Historical motivation for studying Riemann surfaces in the 19th century was the evaluation of integrals. There was a well developed theory of the integral $\int \frac{dx}{\sqrt{1-x^2}}$ ~~am~~ but no the integral $\int \frac{dx}{\sqrt{1-x^3}}$ was a mystery.

We will describe the theory of these "elliptic integrals". Involves the study of "elliptic curves" like

~~$w^2=$~~ $w^2 = 1 - z^3$ and $W = (z^2-1)(z^2-k)$.

C. varieties in $\mathbb{C}^2$

In the 19th-20th centuries there was a shift in mathematical style away from concrete description of ~~solve~~ polynomial equations to abstract definitions like Riemann surfaces. The question then ~~becomes~~ ~~given~~ arises given an abstract object when can you embed it as a variety? The route to answering this question is through studying the collection of meromorphic functions on our abstract object. Genus, p

What can you say about their zeros, poles? ②

~~Theorem~~ We use these functions to embed our abstract Riemann surface in $\mathbb{C}^n$ or $\mathbb{CP}^n$.
Weierstrass P-functions

In the 20th-21st centuries Riemann surfaces have been applied to new areas of study
hyp. 3-manifolds, polygonal billiards
One of the outcomes is new geometric language for ~~descr~~ describing ~~was~~ them. I will try to convey ~~some~~ ~~hints of the~~ some of this as well. Language of "geometric structures".

Connections between translation structures and holomorphic differentials, half-translation structures and quadratic differentials.

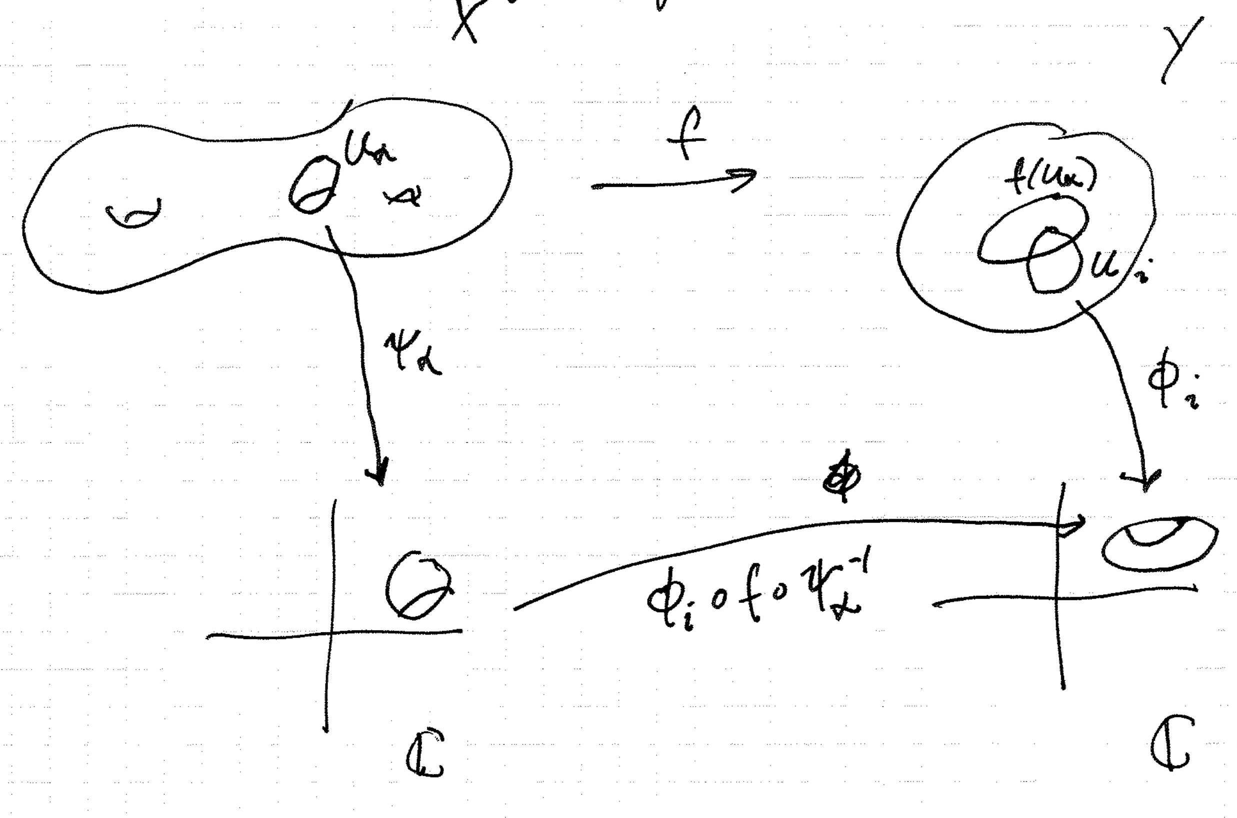
~~Overview? complex analysis. Repackaging of an old notion.~~

Returning to Riemann surfaces.

Definition. Let X be a Riemann surface with an atlas $(U_\alpha, \tilde{U}_\alpha, \psi_\alpha)$ and Y another Riemann surface with an atlas $(V_i, \tilde{V}_i, \phi_i)$. A map $f : X \longrightarrow Y$ is said to be holomorphic if for each α and i the composition

$$\phi_i \circ f \circ \psi_\alpha^{-1} \text{ is holomorphic}$$

on its domain of definition.



Note that $\phi_i \circ f \circ \psi_\alpha^{-1}$ is mapping an open set in $\mathbb{C}$ into another open set in $\mathbb{C}$ so that the notion of "holomorphic" is the usual notion.

~~Using the same formalism we can talk about smooth maps between smooth surfaces etc.~~

Definition. We say that two Riemann surfaces X and Y are *equivalent* if there is a holomorphic bijection $f: X \to Y$ with holomorphic inverse.

Definition. Two atlases $\mathfrak{a} = (U_\alpha, \tilde{U}_\alpha, \psi_\alpha)$ and $\mathfrak{a}' = (V_i, \tilde{V}_i, \phi_i)$ on the same space X are equivalent if the identity map ~~is~~ $\mathrm{id}: \underset{\mathfrak{a}}{X} \longrightarrow \underset{\mathfrak{a}'}{X}$ is an equivalence.

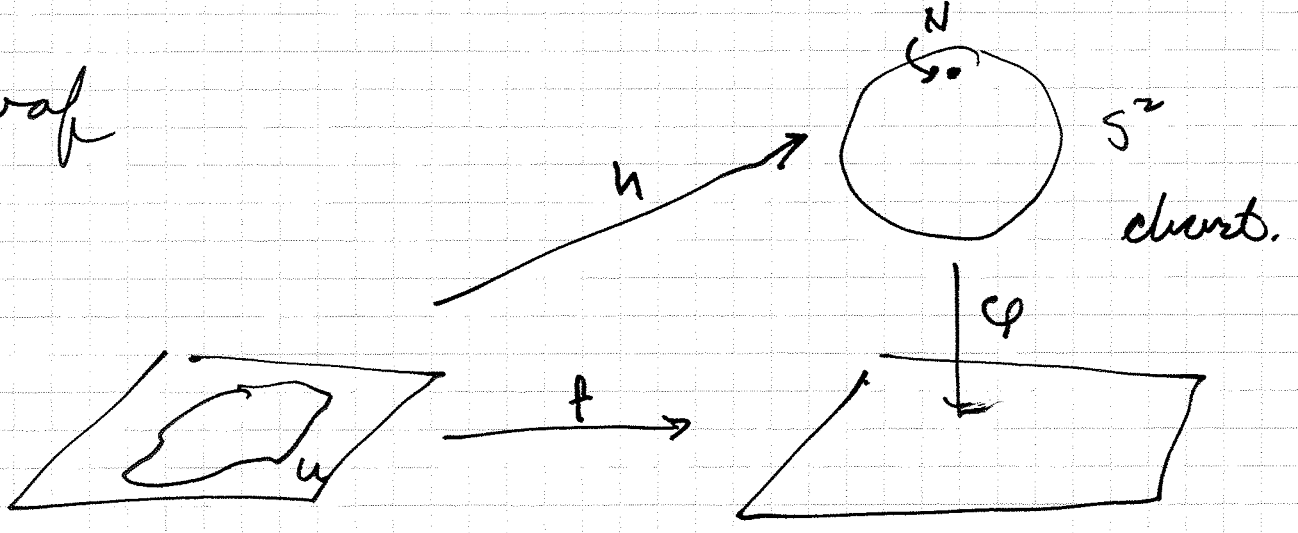
Recall that a function $f : U \subset \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ is meromorphic if it is holomorphic where it is finite valued ~~at a~~ and at each $z_0 \in U$ where ~~it is~~ $f(z_0) = \infty$ the function can be written as a convergent power series

$$f(z) = \sum_{n=-k}^{\infty} a_n (z - z_0)^n \qquad (\text{say } a_k \neq 0).$$

Equivalently $f(z) = \dfrac{g(z)}{(z-z_0)^n}$ where g is holomorphic and $g(z_0) \neq 0$.

Proposition. A holomorphic map $h : U \overset{\subset \mathbb{C}}{\longrightarrow} S^2$ which is not the constant ∞ corresponds to a meromorphic function and conversely.

Proof



Say we are given a meromorphic function f then we define h so that $h(z) = \varphi^{-1} f(z)$ where $f(z)$ is finite and $h(z) = N$ where $f(z) = \infty$.

Want to check that h is holomorphic at those points z_0 where $h(z_0) = N$.

Use the other chart $\psi : S^2 - \{S\} \to \mathbb{C}$.

For z near z_0 $\quad \psi \circ h(z) = \psi \circ \varphi^{-1} f(z)$

$$= \frac{1}{f(z)}$$

$\psi \circ h(z_0) = \psi(N) = 0.$

now $\cancel{f(z)}$ $f(z) = \dfrac{g(z)}{(z-z_0)^n}$ so

$$\varphi \circ h(z) = \frac{1}{f(z)} = \frac{(z-z_0)^n}{g(z)}$$

is holomorphic.

Converse is similar.

~~Example: Given two translation surfaces we can ask whether they are equivalent as Riemann surfaces or ~~whether they are equivalent as translation surfaces~~.~~

For open sets in $\mathbb{C}$ new style holomorphic is old style holomorphic (one chart.)

Example: The unit disk is equivalent to the upper half-plane by way of the map

$$\begin{array}{ccc} z & \longmapsto & \dfrac{w-i}{w+i} \\ \uparrow & & \uparrow \\ \mathcal{U} & & D \end{array}$$

Example: The unit disk D is ~~[illegible]~~ not equivalent to the complex plane.

Proof. If they were equivalent we could find a ~~[illegible]~~ holomorphic map $f: \mathbb{C} \to D$ which was surjective and invertible.
By Liouville's theorem a bounded holomorphic function is constant so f would have to be constant. In particular, it is neither injective nor surjective.

This is a very flexible and powerful idea. If we take the word "holomorphic" in our discussion and replace it with another class of maps between open sets in $\mathbb{C}$ we get another type of mathematical object.

We can consider, for example, the class of smooth (C^∞) functions.
~~In this case we~~ If we replace the word "holomorphic" by the word "smooth" we have defined a "C^∞ atlas" or "smooth atlas" on a surface and hence the notion of a smooth surface.

If we replace $\mathbb{C}$ by $\mathbb{R}^n$ we get the notion of a "smooth n-dimensional manifold".

If we ~~repl~~ ~~rel~~ replace ~~$\mathbb{C}$~~ $\mathbb{C}$ by $\mathbb{C}^n$ and holomorphic ~~map~~ functions by the notion of holomorphic maps used in several complex variables we get the notion of a "complex n-manifold".

From this ~~vantage~~ point of view we see that a Riemann surface is a complex 1-manifold. Call S^2, $\mathbb{P}^1$ or $\mathbb{P}$ because it corresponds to complex lines in $\mathbb{C}^2$. Define $\mathbb{P}^n$ as space of lines in $\mathbb{C}^{n+1}$.

Example. ~~A Riemann surface has a~~

~~transla~~ A "translation structure" on a Riemann surface is given by an atlas where all change of coordinate maps have the form $\varphi_\alpha \circ \varphi_\beta^{-1}(z) = z + c$.

A "half translation structure" is given by an atlas where all change of coordinate maps have the form

$\varphi_\alpha \circ \varphi_\beta^{-1}(z) = \pm z + c$.

Note that in the case of translation structures the ~~fact that~~ statement that $f' = g$ does make sense independently of the coordinate chart.

Remark: The terminology "translation structure" and "half-translation" structure was invented relatively recently but it corresponds to classic notions in

Atlases and geometric structures.

~~Examples~~

We can use an atlas to give our Riemann surface additional ~~structures~~ geometric structure (or to recognize additional structure that our surface already has).

This is a viewpoint which was popularized by Thurston ~~(The geometry and topology of three manifolds)~~ [illegible] but it is of historical and contemporary importance in Riemann surface theory. We will see a

~~Clarifies some~~ Gives a nice perspective on classical work. Connection with elliptic integrals for example. (Gauss, Riemann, Weierstrass)

Here is the idea.

Just for geom. structure: ~~Without loss of generality we can~~ Say that ~~assume~~ that our system of charts lie in some Riemann surface $M = \mathbb{C}, \mathbb{P}^1, D$.

Choose a group G that acts on M by ~~group invertible~~ conformal automorphisms.

eg. ① G = group of translations acting on $\mathbb{C}$. translation structure

half translation structure ② $G = \{z \mapsto az+b : a = \pm 1$ acting on $\mathbb{C}\}$

③ $G = \{$ linear fractional transformations acting on $\mathbb{P}^1$

(4) $G = \{$linear fractional transformations acting on $D\}$.

~~We say~~

Let R be a Riemann surface with an atlas $\mathcal{A}$. We say that $\mathcal{A}$ gives R a G-structure if all of the change of coordinate functions $\phi_\beta \circ \phi_\alpha^{-1}$ ~~are restrictions of~~ have the form $\phi_\beta \circ \phi_\alpha^{-1}(z) = g(z)$, where defined for $g \in G$.

The names ~~associated~~ to the examples are:

1. translation structure
2. ~~half-translation structure~~
3. ~~complex projective structure~~
4. ~~hyperbolic structure.~~

similarity structure,

I will give some examples of Riemann surfaces having some of these structures.

Torus, cylinder

Some G structures have compatible metrics.

Connection ~~with~~ between translation structures and integration.

half-translation structures

Pillowcase, half cylinder.

$\int \sqrt{q(z)}\, dz$.