

sides). and as elliptic curves $w^2 = f(z)$ where

$f$ has degree 3 or 4.

We want to identify these constructions.

We start with a $T^2 = \mathbb{C}/\Lambda$ and we want to find

an ~~other~~ eliptic curve which is conformally

equivalent.

Though we haven't proved it there is a unique

conformal structure on $S^2$. The torus is different

(as are all curves of higher genus) in that

we can vary the conformal structure.

Hence 2 families of tori.

~~As~~ The first thing we want to is consider

the correspondence ~~between~~ from $\mathbb{C}/\Lambda$ ~~an~~ to elliptic

curves. We have seen that if $R \subset \mathbb{CP}^2$ then $R$

has many meromorphic functions.

In fact we can also work in the other direction

& as well. Constructing meromorphic functions

can give us an embedding: Need 2 meromorphic fcns. ~~relation algebr~~ polynomial relation between them

~~Opma~~ We will ~~soon~~ see that we can use meromorphic functions to get a map from $R$ to ~~$\mathbb{CP}^1$~~, $S^2$.

Want to build meromorphic functions on $\mathbb{C}/\Lambda$.

~~Note that $\mathbb{C}/\Lambda$ comes with a specif~~

In the case of $S^2$ ~~there~~ it was easy to construct meromorphic functions.

For $T^2$ it is harder. For a general Riemann surface it is harder still.

~~For $\mathbb{C}$~~ When $T^2$ is given as $\mathbb{C}/\Lambda$ we are starting with some geometric information about it and we will exploit this geometric information in our construction.
(translation structure or non-zero holomorphic 1-form $dz$)

complete

$\mathbb{C} \rightsquigarrow \mathbb{C}/\Lambda \xrightarrow{f} \mathbb{CP}^1$ $S^2$.

~~a meromorphic function~~

If $f$ is meromorphic it has a degree which is the # of zeros or the # of poles.

Recall that $d \geq 2$.

Let's attempt to find a meromorphic function with a single pole of degree $k \geq 2$ at $0$.

$f$ gives rise to a doubly periodic function on $\mathbb{C}$ ie a function $\tilde{f}$ so that $\tilde{f}(z) = \tilde{f}(z+\omega)$ for $\omega \in \Lambda$.

A function $f$ with a pole at $0$ gives us a function $\tilde{f}$ with poles at the points of $\Lambda$.

The simplest way to construct such a function is to write $\tilde{f}(z) = \sum_{\omega \in \Lambda} (z-\omega)^{-k}$ (for $k \geq 2$)

If this expression makes sense it should have poles at pts. of $\Lambda$ of order $k$

We need to deal with the convergence question.

Note that we are not starting with a a Riemann surface of genus 1 and describing which curve it belongs to.

We are starting with $\mathbb{C}/\Lambda$ which has some extra structure and we are using that extra structure when we say what curve we are equivalent to. This construction is ~~not~~ not general to all genera but it gives more information than ~~the known~~ most general constructions.

($\mathbb{C}/\Lambda$ comes with a <u>translation</u> atlas.)

$\mathbb{C}/\Lambda$ has a non-vanishing holomorphic 1-form $dz$.

~~Recall $f(z) = \oint_{\partial D}$~~

~~Furthermore if $f_j \to f$ uniformly then $f_j' \to f'$ uniformly~~

Thm. (Weierstrass M-test) Let $f_n : W \to \mathbb{C}$ be a sequence of holomorphic functions on $W$. Suppose there is a sequence of positive real numbers $M_n$ such that $|f_n(z)| \leq M_n$ and $\sum M_n < \infty$ then $\sum f_n(z)$ converges uniformly to a holomorphic function, call it $f$, and

$$f'(z) = \sum f_n'(z).$$

(Works with 2 indices of summation)

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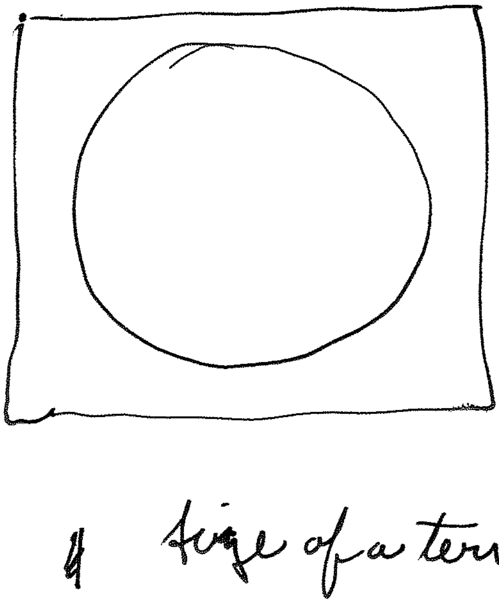

Now consider

$$\sum_{\omega \in \Lambda - \Lambda'} (z-\omega)^{-k} \qquad |z| < R$$

Assume $\Lambda = \{ m \omega_1 + n \omega_2 \}$

Assume $\Lambda' = \{\omega : |\omega| < 2R\}$

~~# of terms of~~

Sum over squares where $\max\{m, n\} = k$. # of terms $= 8k$.

~~#~~ Size of a term ~~$\frac{1}{}$~~ $\frac{1}{(z-\omega)^k}$

$\boxed{\text{assume } \Lambda' \text{ ~~contains~~ all } \omega \text{ with } |\omega| < 2R}$

$k \leq \sqrt{m^2+n^2} \leq \sqrt{2}\, k$ $\quad |\omega| - |z| \leq |z-\omega| \leq |\omega| + |z| \leq |\omega|.$