Riemann surfaces and the grand synthesis.

Algebraic geometry, topology, geometry, commutative algebra (field of meromorphic functions), complex analysis, number theory, discrete groups

We can see many of these themes coming together in studying tori.

~~I used the fact that the product of the~~

~~de~~

I used the fact that the degree of a composition is the product of the degrees.

$$\begin{tikzcd} \mathbb{C}/\Lambda \arrow[d, "\pi"] \arrow[dr, "P"] & \\ \mathbb{C}/\Gamma \arrow[r, "P_0"'] & \mathbb{CP}^1 \end{tikzcd}$$

π is a branched cover. Critical points are $\alpha, \beta, \gamma, \delta$.

π is a regular cover and the ~~critical values~~ deck group is $\langle L \rangle$. Critical points are the fixed points of L.

$\boxed{\text{$P$ is a branched cover of $\mathbb{CP}^1$. Branch pts. are images of $\alpha, \beta, \gamma, \delta$.}}$

Prop. Degree $P_0 = 1$.

1 ~~Follows from~~

Cor. P_0 is a conformal isomorphism.

P_0 is bijective.

Cor. ~~$P(\alpha)$, $P(\beta)$~~ $a = P(\alpha)$, $b = P(\beta)$, $c = P(\gamma)$, $d = P(\delta) = \infty$ are all distinct.

Follows from the fact that P_0 is 1-1.

We have a geometric model for the function P.

Cor. $\wp(z) = \wp(w)$ iff $z = \pm z + \omega$ $\omega \in \Lambda$. In particular

$\wp(z) = \wp(L(z))$ (For pts in $\mathbb{C}/\Lambda$, ~~$\wp(z) = \wp(L(z))$~~ $\wp(z) = \wp(w)$ iff $z = w$ or $z = L(w)$)

Cor. $\wp(\alpha) \neq \wp(\beta) \neq \wp(\gamma) \neq \wp(\delta)$. Proof. These are distinct points of $\mathbb{C}/\Gamma$.

Cor. The surfaces $\mathbb{C}/\Gamma$ which are a priori Riemann surfaces and topological spheres are, in fact, ~~standard~~ conformally the standard sphere $\mathbb{CP}^1$.

Proof. The $\wp$ function gives the conformal equivalence.

Lemma.

$\wp'(L(z)) = -\wp'(z)$. $\qquad \wp' : \mathbb{C} \to \mathbb{C}$.

Proof. ~~If~~ $\wp$ is an even function so $\wp'$ is an odd function. $\wp'(-z) = -\wp'(z)$.

If ~~$z \in \mathbb{C}$~~ $z + \Lambda \in \mathbb{C}/\Lambda$ represents a point then ~~L~~ $\rho(z)$ is represented by $-z$.

$\wp'(z) = \wp'(L(z))$ iff $\wp'(z) = 0$.

(A)

Prop.
$$(\wp'(z))^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$$

where
$$\boxed{\begin{aligned} g_2 &= g_2(\Lambda) = 60 \sum_{\omega\in\Lambda-\{0\}} \omega^{-4} \\ g_3 &= g_3(\Lambda) = 140 \sum_{\omega\in\Lambda-\{0\}} \omega^{-6}. \end{aligned}}$$

Delay this.

Proof. $\wp(z) - \frac{1}{z^2} = \sum_{\omega\in\Lambda-\{0\}} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}$

vanishes at 0 and restricts to a holomorphic function in a nbd. of 0 in $\mathbb{C}$. Moreover $\wp(z)$ is an even function so

$\wp(z) = \wp(-z)$

Check $\wp(0) = 0$. missing term

Construct a poly. in $\wp, \wp'$ with no pole at 0.

$$\wp(z) = z^{-2} + \lambda z^2 + \mu z^4 + z^6 h(z)$$

$$\wp'(z) = -2z^{-3} + 2\lambda z + 4\mu z^3 + 6z^5 h(z) + z^6 h'(z)$$

$2\cdot(-2)\cdot 4$

$$(\wp'(z))^2 = 4z^{-6} - \cancel{8}\lambda z^{-2} - \cancel{8}16\mu + \dots$$

$2(-2)\cdot 4$

$(a+b+c\dots)^2$

$(a^2+2ab+\dots)$

Double the cross terms.

$$(\wp(z))^3 = z^{-6} + 3\lambda z^{-2} + \cancel{3\lambda}\, 3\mu$$

$(1\cdot1\cdot\lambda)\cdot 3$ $\quad 3\cdot(1\cdot1\cdot\mu)$

$$(\wp'(z))^2 - 4(\wp(z))^3 = \cancel{-8\lambda z^{-2} - 20z^{-2} - 16\mu - 12\lambda}$$

$$-20\lambda z^{-2} - 28\mu$$

$(a+b+c\dots)^3$

$a^3 + 3ab^2 + 6abc + \dots)$

$+ 20\lambda\wp(z) =$

$+ \cancel{8}\, 28\mu = 0.$

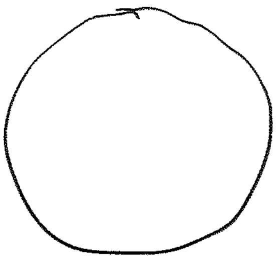
$(\wp'(z))^2 \in \cancel{\wp(z)}$

~~So then~~

The function $k(z) = (\wp'(z))^2 - 4\wp(z)^3 + g_2\wp(z) + g_3$

has no pole at 0. ~~So~~ Recall that ~~this function~~ $\wp$ only has poles at points of Λ and $\wp'$ ~~only~~ $= -2\sum_{\omega\in\Lambda}(z-\omega)^{-3}$ only has poles at points of Λ.

(After subtracting a meromorphic function with poles at pts. of Λ in the disk of radius R, $\wp$ and $\wp'$ are uniform limits of holomorphic functions)



$k(z)$ is Λ periodic.

Now $k(z)$ can only have poles at pts. of Λ but we have created k so that it has no poles at 0. Conclude that, ~~$k(z)$~~ not being surjective, $k(z)$ is constant. $k(0)=0$

~~so then~~ $k(z) \equiv 0$.

Lemma. $g_2 = 60 \sum_{\omega \in \Lambda} \omega^{-4}$, $g_3 = 140 \sum_{\omega \in \Lambda} \omega^{-6}$ Eisenstein series.

$$P(z) - z^2 = \lambda z^2 + \mu z^4$$ where $g_2 = 20\lambda$, $g_3 = 28\mu$.

$$= \sum_{\omega \in \Lambda - \{0\}} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}$$

evaluated at 0,

$$(P(z) - z^2)' = -2 \sum_{\omega \in \Lambda - \{0\}} (z-\omega)^{-3} = 2\lambda z + 4\mu z^3$$

$$(P(z) - z^2)'' = 6 \sum_{\Lambda - \{0\}} (z-\omega)^{-4} = 2\lambda + 12\mu z^2$$

$$(P(z) - z^2)''' = -24 \sum (z-\omega)^{-5} = 24\mu z$$

$$(P(z) - z^2)'''' = 120 \sum (z-\omega)^{-6} = 24\mu$$

$$120 \sum_{\omega \neq 0} \omega^{-6} = 24\mu$$

$$6 \sum_{\omega \neq 0} \omega^{-4} = 2\lambda$$

Let $g_2 = \cancel{24\lambda}\ 20\lambda$

$g_3 = 28\mu$.

$g_2 = 20\lambda = 60 \sum \omega^{-4}$

$g_3 = 28\mu = \cancel{120}$

$28 \cdot \frac{120}{24} \sum \omega^{-6}$

$= 28 \cdot 5 \cdot \sum \omega^{-6}$

Define g_2, g_3

$$k(z) = (\wp'(z))^2 - 4\wp^3(z) + \underbrace{20\lambda}_{g_2}\wp(z) + \underbrace{28\mu}_{g_3}.$$ ⑥

~~Let $g_2=$~~ $\lambda = 3\sum \omega^{-4}$

$\mu = 5\sum \omega^{-6}$

$g_2 = 60\sum \omega^{-4}$

$g_3 = 140\sum \omega^{-6}.$

Recall that in order to show that $\mathbb{C}/\Lambda$ is
an algebraic curve we need two non-constant meromorphic fcns. and an algebraic relation between them. We now have that for $\wp_\Lambda$ and $\wp'$.

~~Then~~ still need to show that the resulting curve is non-singular.

4d
(8)

Definition. Let C_Λ be the projective curve in P_2 defined by the polynomial

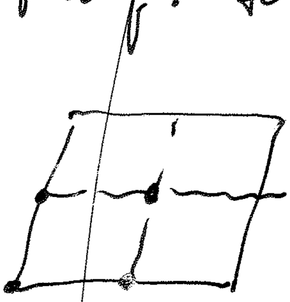
$$Q_\Lambda(X,Y,Z) = Y^2Z - 4X^3 + g_2XZ^2 + g_3Z^3$$

with $g_2 = g_2(\Lambda)$ and $g_3 = g_3(\Lambda)$ defined as before.

~~Prop~~ Note that this is the homogenization of the curve defined before.

~~Lemma~~. Prop. C_Λ is non-singular.

Proof. Let $\alpha = \wp(\frac{1}{2}\omega_1)$ $\beta = \wp(\frac{1}{2}\omega_2)$ $\gamma = \wp(\frac{1}{2}(\omega_1+\omega_2))$



~~Vertex~~ "Vertices of the pillowcase".

~~α, β, γ are distinct~~

$\wp(\alpha)$, $\wp(\beta)$, $\wp(\gamma)$ are distinct points in $\mathbb{C}$ by the previous prop.

~~We also have~~ 4Σ

Since $\wp$ is even and doubly periodic, $\wp'$ is odd and doubly periodic with the same periods.

Prop. The cubic curve C_Λ is non-singular

Proof. Let $a = \wp(\frac{1}{2}\omega_1)$, $b = \wp(\frac{1}{2}\omega_2)$, $c = \wp\left(\frac{\omega_1+\omega_2}{2}\right)$.

~~We know that $\iota(\alpha) =$~~

$\wp'(\iota(z)) = -\wp'(z)$ so $\wp'(\alpha) = -\wp'(\iota(\alpha)) = -\wp'(\alpha)$.

and $\wp'$ vanishes at α, β, γ.

Now $(\wp'(z))^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$

so a, b, c are roots of $4x^3 - g_2 x - g_3$ and they are distinct.

$$4x^3 - g_2 x - g_3 = 4(x-a)(x-b)(x-c)$$

$$y^2 = 4x^3 - g_2 x - g_3$$

~~$y^2 z = 4($~~

Homogenize $\quad y^2 z = 4(x-az)(x-bz)(x-cz)$

Distinct roots $\Rightarrow C_\Lambda$ smooth.