Consequences of the Euler char. formula. $\chi(R) = -\sum \mathrm{Ord}(z_i)$

① There are no hol. 1-forms on S^2; $\mathrm{Ord}(p_i) \geq 0$.

Curvature at isolated pts.

② ~~Every~~ A hol. 1-form on T^2 has no zeros.

(2a) Any 2 hol 1-forms on ~~the~~ a given T^2 are scalar multiples of one another.
~~Quotient is a non-~~ Ratio is a non-zero hol. fcn hence constant

③ 2-types of hol. 1-forms in genus 2:
R has genus 2 then $\chi(R) = -2$. Either there are 2 pts. with simple zeros at each pt. ~~or there~~ (cone angle 4π singular pts).
or there is 1 pt. with a double zero.
Double zero example: octagon.
2 simple zeros: decagon.

Quadratic differentials.
Recall that ~~as~~ in studying elliptic integrals we deal with expressions such as.

$$\int \frac{dz}{\sqrt{x^3-1}} \qquad \int \frac{dz}{\sqrt{f(z)}}$$

~~&~~ ① Expressions $\frac{dz}{\sqrt{f(z)}}$ give rise to half-translation structures

$$\frac{dz}{\sqrt{f(z)}} \quad \frac{dz}{-\sqrt{f(z)}} \text{ both give charts.}$$

The classical way of ~~descr~~ describing these is as quadratic differentials.

$$u \xrightarrow{\varphi} z$$

$$\left(\frac{dz}{\sqrt{f(z)}}\right) \longmapsto \frac{dz^2}{f(z)}.$$

$$\varphi^*(g(z)\,dz^2) = g(\varphi(u))\left(\frac{d\varphi}{du}\right)^2 du^2$$

This makes the denominator a well defined holomorphic (or meromorphic) function.

$g(z)dz^2$ gives rise to a half trans structure.

zeros of order n of g correspond to singularities of the half-trans. structure which are cone type singular ~~points~~ with cone angle

$$\frac{2\pi(n+2)}{2} = \pi(n+2). \quad 2\pi\left(\frac{n}{2}+1\right)$$

(Division by 2 corresponds to the fact that we take square root to get charts.)

Construct local charts by taking square root and integrating if poles of even order. Take branched double cover

~~Note that we can get a cone angle less the~~

For holomorphic 1-forms the cone angle is always a multiple of 2π. For quadratic differentials the cone angle is a multiple of π. Special case $n=-1$ corresponds to a cone angle of π.

$$\int \frac{dz}{\sqrt{x(x-1)(x-\lambda)}}$$ ~~gives rise to a flat~~

corresponds to the QD $\frac{dz^2}{x(x-1)(x-\lambda)}$ has poles at $0, 1, \lambda$. Half-trans. surface has cone angle π singularities at $0, 1, \lambda$ (and ∞).

Euler char. formula: If α is a QD on R then

$$\chi(R) = -\sum \frac{\mathrm{Ord}(z_j)}{2} \quad \text{where } \mathrm{Ord}(z_j) \geq -1.$$

Example: S_2 with 4 simple poles:

$$2 = -\sum_{j=1}^{4} \frac{-1}{2} = 4 \cdot \frac{1}{2}.$$

QD lifts to a hol 1-form on the
(Branched double cover is T^2 & at the branch points the cone angle doubles. Get now cone angle 2π ie non-singular points.

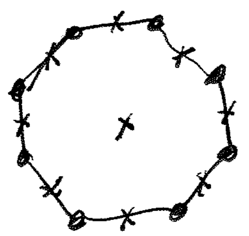
~~Families of Riemann surfaces~~ We have explored genus 1 to some extent.

What happens in genus 2?

Surfaces in genus 2 are hyper-elliptic which means that they can be realized as Riemann surfaces branched double covers of S^2. (like T^2).

Consider the octagon. We showed that there was an involution ι so that the quotient was a (topological) sphere. (Now we know it is really $\mathbb{CP}^1$.)

M_O

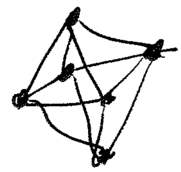


$\iota(z) = -z$. ι has 6 fixed points: ~~with con~~ 5 with cone angle π, 1 with cone angle 4π.

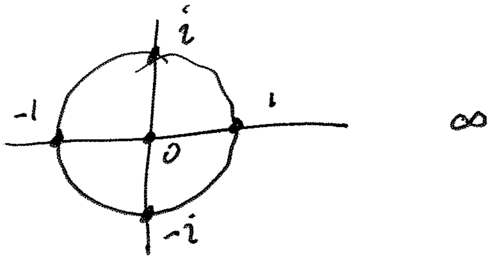
We can identify the Riemann surface M_O if we can identify the images of these points in $\mathbb{CP}^1$. (These will be the branch pts. for a branched double cover.)

Note that M_O has a rotation ρ with $\rho^8 = 1$. ρ commutes with ι (any matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ commutes with $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$) so ρ induces an automorphism of $\mathbb{CP}^1$ which ~~leaves~~ the

~~branch points~~ permutes the branch points.
Now $\rho^4 = L$ so ρ induces an automorphism of order 4 on $\mathbb{CP}^1$ which cyclically permutes the edge points and fixes the center point and the vertex. The only possibility is



that the points are arranged on $\mathbb{CP}^1$ like the corners of a tetrahedron.



These are the roots of $f(z) = z(z^4-1)$. An equation for an algebraic curve corresponding to M_0 is $y^2 = z(z^4-1)$.
(Remark: singular projective curve)

We can also describe the translation structure on M_0. This corresponds to a half-translation structure on $\mathbb{CP}^1$.
Since the points $0, 1, -1, i, -i$ are non-singular in M_0 they correspond to points of cone angle π in the quotient.

The point at ∞ ~~given~~ has cone angle ~~...~~ 6π and corresponds to a point of cone angle 3π in the quotient.

The quadratic differential should have simple poles at the roots of $z(z^4-1)$ and a zero of order n ~~where~~ at ∞ where $2\pi(\frac{n}{2}+1) = 3\pi$ so $\frac{2n}{2}+2=3$ $n=1$.

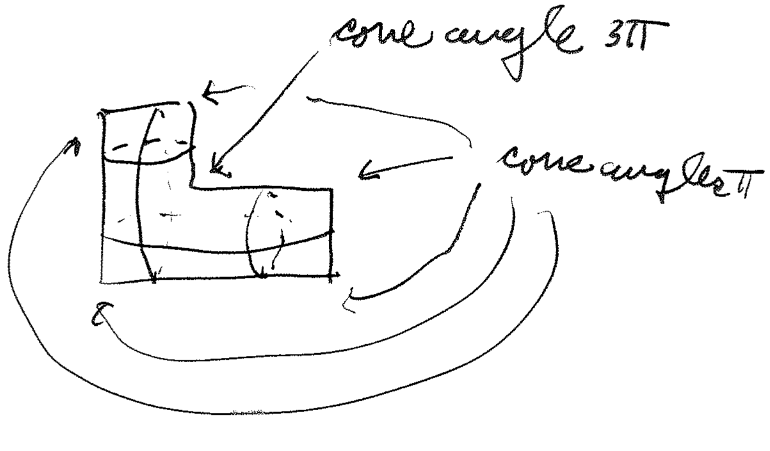
$$q = \frac{dz^2}{z(z^4-1)}$$

Hyper-elliptic integral:

$$\int \frac{dz}{\sqrt{z(z^4-1)}}$$

(We can now construct by integration a parametrization of M_0 so we ~~...~~ do not need to appeal to the Uniformization Theorem.)

Geometric picture



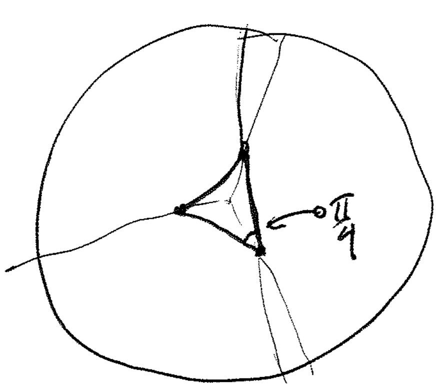
Not geometrically correct. Should shear.

The surface

Recall that for cubic curves in $\mathbb{CP}^2$ we showed that they have non-vanishing hol. 1-forms and we used that to show that they were $\cong \mathbb{C}/\Lambda$.

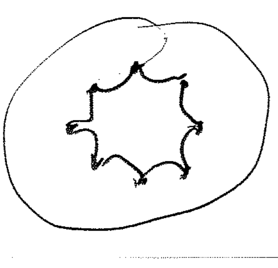
1. For a general ~~curve~~ projective curve C ~~no~~ of degree d in $\mathbb{CP}^2$ we can do a similar calculation to ~~show~~ that $g(C) = \frac{(d-1)(d-2)}{2}$.
   (See Donaldson ~~p. 102~~ p. 83-84)

2. M_0 ~~obviously~~ obviously does not have a complete translation structure.
   It does have a hyperbolic structure (by the UT) which is given as follows.

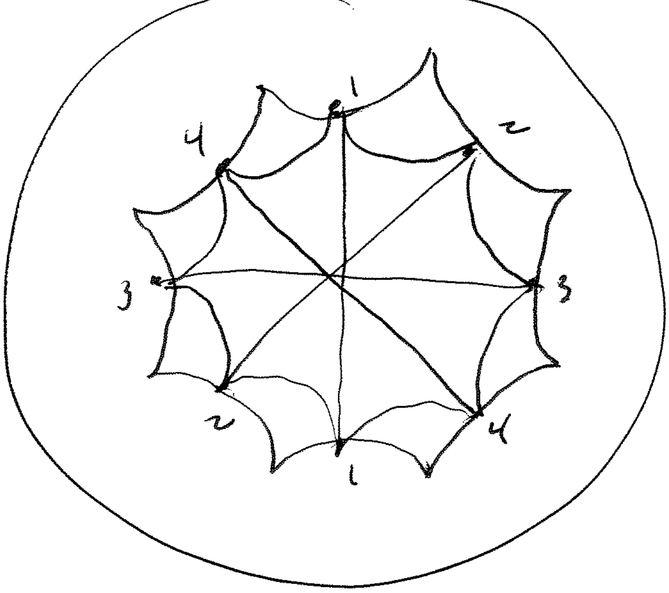


Construct an equilateral hyperbolic triangle with 3 angles equal to $\frac{\pi}{4}$.
Any number between 0 and $\frac{\pi}{3}$ is possible.

Assemble 8 of these to build a regular polygon.

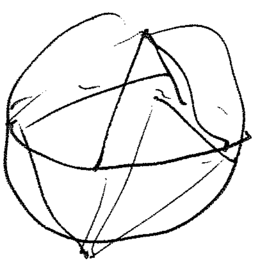


Use 8 more to build the next level.



Now glue opposite sides. Note that the angles at the vertex add up to $8 \cdot \frac{\pi}{4} = 2\pi$.
We get a hyperbolic surface which is ~~In fact the uniformizing~~
conformally equiv. to a branched double cover of the spherical regular tetrahedron.

Uniformizing map takes hyperbolic triangles to spherical triangles.



Automorphism of surface corresponding to $x \mapsto \frac{1}{x}$ takes hol. differential to a distinct hol. differential
Space of hol. diffs. has dimension 2.
~~Can~~ Construct an analog of the integration map into a 2 complex dimensional surface $\mathbb{C}^2/\Lambda$.
The Jacobian variety. Inclusion not isomorphism.