Support class: Monday 12-1 in MA_B3.01

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Remarks. Donaldson's book is for a 2-term course.

Though geometric structures do not figure explicitly in Donaldson's book he is very interested in differential equations. ~~Ends the book~~ (partly in connection with mirror symmetry)
Geometric structures are closely connected to differential equations.

We only look at one simple case: flat structures $\Leftrightarrow$ $\underbrace{z' = f(z)}_{\text{D.E.}}$ (ie integration)

~~Donaldson~~

Connection between translation structures and Abelian differentials.

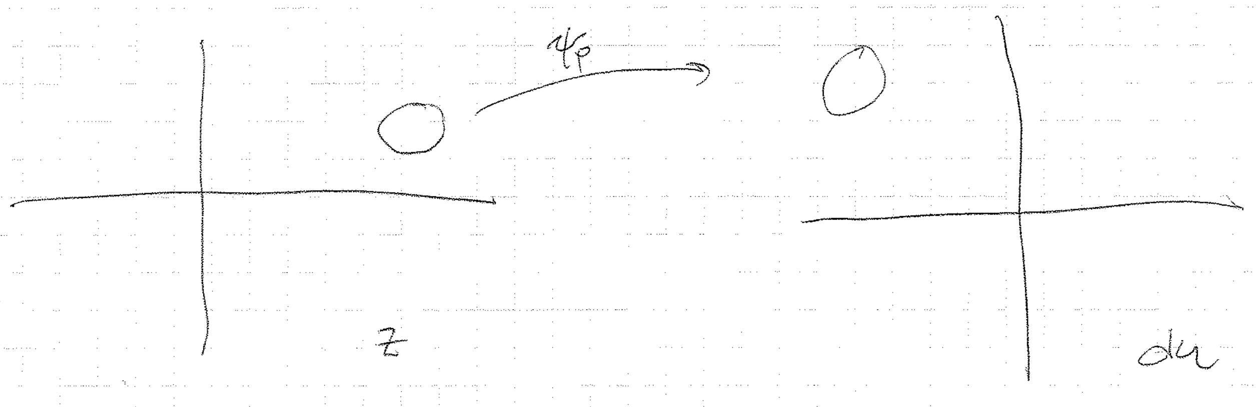
Showed that a translation structure gives an abelian differential. Also true that an Abelian differential gives a (singular) translation structure.

~~Say $V \subset \mathbb{C}$ is the domain of f~~
Consider $f\,dz$ where
~~Say~~ f is holomorphic in $V \subset \mathbb{C}$. Let V' be the set where f is holomorphic and non-zero.

V has a conformal structure, already, a with one chart

We will construct a translation atlas $\mathcal{A}'$ on V'.

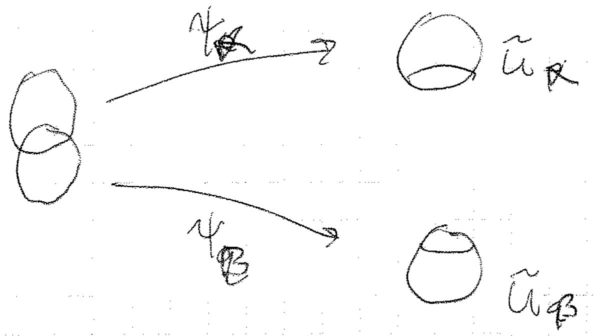
Say $p \in V'$ consider an anti-derivative F defined in a nbd of p. $F'(p) = f(p) \neq 0$ so F is an injection in a nbd U_p of p. Let $\psi_p : U_p \to \mathbb{C}$ be equal to F.



Claim that $(\psi_p)^* du = f(z)\,dz$:

$$(\psi_p)^* du = \frac{d\psi}{dz} \cdot dz = F'\,dz = f\,dz.$$

What do the overlap functions look like?



Note that the antiderivative is defined up to a constant so $\psi_\alpha = \psi_\beta + C$ Write $z \in \tilde{U}_\beta$ as $\psi_\beta(p)$

$$\psi_\alpha \circ \psi_\beta^{-1}(z) = \psi_\alpha \circ \psi_\beta^{-1}(\psi_\beta(p))$$
$$= \psi_\alpha(p)$$
$$= \psi_\beta(p) + C$$
$$= z + C.$$

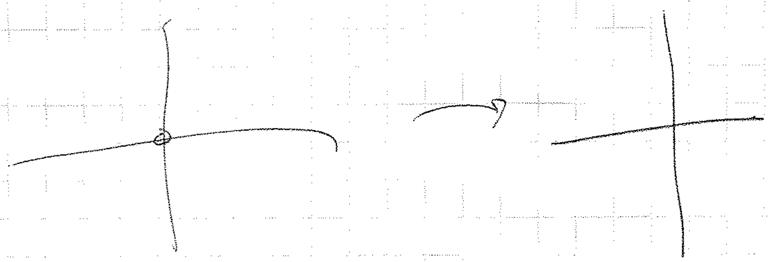
zeros of f give singularities for translation structure, "cone type". (not for ~~holom~~ conformal structure)

Simple example:

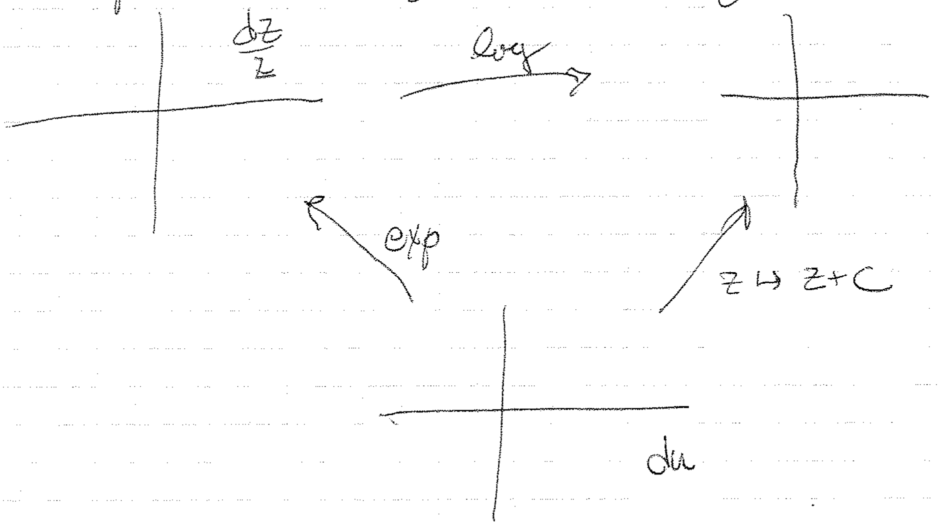
Translation structure associated to

$\frac{dz}{z}$ on $\mathbb{C} - \{0\}$.

Charts are branches of $\int \frac{dz}{z} = \log(z)$.



The exponential ~~fuaction~~ function



gives a translation map from $\mathbb{C}$ to $\mathbb{C}-\{0\}$,

This map is not injective since z and $z+2\pi i$ map to the same point.

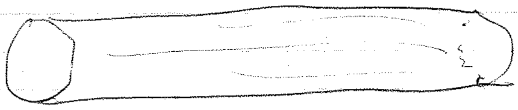
It induces a map from $\mathbb{C}/\Lambda$ $\quad \Lambda = \{2\pi i n : n \in \mathbb{Z}\}$

$\subset$ subgroup of $\mathbb{C}$

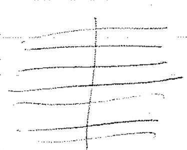
and this map is a translation equivalence.

2 ways of looking at a translation surface

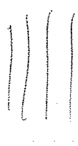
metric and $\underset{\text{real}}{\text{hor}}$ + $\underset{\text{im}}{\text{vert}}$ trajectories.



$\frac{dz}{z}$ metrically

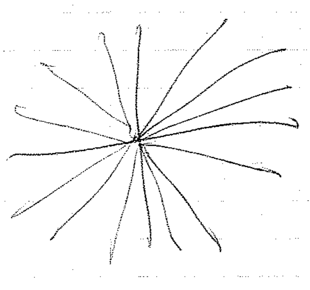


hor

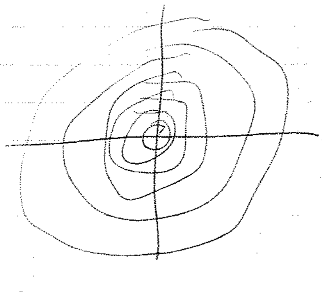


trajectories

"hor" ~~real~~



~~im~~ "vert"



Note value of an integral over a closed loop of $f(z)dz$ is called a "period". Periods of $\frac{dz}{z} = \Lambda$.

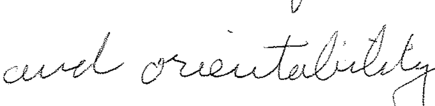
Period map: $\pi_1(V) \to \mathbb{C}$ homomorphism. $(\in H^1(V;\mathbb{C}))$.

Note: Residue around a pole makes sense (in a coordinate free way) for Abelian differentials (but not functions). ~~Integral around a small loop of index 1.~~ (

$\boxed{\text{Cut and paste construction of Abelian differentials}}$

Example. Torus. (Recall that ~~closed~~ compact surfaces are classified by genus and orientability.

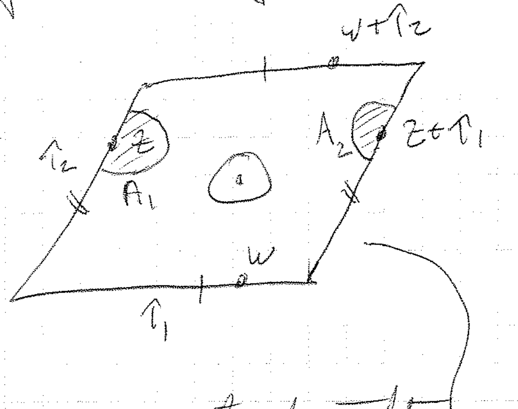
All Riemann surfaces are orientable.



~~Surfaces~~ ~~Non~~ ~~M~~ ~~The~~ Typical non compact surfaces ~~of finite~~ can be described ~~as~~ topologically as surfaces of genus g minus K disks (or points)

$\left(\begin{array}{l} \tau_1 = (1,0) \\ \tau_2 = (0,1) \end{array} ? \right)$

Cylinder = genus 0 minus 2 pts,

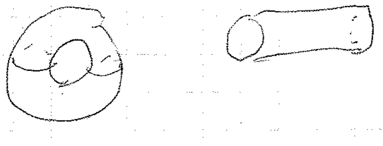


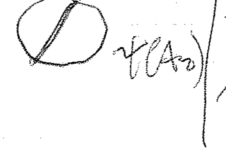
Glue opposite sides by isometries

$\psi_z(A_1 \cup A_2) \to \mathbb{C}$

$\psi_z|A_1(z) = z \qquad \psi_z|A_2(z) = z - \tau_1$

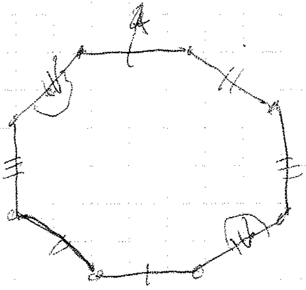
We can paste together partial disks to get charts





We will show that this translation structure is associated to elliptic integrals $\int \frac{dz}{\sqrt{P(z)}} \leftarrow$ deg 3 or 4.

Genus 2 example.

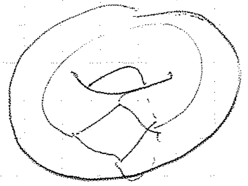


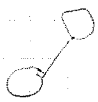
Regular octagon. Glue together opposite sides.

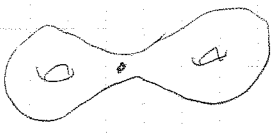
~~For the moment remove the vertex.~~ ~~vertices~~

~~We get an~~

topologically we get a surface of genus 2.







~~We can put~~

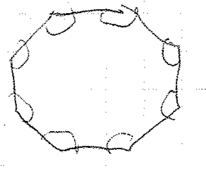
Note that all vertices of the octagon get identified with a single point in the quotient.

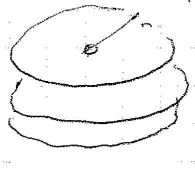
We can put a translation structure on the surface if we remove the vertex.

What does this translation structure look like near the vertex?

~~Total angle is $3\pi/2$ at ea~~

Angle is $3\pi/2$ at each vertex. Total cone angle is $8 \cdot 3\pi/2 = 12\pi = 3(2\pi)$. ~~Pro~~ If we glue ~~these~~ a nbd. of the vertices together it looks like





a spiral ramp going around the circle 3 times.

This is a typical singularity ~~for the trans~~ of the translation structure. Claim that it is not a singularity for the conformal structure. Construct a chart around the vertex so that change of coordinate charts are conformal (but not translations).

Use polar coordinates:

$$\{(r,\theta) : r<\varepsilon,\ 0\le\theta\le 6\pi\}.$$

Consider the chart $\psi((r,\theta)) = (r^{\alpha}, \alpha\theta)$ where $\alpha=\frac{1}{3}$.

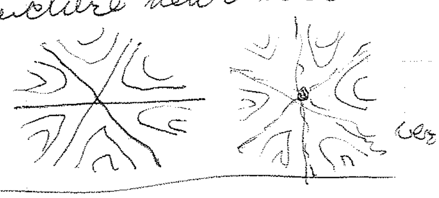
Image of ψ is $\{(r,\theta) : r<\varepsilon^{\alpha},\ 0\le\theta\le 2\pi\}$ disk of radius ε^{α}.

Change of coordinate charts have the form

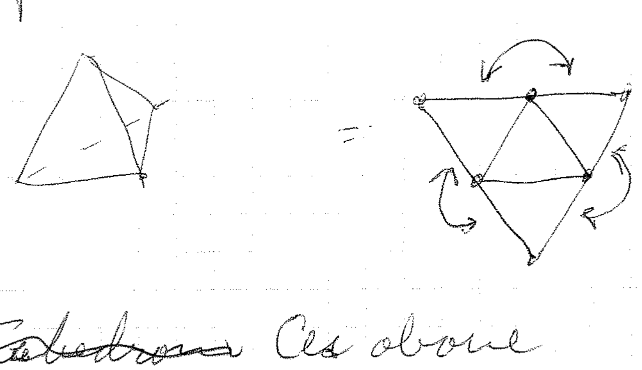
$$\left.\begin{array}{l} z\mapsto z+c \\ z\mapsto z^{3} \\ z\mapsto z^{1/3} \end{array}\right\}\ \text{holomorphic}$$

in polar coordinates $(r,\theta)\mapsto(r^{\alpha},\alpha\theta)$ $\alpha=3$ or $\frac{1}{3}$.

Trajectory structure near the singularity:



Example: Tetrahedron

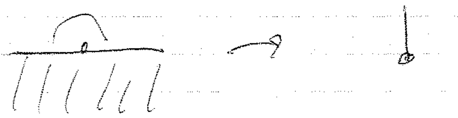


At the top vertex the cone angle is $3\cdot\left(\frac{\pi}{3}\right)=\pi$.
At the side vertices cone angle is π.

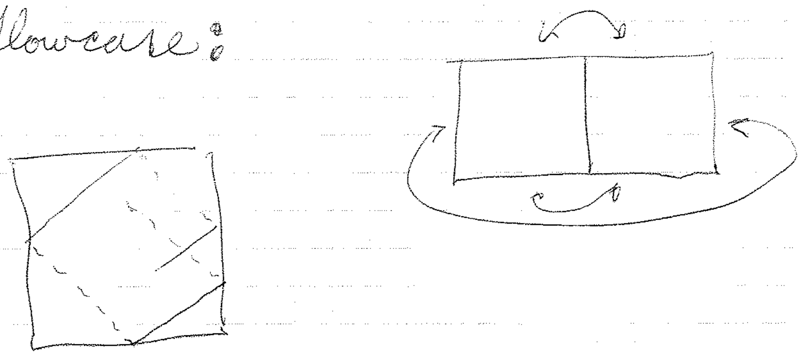
~~The tetrahedron~~ As above we can construct an atlas away from the vertices. Half-translation surface.

We can extend this atlas to a conformal atlas at the vertices

$$(r,\theta) \mapsto (r^2, 2\theta)$$

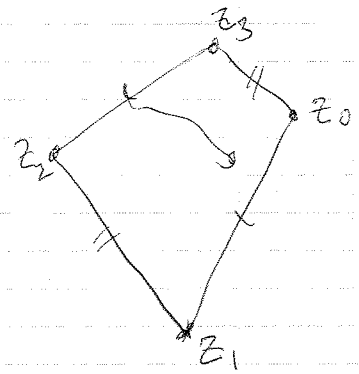


Pillowcase:



Trajectories correspond to billiard paths.

Similarity structure:



There is a transformation of the form $z \mapsto az+b$ that takes $z_2 \mapsto z_1$

$z_3 \mapsto z_0$

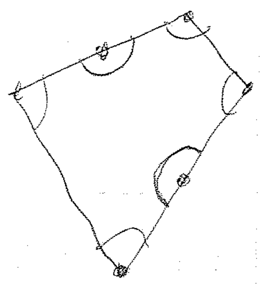
We can compute a, b:

$$a = \frac{z_0 - z_1}{z_3 - z_2}$$

$$b = \frac{z_1 z_3 - z_0 z_2}{z_3 - z_2}$$

Glue these sides together by means of this transformation

Do the same thing for sides (z_3, z_0) and (z_2, z_1).



We can glue ~~tble~~ together half-~~dd~~ disks on the edges to get charts at points along the edges. Note gluing has the form $z \mapsto az+b$.

~~We can glue~~

We can glue sectors at the vertices together to obtain ~~charts~~ a chart at the vertex. Angles add up to 2π. Non-singular similarity structure.