①

Recall that a "hyperbolic" surface is a Riemann surface R so that $\tilde{R} \simeq$ ~~UHP~~ Δ.

"Parabolic surface" $\tilde{R} \simeq \mathbb{C}$

This is a holomorphic invariant.

~~We emphasize that the~~

A hyperbolic surface has a unique conformal metric of curvature -1.

Follows from the Schwartz lemma that any holo. map between hyperbolic surfaces is distance non-increasing. Any hol. automorphism is distance preserving. ~~(though I did not prove this)~~

metric is a holomorphic invariant. global invariant

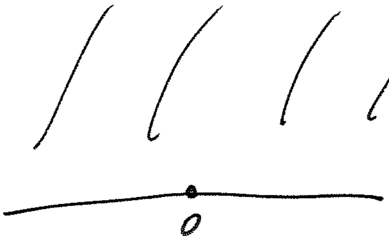
This metric is a tool that can be used to study the Riemann surface structure of R. ~~(not the only to~~ (not the only possible tool).

~~We saw that~~ Questions about equivalence of hyp. surfaces reduce to questions about subgroups of $PSL(2,\mathbb{R})$.

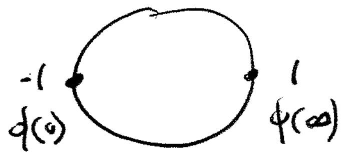
Theme is using geometry to help us understand complex structure.

Naive discussion of ~~gen~~ hyperbolic geometry.

2 models for the hyperbolic plane



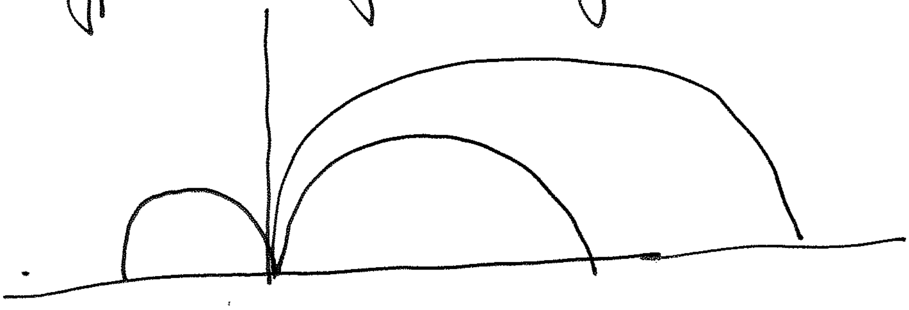
$z \mapsto \frac{z-i}{z+i}$

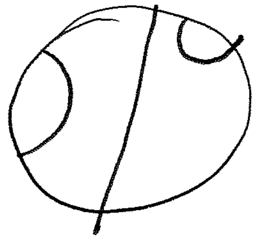


circle at ∞.

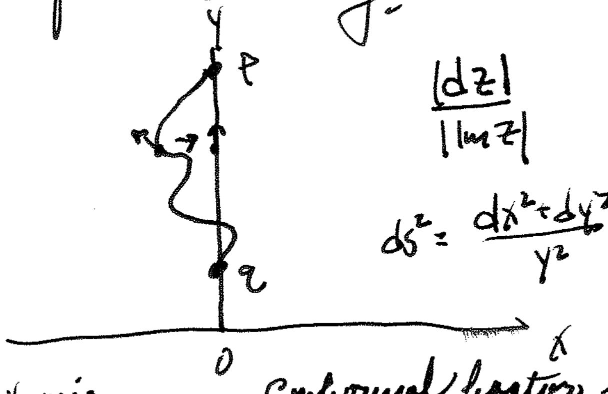
Some properties are easier to see in one model some in the other.

Hyperbolic geometry:





Prop. In the ~~upper~~ UHP model or the disk model hyperbolic geodesics correspond to circles/lines which intersect the boundary perpendicularly.



$\frac{|dz|}{|\mathrm{Im}\, z|}$

$ds^2 = \frac{dx^2+dy^2}{y^2}$

Φ

~~Projection~~

Say p, q lie on the ~~two~~ pos. imaginary axis. Let γ be a path from p to q.

$\pi: \mathbb{R}^2 \to Y$-axis decreases distance.

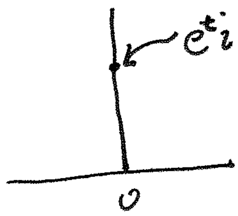
Conformal factor is the same for both pts.

Claim that the path of minimal length must lie on the imaginary axis since the projection $\pi_y (x+iy) \mapsto y$ is distance ~~non-increasing~~ ~~dec~~ non-increasing.

Conclude that this half-line is a geodesic. Geodesics are preserved by isometries. All isometries extend to conf. automorphisms of $\mathbb{CP}^1$ so they preserve the intersection angle with the boundary. Can compute that the image of a circle/line is a circle/line.

~~Since we are in PSL(2, ℂ) we can assume that the product of the eigenvalues is 1. and replacing since we mod out by $\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$ we may assume that eigenvalues are positive when real.~~

Parametrization of the geodesic:



~~$t \mapsto e^t$~~ $\phi(t) = e^t i$

$\phi'(t) = e^t i$

$$ds^2 = \frac{|dz|^2}{|\mathrm{Im}\, z|^2} \qquad ds = \frac{|dz|}{|\mathrm{Im}(z)|} \qquad |\phi'(t)| = \frac{|i \cdot e^t|}{e^t} = 1.$$

~~$z \mapsto az$~~

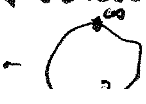
The isometry $z \mapsto az$ preserves this geodesic and takes $i \mapsto ai$. $\phi(0)$ to $\phi(\log a)$.

Translation by $\log a$.

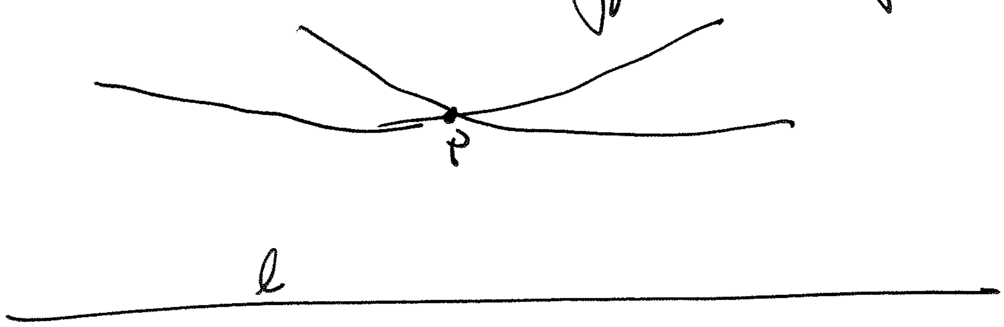
$$\phi(t) = e^t i \mapsto a \cdot e^t i = e^{t + \log a}\, i = \phi(t + \log a).$$

$z \mapsto az$ preserves a unique geodesic. Geodesic connects 2 pts on the circle. } action on the circle at ∞ has exactly 2 fixed pts.

$\Delta =$ 



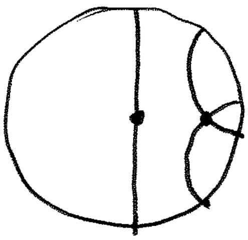
Note that then Euclid's parallel postulate does not hold in hyperbolic geometry.



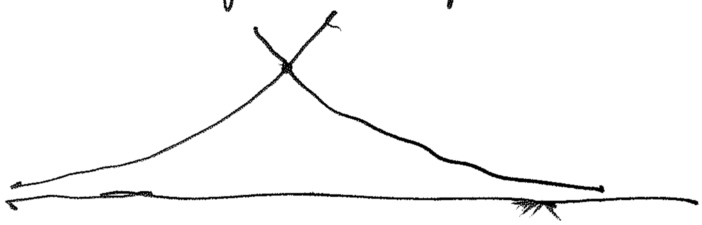
geodesic

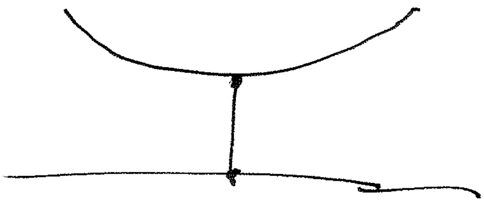
(independent model free picture.)

Hyperbolic geometry was introduced by to show the ~~indep~~ logical independence of the parallel postulate.



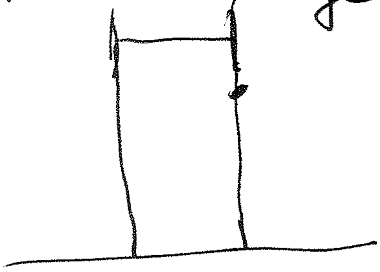
Among the the family of parallels there are two limit ~~parallels~~ parallels.

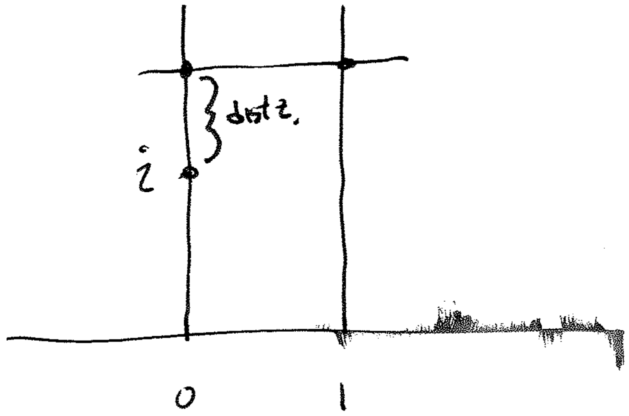




These converge exponentially fast.

UHD





$$\frac{\sqrt{dx^2 + dy^2}}{y}$$

⑤

$d(ie^t, ie^t + 1)$

$\leq \frac{1}{e^t}.$

Conjugacy classes of elements of $PSL(2,\mathbb{R})$.

Prop. An isometry of hyperbolic space preserves a line, pt. or horocycle.

Proof. $\alpha \in PSL(2,\mathbb{R})$ corresponds to a matrix A with real entries. Jordan form of A is:

$A: \mathbb{C}^2 \to \mathbb{C}^2$
$A(\mathbb{R}^2) = \mathbb{R}^2$.
$\mathbb{R}^2 \sim \mathbb{R} \subset$ UHP.

① A has 2 distinct eigenvalues that are real.

② that are complex conjugate

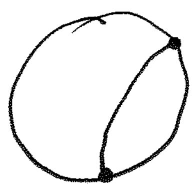
③ A has a unique real eigenvalue and 1 eigenvector

④ $A = I$. [Eigenvalues themselves are not conj. invariant but the ratio of eigenvalues is $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} = \begin{pmatrix} \lambda\alpha & 0 \\ 0 & \lambda\beta \end{pmatrix}$ α/β.

Case 1a. In case 1. α has 2 fixed pts on $\mathbb{CP}^1$. If these are not on $\mathbb{R}$ then they are complex conj. one in upper disk one in lower disk.

Fixes a pt. in UHP.

If these are real then there are 2 pts on $\partial\Delta$. ~~Affine~~ A leaves invariant the line between them.



If there is a single fixed pt. it must correspond to a real eigenvector. Choose coordinates to ~~put~~ put it at ∞ in the UHP model.

$z \mapsto z + c$. Preserves horizontal lines. These are horocycles.

~~Classification of annuli~~

The holomorphic ~~clas~~ classification of annuli.

Let ~~...~~ R be an annulus. $\pi_1(R) = \mathbb{Z}$.

$R = {}^{UHP}/_{\Gamma}$ where Γ is generated by a single element.

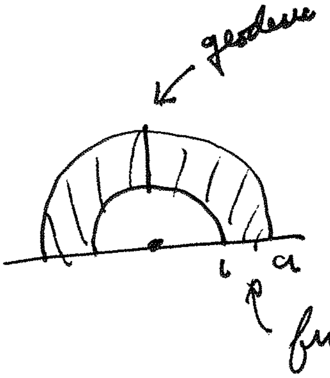
$R \sim R'$ iff Γ is conjugate to Γ'.

~~Then~~ Say γ generates Γ.

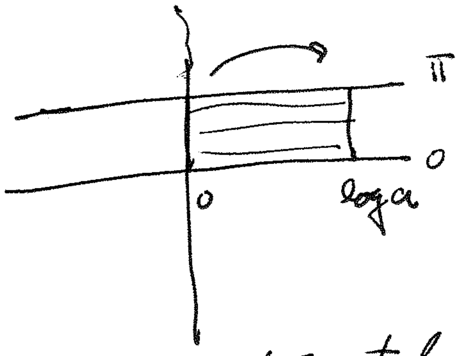
γ cannot fix a point in the hyp. plane so γ is translation along a geodesic or

~~γ is parab~~

$\gamma \sim z \mapsto az$ or $\gamma \sim z \mapsto z+1$



$e^z \longleftarrow z$



mult. by a here

corresponds to translation by log a here.

Hyperbolic picture

metrically correct picture

length of geodesic is log a.

Flat picture:

Cylinder of ht. π, and circum. log a.

log a is a <u>conformal</u> invariant of the
(holomorphic)
annulus. The metric here is a tool.
Both flat and hyperbolic metric pictures
are interesting.

Parabolic transformation.

$z \mapsto z + 1$

Consider the function $e^{2\pi i z}$: UHP $\longrightarrow \Delta - \{0\}$.

$\mathrm{Im}(z) > 0 \iff \cancel{\mathrm{Im}(2\pi i z)}\ \mathrm{Re}(2\pi i z) < 0$

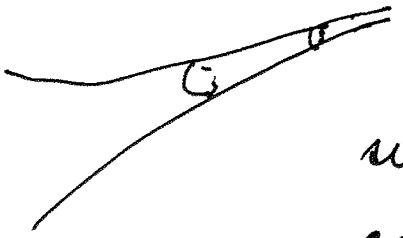
So $\mathrm{UHP}/_{z \mapsto z+1} \simeq \Delta - \{0\}$.

Hyperbolic surfaces of finite type.

~~If R is a surface with finite~~

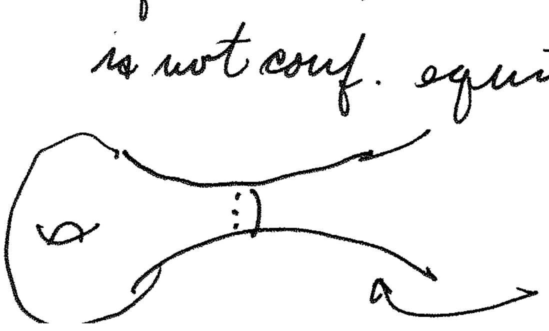
Say R is a surface with finitely generated fundamental group. ~~A~~ This is equivalent to saying that $R = X \cup A_i$ where X is compact and A_i are annuli. X is a compact ~~[illegible]~~ manifold with boundary and $R - X$ is a union of annuli.

Say R is a hyperbolic Riemann surface then we can take a covering space of R corresponding to ~~can~~ $\pi_1(A_i)$. If R has finite type (as a Riemann surface each A_i is conformally equivalent to a punctured disk.



So R viewed as a hyperbolic surface has ~~a~~ finitely ~~many~~ many cusps.

If some ~~each~~ annulus ~~does not have finite~~ is not conf. equiv. to a punctured disk then it looks like a flared end, ~~[illegible]~~ cut off by a geodesic.



Proposition. A subgroup of $PSL(2,\mathbb{R})$ which acts discretely and with no fixed points does not contain $\mathbb{Z}\oplus\mathbb{Z}$.

Proof. If matrices A and B commute then B permutes the ~~eigenvs~~ eigenvectors of A:

$AV=\lambda V$ $\quad$ ~~BV~~ $\cancel{BAV=B\lambda V=BV}$

Claim BV is an eigenvector of A:

$ABV = BAV = B\lambda V = \lambda BV$. (with same eigenvalue)

If A is a hyperbolic trans then A has 2 eigenvectors associated to different real eigenvalues. B preserves these eigenvectors so B takes the corresponding geodesic to itself. 2 generator ~~group~~ cannot act discretely ~~on~~ by isometries on a line.

~~If A is elliptic~~

If A is parabolic then B fixes the unique eigenvector of A. If B had a 2nd eigenvector then A would be hyperbolic so B is parabolic. A, B both ~~fix~~ / preserve the ~~horocycles corresponding~~ same set of horocycles.

Corollary. If R is a Riemann surface (compact or not) and $\pi_1(R) = \mathbb{Z} \oplus \mathbb{Z}$ then ~~$R = \mathbb{C}/\Lambda$~~ R is not hyperbolic so ~~$R \neq \mathbb{C}$~~ R is conf. equiv. to $\mathbb{C}/\Lambda$. If $\pi_1(R) = \mathbb{Z}$ then R can be hyperbolic or parabolic. If $\pi_1(R)$ is not trivial, $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}$ then R is hyperbolic.

Does not use Gauss-Bonnet. Takes care of the non-compact case.

Corollary. $\chi(R) < 0 \Rightarrow$ then R hyperbolic.

~~$\chi(R) = 0$ R. the torus is parabolic.~~

~~$\chi(R) \ldots$~~

T^2 is parabolic

annulus can be either

disk can be either.