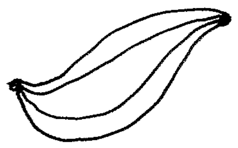
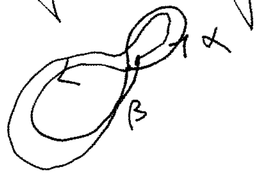
Covering spaces and fundamental groups.



Recall that the fundamental group of a space X based at a point $p \in X$, $\pi_1(X,p)$ is the collection of homotopy classes of parametrized loops based at p.

The group operation is given by concatenation:

Example.



$\alpha\beta$ is the path obtained first by following α then following β.

Let $\mathbb{C}^* = \mathbb{C} - \{0\}$.

$\pi_1(\mathbb{C}^*, 1)$ is generated by γ where $\gamma(t) = e^{2\pi i t}$ $t \in [0,1]$.

There is a close connection between covering spaces and fundamental groups.

Let $f: X \to Y$ be a covering space. Pick a point $q \in Y$ and a point $p \in X$ with $f(p) = q$.

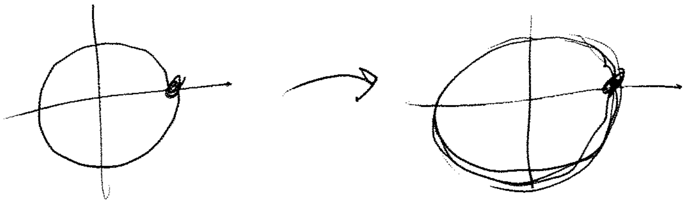
f induces a map $f_*: \pi_1(X, p) \to \pi_1(Y, q)$.

In an appropriate sense the covering space is determined by knowing the image of f_*, it corresponds to $f_*(\pi_1(X, p)) \subset \pi_1(Y, q)$.

Example: $f: \overset{X}{\underset{\shortparallel}{}}\mathbb{C}^* \to \overset{Y}{\underset{\shortparallel}{}}\mathbb{C}^*$ $f(z) = z^3$.

Choose $p = q = 1$.

$f_*(\gamma) = \gamma^3$. If we identify each group with $\mathbb{Z}$ then $f_*(n) = 3n$, $f_*(\pi_1(\mathbb{C}^*, 1)) = 3\mathbb{Z} \subset \underset{\underset{\pi_1(\mathbb{C}^*, 1)}{\shortparallel}}{\mathbb{Z}}$.



~~[illegible]~~

$\# f^{-1}(z) = 3$,

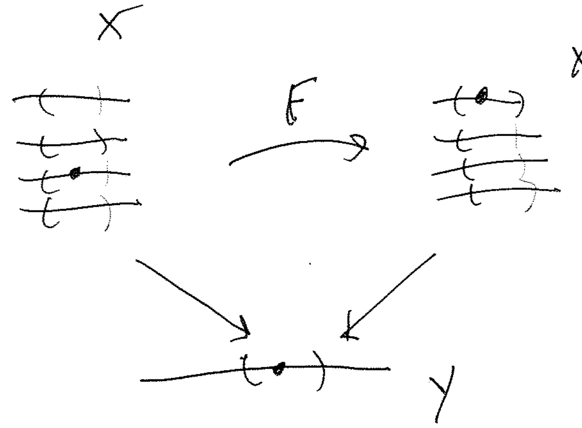
$f_*(\pi_1(\mathbb{C}^*, 1))$ has index 3 in $\pi_1(\mathbb{C}^*, 1)$

Define a relation on <u>pointed</u> covering spaces.

Let $f:(X,p)\to(Y,q)$ and $f':(X',p')\to(Y,q)$ be connected covering spaces. We say f and f' are equivalent if there is a homeomorphism $F:(X,p)\to(X',p')$ so that

$$\begin{tikzcd}
(X,p) \arrow[rr] \arrow[dr, "f"'] & & (X',p') \arrow[dl, "f'"] \\
& (Y,q) &
\end{tikzcd}$$

commutes.



Over any evenly covered set F is permuting the inverse images "the sheets".

Example: Consider $f:(\mathbb{C}^*,1)\to(\mathbb{C}^*,1)$ and $f':(\mathbb{C}^*,\xi)\to(\mathbb{C}^*,1)$. These are distinct as pointed covering spaces but they are equivalent where

$F:\mathbb{C}^*\to\mathbb{C}^*$ $\quad F(z)=\xi z$.

$$\begin{tikzcd}
(\mathbb{C}^*,1) \arrow[rr, "z\mapsto\xi z"] \arrow[dr, "z\mapsto z^3"'] & & (\mathbb{C}^*,\xi) \arrow[dl, "z\mapsto z^3"] \\
& (\mathbb{C}^*,1) &
\end{tikzcd}$$

Thm.
(Galois correspondence).

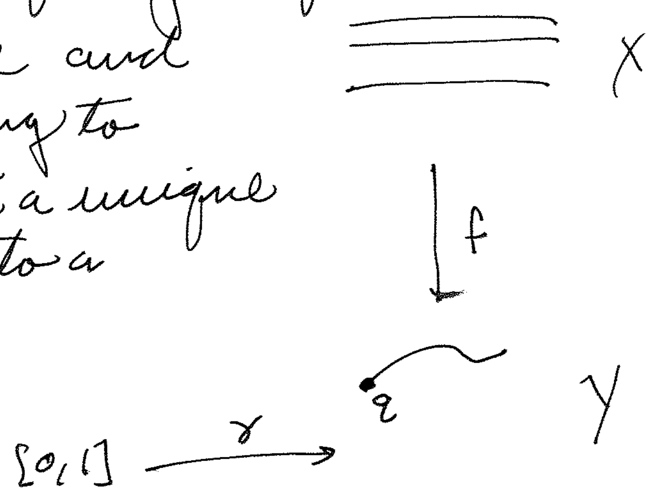
Let X, Y be connected;

Let $f:(X,p) \to (Y,q)$ be a covering space then f_* induces an injective map $f_*: \pi_1(X,p) \to \pi_1(Y,q)$.

Covering spaces are determined up to equivalence by the corresponding subgroups. $f:(X,p) \to (Y,q)$ and $f':(X',p') \to (Y,q)$ are equivalent if and only if $f_*(\pi_1(X,p)) = f'_*(\pi_1(X',p'))$.

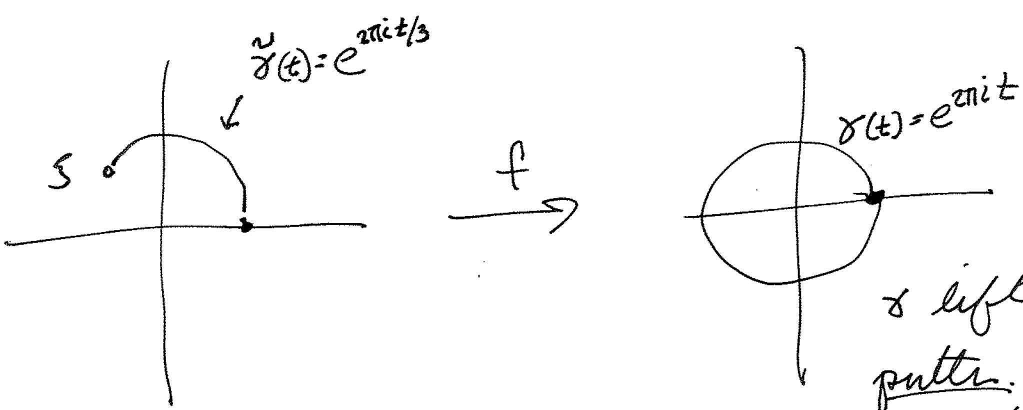
If Y is sufficiently nice (eg a manifold) then for every subgroup $\Gamma \subset \pi_1(Y,q)$ there is a covering space $f:(X,p) \to (Y,q)$ with $f_*(\pi_1(X,p)) = \Gamma$.

A basic construction in analyzing covering spaces is path lifting. If we have a path in the base and a point p mapping to $\gamma(0)$ we can construct a unique lift of this path to a



$\tilde{\gamma}:[0,1]\to X$ with $\tilde{\gamma}(0)=p$ and $f\circ\tilde{\gamma}=\gamma$.

If you start with a loop downstairs it can lift either to a path or a loop. $f^*(\pi_1(X,p))$ tells you which loops at q lift to loops at p and which lift to paths

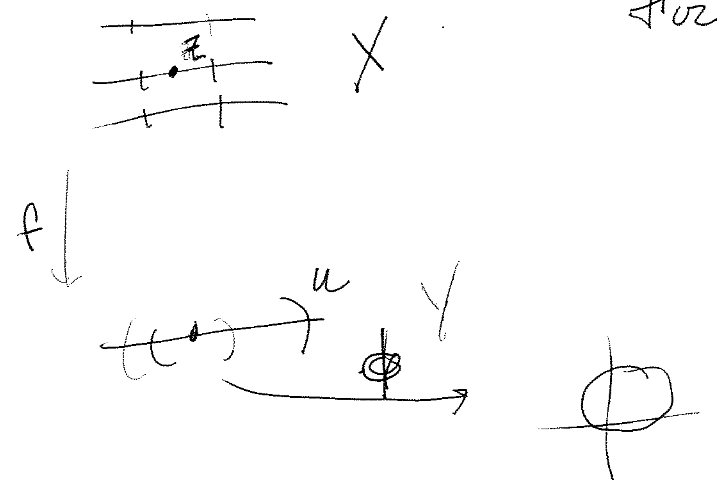


γ lifts to a path.
γ^3 lifts to a loop

This construction allows us to build topological spaces abstractly in terms of subgroups of $\pi_1(Y,q)$.

Thm. If (Y,q) is a Riemann surface then any covering space $f:(X,p)\to(Y,q)$ of (Y,q) is a Riemann surface and f is holomorphic. Two topologically equivalent covers are are conformally equivalent.

Proof. How do we construct the atlas?



For each chart ϕ downstairs defined on $U \subset Y$ and each $z \in X$ with $f(z) \in U$ there is an open set $U' \subset U$ which is evenly covered and contains $f(z)$.

z is in a component U'_z of $f^{-1}(U')$. Define $\phi_z : U'_z \to \mathbb{C}$ by $\phi_z = \phi \circ f$.

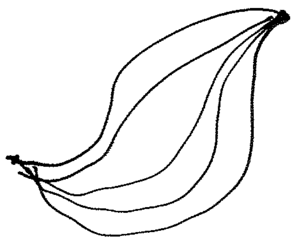
This construction allows us to build Riemann surfaces abstractly.

One of the motivations for the development of the theory of Riemann surfaces is the desire to deal with "multivalued functions" such as the logarithm which ~~do no~~ have ambiguous values.

One way in which multivalued functions arise is through integration. Let me relate this to the theory of covering spaces.

Say that f is a holomorphic function on $U \subset \mathbb{C}$. We want to define a Riemann surface on which $\int f dz$ is well defined.

Pick $q \in U$ and define $h: \pi_1(U, q) \to \mathbb{C}$ by $h(\gamma) = \int_\gamma f\,dz$. We note that h is actually well defined since 2 homotopic paths have the same integrals.



We also note that h is a homomorphism $h(\alpha\beta) = h(\alpha) + h(\beta)$ to the additive group of $\mathbb{C}$.

Let $\Gamma = \ker(h)$ and let X_Γ be the covering space corresponding to Γ under the Galois correspondence.

Example: $U = \mathbb{C}^*$ $f(z) = \frac{1}{z}$. $\gamma : [0,1] \to U$.

$h(\gamma) = \int_\gamma \frac{dz}{z}$. $h(\gamma)$ is 2π times ~~the winding~~ "
with respect to 0.

$w(\gamma, 0)$ the winding # of γ

$\ker h$ = is the trivial subgroup. X_Γ is the covering space corresponding to the trivial subgroup which is the universal cover of $\mathbb{C}^*$.

More generally:

$h(\alpha\beta) = h(\alpha) + h(\beta)$ whenever you can compose paths α, β

$\pi : U_p \to U$.

For $w \in U_p$ define $F(w) = \int_\gamma f(z)\,dz$ where

$\gamma = \pi(\tilde\gamma)$ and $\tilde\gamma$ is a path from p to w.

Claim that F is holomorphic and well defined on U_p

Proof. Say that α and β are paths from p to w in U_p. We want to show that

$$\int_{\pi(\alpha)} f(z)\,dz = \int_{\pi(\beta)} f(z)\,dz \quad \text{or} \quad \int_{\pi(\alpha)\cdot\pi(\beta^{-1})} f(z)\,dz = 0$$

$\boxed{\text{Choice of constant of integration has an effect.}}$

or $\int_{\pi(\alpha\beta^{-1})} f(z)\,dz = 0$. By construction

$\alpha\beta^{-1}$ is a path in U which lifts to a loop in U_p so $\alpha\beta^{-1} \in \Gamma$ and

$$0 = h(\alpha\beta^{-1}) = \int_{\pi(\alpha\beta^{-1})} f(z)\,dz$$

Conclude that anti-derivatives exist though perhaps on a different surface. Parking garage picture.

Further Applications of covering space theory to Riemann surfaces

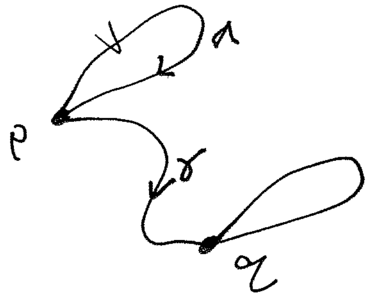
How does $\pi_1(X, p)$ depend on p?

Assume X is path connected.

Let p, q be points and γ a path between them.

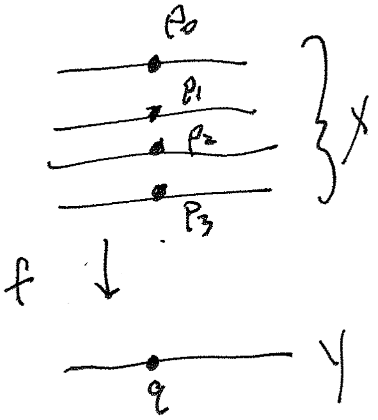
We can define a homomorphism $L_\gamma : \pi_1(X, p) \to \pi_1(X, q)$

$$L_\gamma(\alpha) = \gamma^{-1}\alpha\gamma.$$



Prop. L_γ is an isomorphism (with inverse $L_{\gamma^{-1}}$).

Remark: In general γ is a path.
If $p = q$ then γ is a loop and represents an element of the homotopy group.



Any $f : X \to Y$ is a covering and X is connected

Prop. The deck group acts transitively on f^{-1} iff $\underline{f_*(\pi_1(X, p_0))}$ is normal in $\pi_1(Y, q)$.

Remark. Key point: $f_*(\pi_1(X, p_0))$ and $f_*(\pi_1(X, p_1))$ differ by an <u>inner</u> automorphism. Choose a path γ from p_0 to p_1. This gives an isomorphism

$$\begin{tikzcd} (X, p_0) \arrow[rr, "F"] \arrow[dr, "f"'] & & (X, p_1) \arrow[dl, "f"] \\ & (Y, q) & \end{tikzcd}$$

F exists iff $f_*(\pi_1(X, p_0)) = f_*(\pi_1(X, p_1))$

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Formula for the deck group action:

Let $f:(X,p_0)\to(Y,q_0)$ be a covering space. Let γ be a loop based at q_0 representing an element of $\pi_1(Y,q_0)$. We define a map $g:X\to X$ in the deck group.

Let $z\in X$. Choose a path α from p_0 to z in X. Consider the composition path $\gamma\cdot f(\alpha)$ in Y. Lift this to a path based at p_0. Define $g(z)$ to be the endpoint of this path.

L_γ from $\pi_1(X, p_0)$ to $\pi_1(X, p_1)$. Downstairs this corresponds to conjugation by $f(\gamma)$ which is a loop. If $\pi_1(X, p_0)$ is normal then $f_*(\gamma)$ is a loop and $f_*(\pi_1(X, p_0))$, $f_*(\pi_1(X, p_1))$ differ by conjugation by the corresponding element of the fundamental group.

$f: X \to Y$

Def. If $\Gamma \subset \pi_1(X, p)$ is normal then $\pi: X_p \to X$ is called a regular cover. * In this case the deck group acts freely on X so that $X/\Gamma = Y$. Deck group is $\Gamma = \pi_1(Y, q)/f^*(\pi_1(X, p))$.

$\boxed{\text{Example } z \mapsto z^n. \text{ Deck group is } \mathbb{Z}/n\mathbb{Z}.}$ acts conformally

Cor. Any Riemann surface S is the quotient of a simply connected Riemann surface R by subgroup Γ of the conformal automorphism group acting freely.

[Example: $\mathbb{C}^*$ is conf. equivalent to $\mathbb{C}/z \mapsto z + 2\pi i$. Deck group $= \mathbb{Z}$ acting conformally by $T_n(z) = z + 2n\pi i$]

Proof. Let R be the universal cover of S and $\pi: R \to S$ the covering map. Let Γ be the deck group. Γ acts freely and conformally.

R inherits a Riemann surface atlas from S.

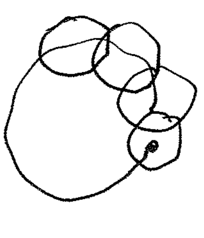
In the last section of the course I will show that the only simply connected Riemann surfaces are S^2, $\mathbb{C}$ and $UHP = \{z : \operatorname{Im} z > 0\}$. This implies that the quotient construction builds all examples

[~~$z \mapsto$~~ $\operatorname{aut}(\mathbb{C}) = \{z \mapsto az + b\}$ $\quad$ $\operatorname{aut} UHP = \{z \mapsto \frac{az+b}{cz+d}\}$

Second application, $\boxed{\text{Problem 8. Example sheet 1.}}$ $\quad$ $\mathbb{C}$ linear fractional trans. a, b, c, d real and $ad - bc > 0$.

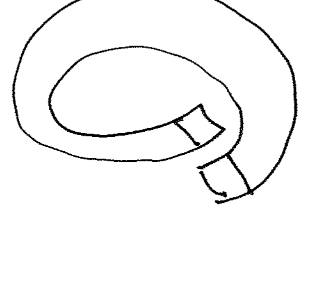
A basic idea leading to the construction of Riemann surfaces is the idea that many of the functions one would like to consider are naturally multiple valued.

This arises for example in solving complex differential equations. Power series methods let you solve them in a disk. We get solutions by piecing disks together. We often find that



in piecing these together along a loop we get a different solution over the initial disk.

Simplest example



$F' = \frac{1}{z}$. Solution is log but log is multivalued, In some cases we can solve this problem by building an appropriate covering space and thinking about our multivalued function as being single valued on the <u>covering space</u>.

find F s.t.

We consider here the simplest case $F' = f$ for f holomorphic in $U \subset \mathbb{C}$. f is a polynom

Let $U = \mathbb{C} - \text{roots of } f$. Let $q \in U$.

Define $h : \pi_1(U, q) \to \mathbb{C}$ by $\gamma \mapsto \int_\gamma f\,dz$.
h is a homomorphism from π_1 to the additive group of $\mathbb{C}$.
Let $\Gamma = \ker h$.

Define $\pi : U_\Gamma \to U$ to be the covering space constructed by the Galois correspondence

Define $F(z)$ on U_Γ by

Let f be a holomorphic function defined on $U \subset \mathbb{C}$. Let $z_0 \in U$

Define a homomorphism h from $\pi_1(U, z_0) \to \mathbb{C}$ by $h(\gamma) = \int_\gamma f(z)\,dz$.

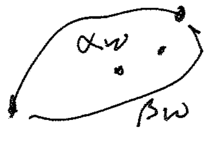
Let $\Gamma \subset \pi_1(U, z_0)$ be the kernel of h.

Let U_Γ be the connected covering space U corresponding to Γ. Then there is a well defined holomorphic function F on U_Γ defined by $F(w) = \int_{\gamma_w} f(z)\,dz$ where γ_w is some path U which lifts to a path connecting z_0 to w.

$\boxed{\text{Parking garage picture.}}$

Proof. We have given a formula for constructing F. We need to check to see that it is well defined. Say we have paths α_w, β_w that connect z_0 to w. Need to show that

$$\int_{\alpha_w} f\,dz = \int_{\beta_w} f\,dz \quad \text{or} \quad \int_{\alpha_w \circ \beta_w^{-1}} f\,dz = 0.$$



But $\alpha_w \beta_w^{-1}$ is a loop based at z_0 which is a projection of a loop in $\tilde{U}$.

By definition $\alpha_w \beta_w^{-1} \in \Gamma$ so $\alpha_w \beta_w^{-1} \in \ker h$

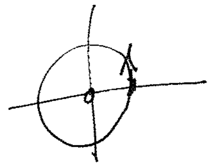
so $\int_{\alpha_w \beta_w^{-1}} f\,dz = 0$.

$\boxed{\text{This uses composition of arbitrary paths.}}$

Example, $f = \frac{1}{z}$ on $U = \mathbb{C}^*$ (attempting to define log, not defined on U.)

$\pi_1(\mathbb{C}^*) = \mathbb{Z}$. h maps

Generator corresponding to the loop going counterclockwise around the

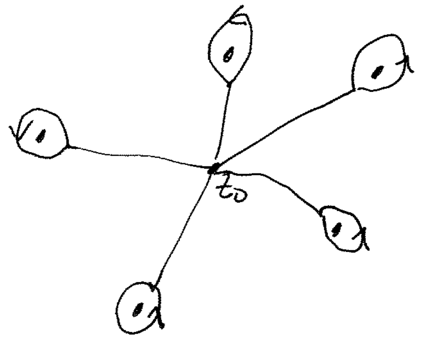


origin, $h(\gamma^n) = n \cdot 2\pi i$.

(attempting to define lo

Example $f = \frac{P'}{P}$ for P a polynomial $P = \prod (z - z_j)^{m_j}$

Say $z_1 \ldots z_d$ (distinct roots with mult m_k.) are the roots of P. Say z_0 is a distinct point.



$U = \mathbb{C} - \{z_1 \ldots z_d\}$

$z_0 \in U$ is the basepoint.

Let $\gamma_1 \ldots \gamma_d$ correspond to loops around $z_1 \ldots z_d$.

Claim $\pi_1(U)$ is the free group on $\gamma_1 \ldots \gamma_d$.

Recall that the residue of $\frac{P'}{P}$ at z_k is the multiplicity of z_k as a root of P. ≥ 1.

Residue formula for integration

Gives $h(\gamma_j) = m_j$.

$$h(\gamma_{j_1}\gamma_{j_2}\cdots\gamma_{j_k}) = 2\pi i\sum_{\ell=1}^{k} m_{j_\ell}.$$

alternatively

$$\frac{(QR)'}{QR} = \frac{Q'R+QR'}{QR} = \frac{Q'}{Q}+\frac{R'}{R}$$

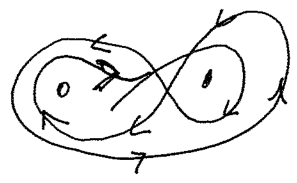
so the integral is the sum of the winding numbers of γ around $\sum_{j=1}^{\#K} w(\gamma, z_j)$,

Equiv. winding # of $P(z)$ around 0.

Constant of integration.

Shows that the covering space U_P is not simply connected.

$\gamma =$  $h(\gamma)=0$.

$\gamma =$  $h(\gamma)=0$.

Define $\sqrt{P(z)}$. Relate this to $\log P(z) = \int \frac{P'}{P}\,dz$.

Image contained in $\mathbb{Z}\cdot\pi i$.

$$\sqrt{P(z)} = \exp\left(\frac{1}{2}\int\frac{P'(z)}{P(z)}\,dz\right)$$

$$\gamma \mapsto \overset{u(\gamma)}{\overset{\shortparallel}{\exp}}\left(\tfrac{1}{2}h(\gamma)\right).$$

This is a multiplicative homomorphism with image contained in $\exp(\mathbb{Z}\cdot\pi i) = \pm 1$.

A loop maps trivially iff the weighted sum of its winding #'s around the zeros of P is ~~even~~.

Let $\Gamma = \ker \mu$.

We can define a parametrization of $y^2 = P(x)$ by U_Γ.

Identification of a an abstract surface U_Γ with a concrete surface.

$$\left(\exp \frac{1}{2} \int_\gamma \frac{P'}{P} \, dz \right)^2 = P(z)$$

Make it true at z_0 with an appropriate choice of constant of integration.

$\int_\gamma \frac{P'}{P} dz$ is a local branch of

$$\left(\exp \frac{1}{2} \log P\right)^2 = e^{2 \cdot \frac{1}{2} \log P} = e^{\log P} = P.$$

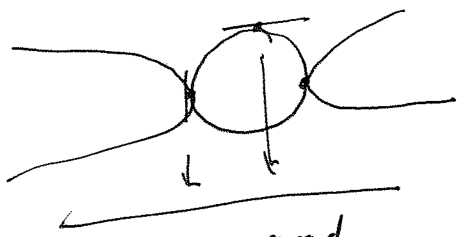
up to a constant

$P(X,Y) = \; -Y^2 + f(X)$.

Let $V = \{(X,Y) : Y^2 = f(X)\}$ where f is a polynomial.

Let $\pi_X : V \to \mathbb{C}$ map (X,Y) to $\mathbb{C}$.

The projection π is a local homeomorphism where $\frac{\partial P}{\partial Y} \neq 0$. This follows from the implicit function theorem as we used. Recall the two types of charts.



$V' = V - \{(X,Y^*) : Y = 0\}$, $\mathbb{C}' = \mathbb{C} - \{X : f(X) = 0\}$,

$\frac{\partial P}{\partial Y}$ and P vanish where $Y=0$ and X is a root of f.

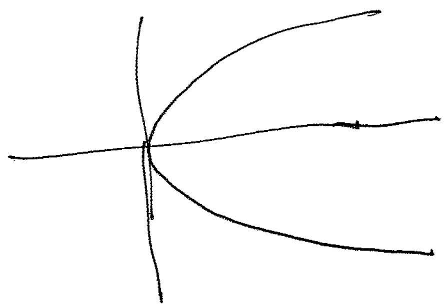
[Claim: $\pi_X : V' \to \mathbb{C}'$ is a covering space.

I want to build this cover explicitly by using a cover associated to $\log f$

or $\int_\gamma \frac{f'}{f} dz$.

Let $Z = \{z \in \mathbb{C} : f(z) = 0\}$.

$V = \{Y^2 = X^3\}$ as a covering space.



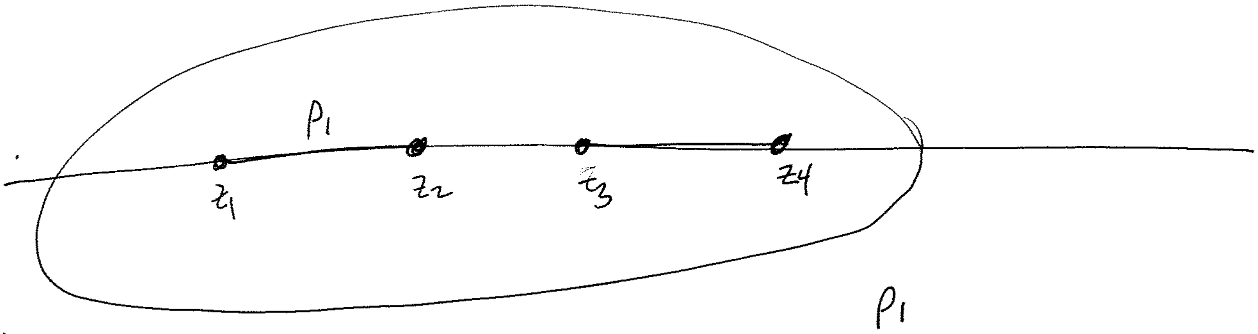
Consider $\pi_X : V' \to \mathbb{C}'$.

Let $f(z)$ be defined on $U \subset \mathbb{C}$. We can define a homomorphism $\pi_1(U, p) \to \mathbb{C}$ taking γ based at p to $\int_\gamma f(z)\,dz$. Consider the (abstract) covering space corresponding to the kernel. Claim $F(u) = \int_p^u f(z)\,dz$ is well defined on this covering space.

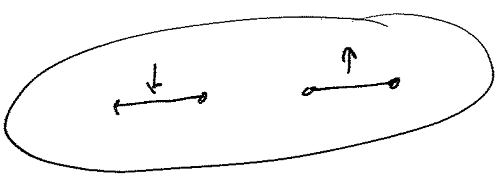
Can't always find a global parametrization for a Riemann surface.

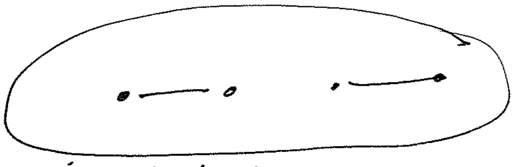
Applications:

Branch cut construction:

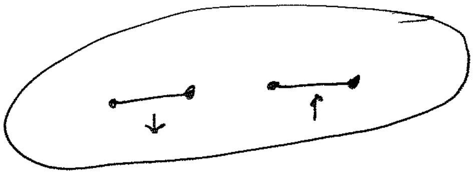


Closed Curves that don't cross lift to curves since each one has the same winding # around z_1 and z_2 hence ~~its~~ its total winding number is even.

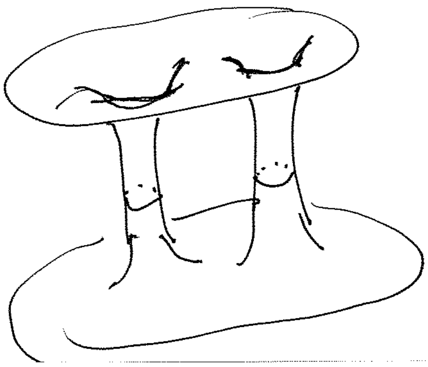


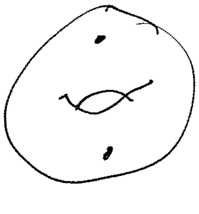


Trivial holonomy around ∞.



affine Hyper-elliptic curve of even degree ~~is a~~ $2d$ is a surface of genus d minus 2 pts.





affine Hyper-elliptic curve of degree $2d-1$ has ~~gen~~ genus d minus 1 pt.

Recall $V=\{Y^2=P(x)\}$

$P(x)=\prod(x-x_j)^{m_j}$

V non-singular all $m_j=1$.

$V'=V-\{(0,x_j)\}$ $\quad U=\mathbb{C}-\{x_j\}$.

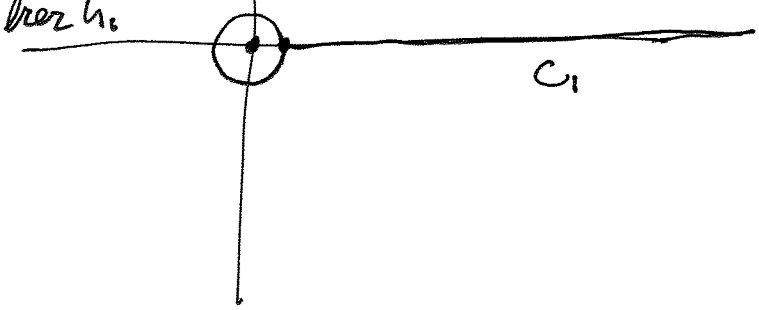
$\pi: V'\to U$ is a covering:

$V'=U_\rho$ normal covering corresponding to $\ker h_1:\pi_1(U)\to\{\pm 1\}$
$=\ker \bar{h}_1$

$$\bar{h}_1 = \frac{h_0}{2\pi i} \bmod 2 \in \mathbb{Z}/2\mathbb{Z}$$

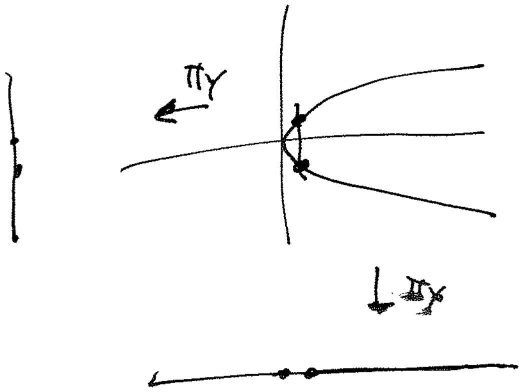
$$h_1 = (-1)^{\bar{h}_1}.$$

$$\bar{h}_1(\gamma) = \sum m_j \cdot \mathrm{wind}(\gamma, x_j) \bmod$$

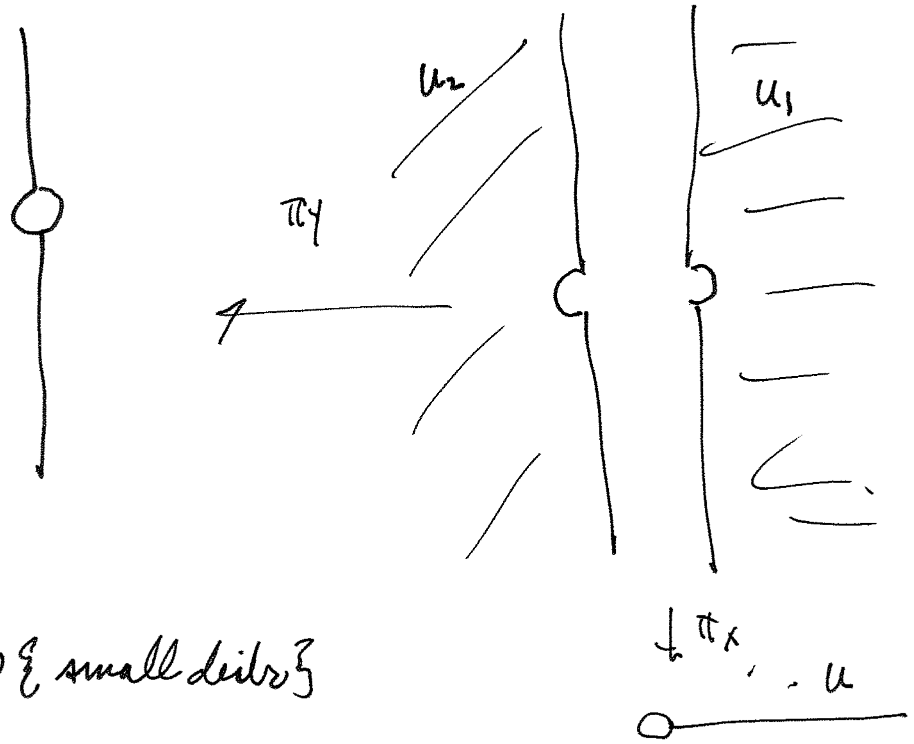


$V=\{Y^2=X\}$

$V'=V-\{0,0\}$.



$\hat{U} = U-\{C_1\cup D_0\}$



For this particular example π_Y provides a global parametrisation of V.

V'

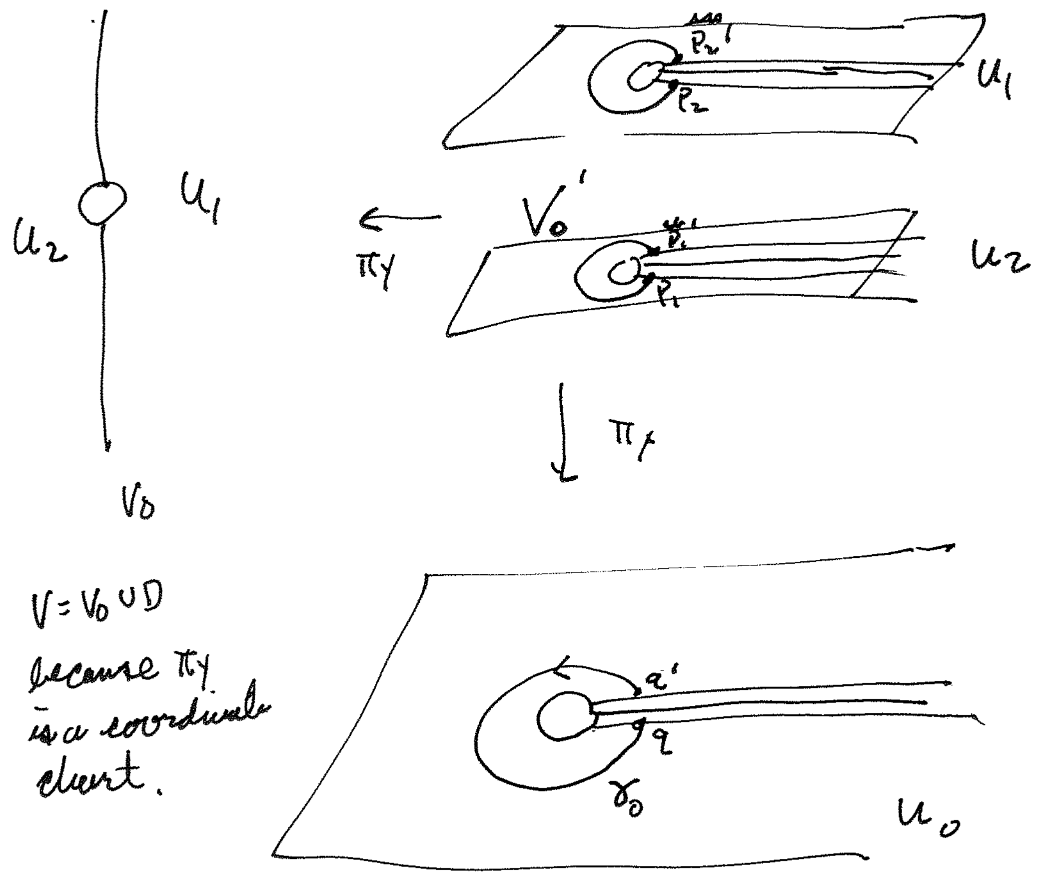
$V=V'\cup\{$small discs$\}$

Objective: Use our information about π_1 to build a topological model of V.

$U_0 = U$ with small disks around the roots removed,

$V_0' = \pi^{-1}(U_0)$,

If V is non-singular then V is V_0' with small disks around the points $\{(0, t_j)\}$ added,



C = slit from ε to ∞.

$V = V_0 \cup D$ because π_Y is a coordinate chart.

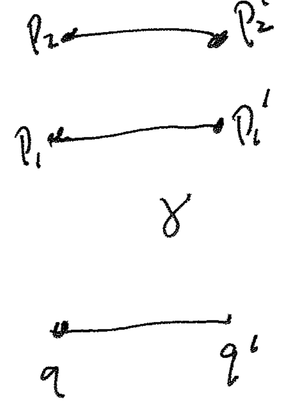
$\pi_x^{-1}(\gamma)$ is a single circle

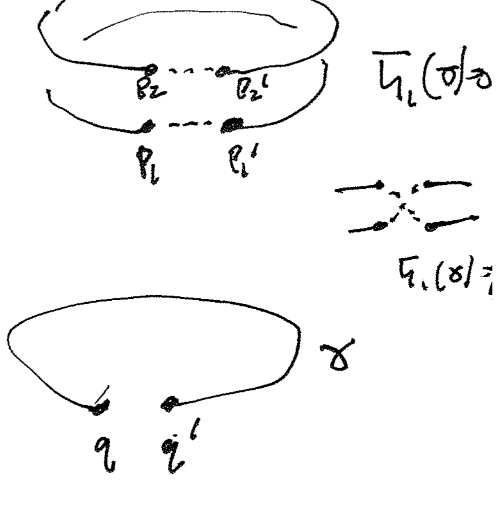
Lift of a short segment downstairs connects p_1 to $\bar{p}_1$ or $\bar{p}_2$, p_2 to $\bar{p}_1$ or $\bar{p}_2$.

Which is it?

It is determined by the monodromy of γ.

If $\bar{h}_1(\gamma)=0$ then p_1 gets connected to $\bar{p}_1$ and





Lemma. Let $p(t) : \{t \geq 0\} \to \mathbb{C}$ be a proper map. Let γ be a parametrized curve disjoint from the image of p. Then $\text{wind}(\gamma, p(0)) = 0$.

(Proper map means that the inverse image of a compact set is compact. Means that $p(t)$ eventually leaves any disk around the origin.)

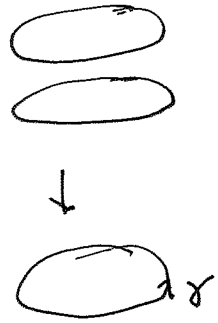
Proof. $\text{wind}(\gamma, p(t)) = \int_\gamma \frac{dz}{z - p(t)}$.

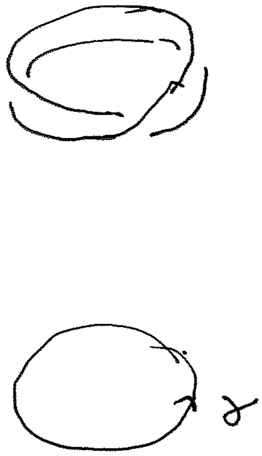
Since $p(t)$ does not lie on γ the integrand varies continuously with t. Thus $\text{wind}(\gamma, p(t))$ is a continuous function of t.

As $t \to \infty$, $p(t) \to \infty$ and for a fixed z value $\frac{1}{z - p(t)} \to 0$. It follows that $\lim_{t \to \infty} \text{wind}(\gamma, p(t)) = 0$.

On the other hand $\text{wind}(\gamma, p(t)) \in \mathbb{Z}$ so $\text{wind}(\gamma, p(t)) = 0$ for all t and $\text{wind}(\gamma, p(0)) = 0$.

Observation: There are exactly 2 degree 2 covering maps of the circle.





trivial monodromy $w_1(\gamma)=0$

2 components.

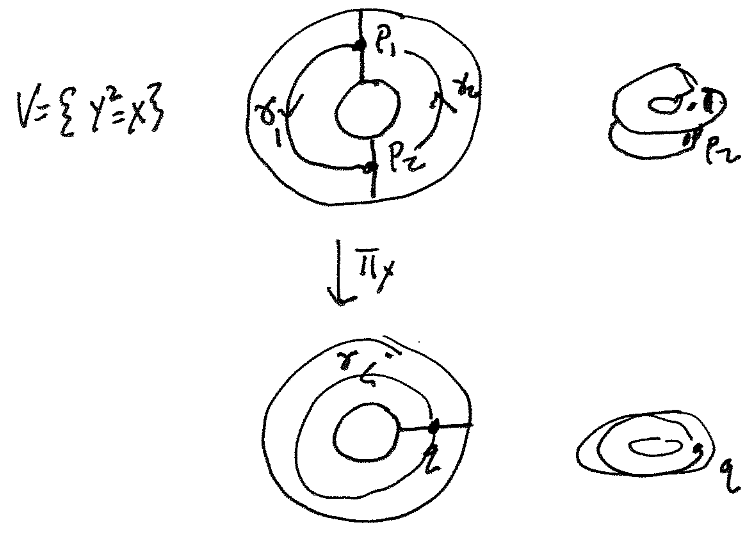
non-trivial monodromy

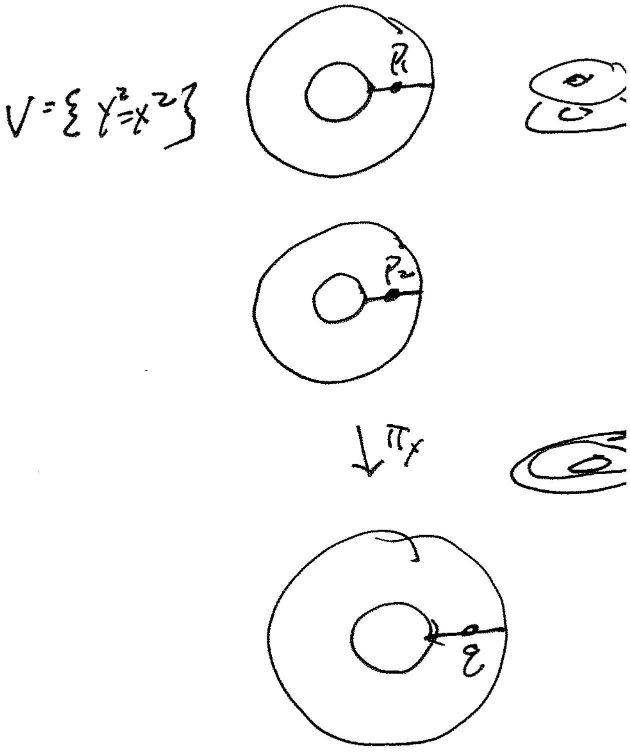
connected.

$w_1(\gamma)=1$

We described a method last time of cutting our region $U \subset \mathbb{C}_x$ by slits as a way of investigating the topology of V'.

As a warm up example let's apply the method to our 2 earlier examples.





$\gamma(t) = e^{2\pi i t}$

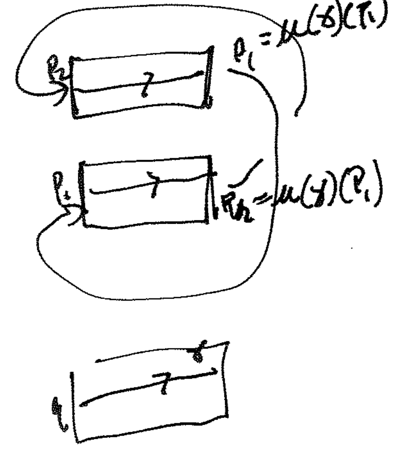
$\mu: \pi_1(U) \to \mathrm{Perm}(\pi_x^{-1}(q))$.

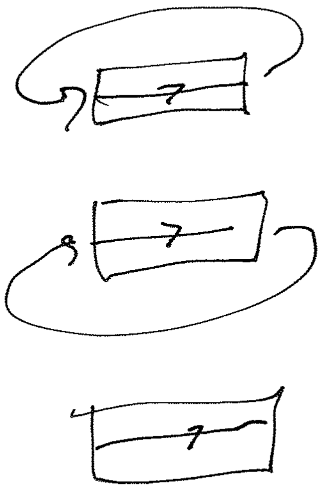
$\gamma \overset{\mu}{\longmapsto}$ $1 \to 2$, $2 \to 1$

$\gamma \overset{\mu}{\longmapsto}$ $1 \to 1$, $2 \to 2$

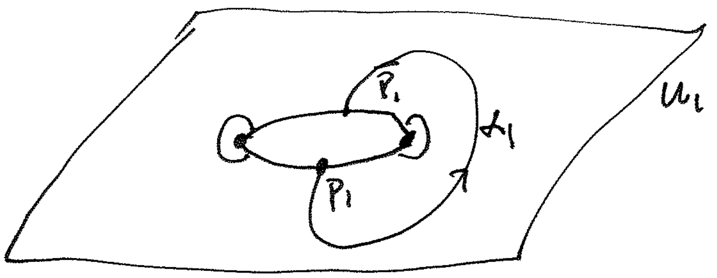
How to build the covering space.

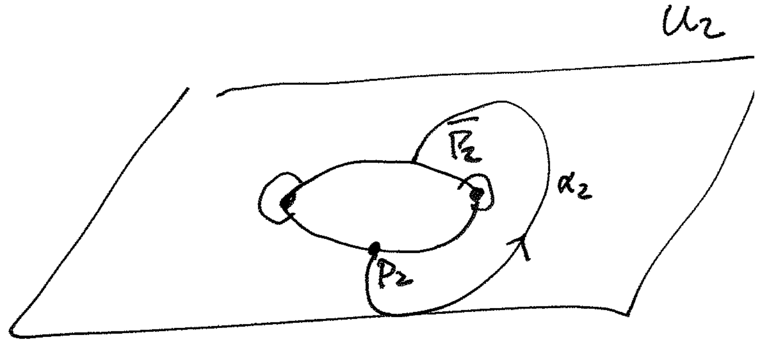
In both cases we get 2 rectangles. μ tells us how the sides are glued.





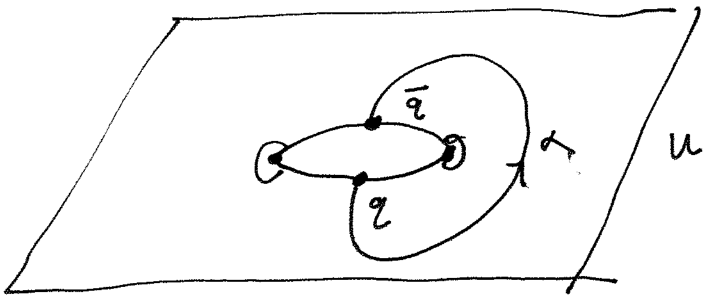
$V = \{y^2 = x^2 + 1\}$

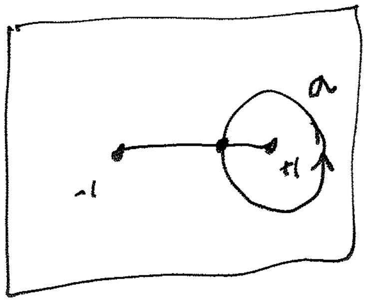




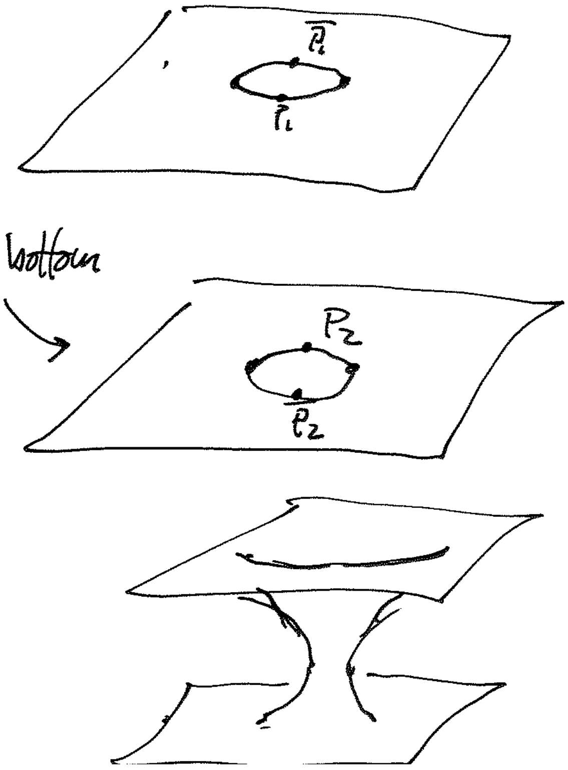
$p_1 \longrightarrow \bar{p}_2$

$p_2 \longrightarrow \bar{p}_1$





Flip the bottom sheet.



Recall $h_1(\alpha) = 1$ so

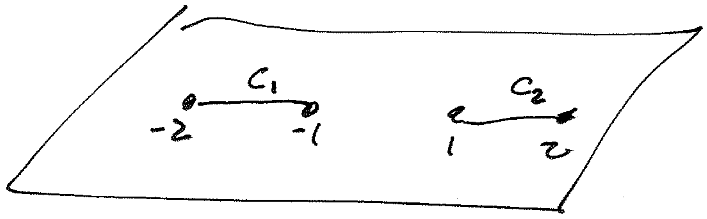
α corresponds to a non-trivial permutation.

Builds a cylinder

Now consider $V = \{ y^2 = (x^2-1)(x^2-4) \}$.
$= \prod_{j=1}^{4}(x - x_j)$

P has degree 4 with 4 real ~~yes~~ distinct zeros at ± 1 and ± 2.

Introduce 2 slits.



Let $\hat{U} = U - \{c_1 \cup c_2\}$.

Let α be a loop in $\hat{U}$.

Since α avoids c_1 and c_2 we have:

$$\text{wind}(\alpha, -2) = \text{wind}(\alpha, -1) \quad \text{and}$$

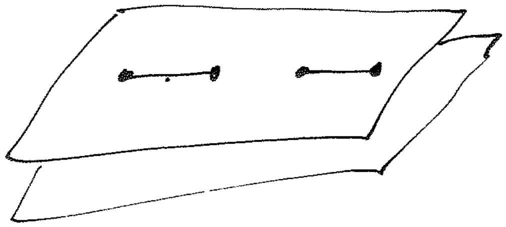
$$\text{wind}(\alpha, 1) = \text{wind}(\alpha, 1) \quad \text{so}$$

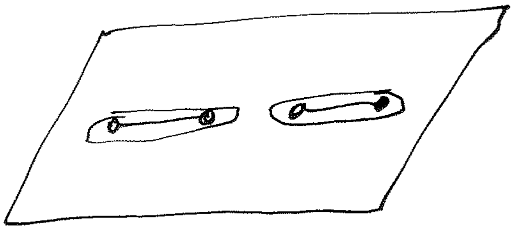
$$\frac{u_1}{2\pi i}(\alpha) \overset{\text{mod } 2}{=} \sum_{j=1}^{4} \text{wind}(\alpha, x_j) = 2\,\text{wind}(\alpha, -2) + 2\,\text{wind}(\alpha, -1) \equiv 0 \text{ mod } 2.$$

So if we restrict the covering map $\pi_x : V' \to U$ to $\pi_x : \hat{V}' \to \hat{U}$ now the monodromy is trivial.

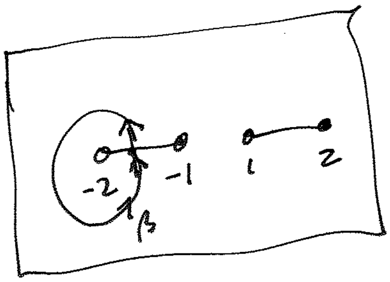
In particular $\pi_x^{-1}(\hat{U})$ is disconnected.

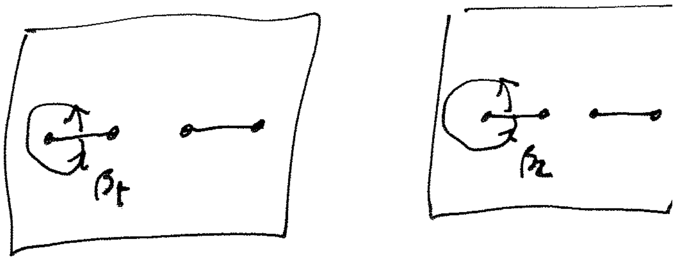
Let $\hat{W}_1$ and $\hat{W}_2$ be the two sheets.



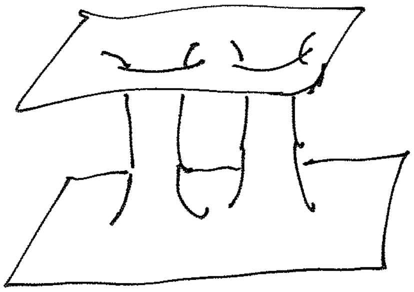


We want to see how the opposite sides of the slits get matched up when we go from $\pi_1(\hat{W})$ to $\pi_1(W)$. As before we study the monodromy of a loop β which crosses a single slit





β has winding # 1 around -2 and winding # 0 around -1, 1, 2 so $\frac{\log}{2\pi i}(\beta) = 1$, $\mu(\beta)$ is non-trivial.

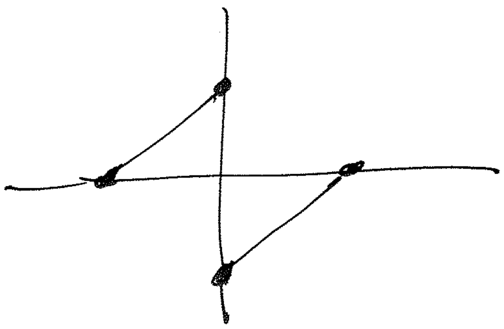


2 disks with 2 handles between them.
Note that the first time that you put in a handle you get a cylinder.

Then still has genus 0. The next time you put in a handle you increase the genus.

Conclusion V' is Topologically a torus minus 2 pts.

If P has 4 arbitrary distinct zeros we can choose a pair of slits and do the same analysis.



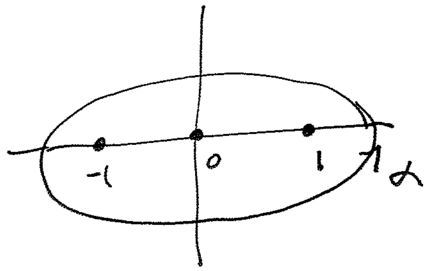
cut along ~~case~~ k slits

If P has degree $d=2k$ (then ~~U~~ $\hat{U}$ consists of ~~we get~~ 2 disks each with k slits. Corresponds ~~to~~ to k handles added to 2 disks. The first handle does not increase the genus but every subsequent handle increases the genus by 1.

Prop. ~~Cy~~ hyperelliptic non-singular affine curve of even degree $d=2k$ $k\geq 1$ is homeomorphic the surface of genus $k-1$ with 2 pts. removed.

If $P(x)$ has $2d+1$ distinct zeros then we have a different phenomenon for $V = \{Y^2 = P(x)\}$

$P(X) = X(X^2-1)$

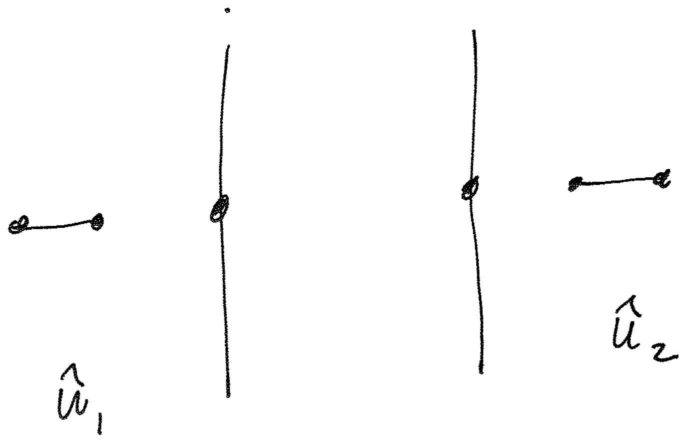


now $\frac{u_0(\alpha)}{2\pi i}$ is odd so

$u_1(\alpha) = \frac{u_0(\alpha)}{2\pi i} \bmod 2 = 1$

and $\mu(\alpha)$ is non-trivial. We introduce a slit connecting 1 to ∞

$\hat{U} = U - \{c_1 \cup c_2\}$.

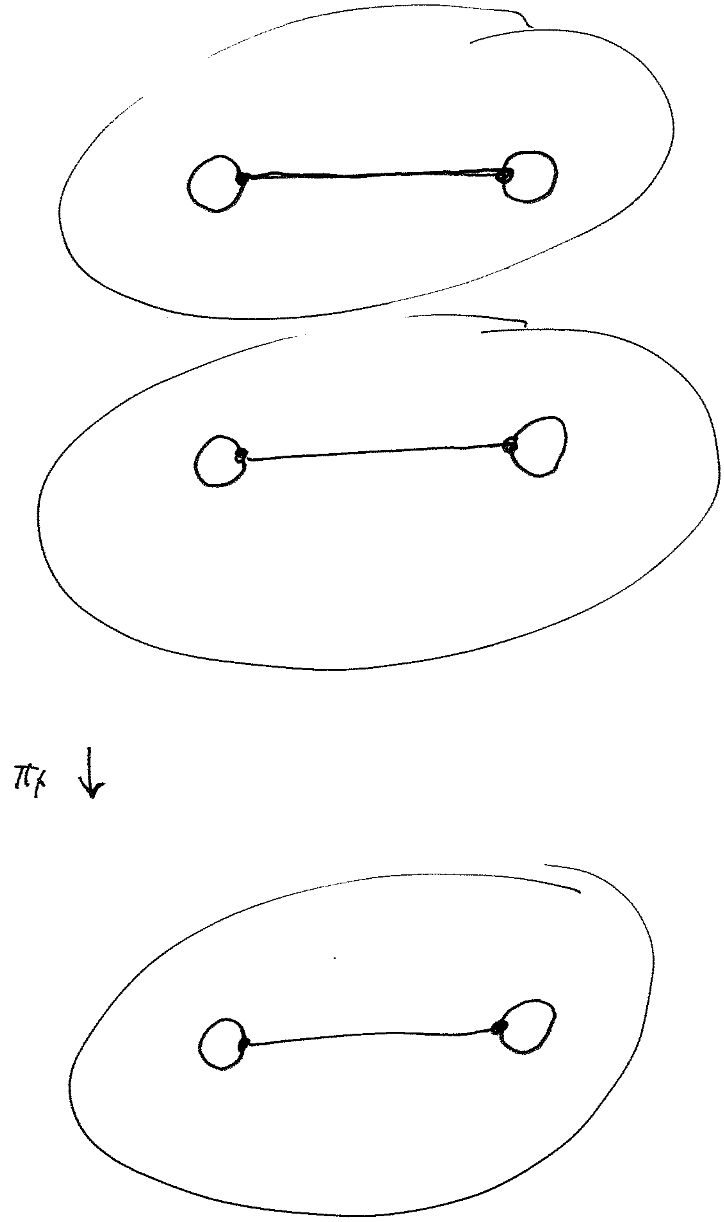


Lemma.

If ~~xx~~ α is any loop that avoids c_2 then $\text{wind}(\gamma, 1) = 0$.

Proof. Let $t \mapsto c_2(t)$ be a parametrization of c_2. ~~Say c_2~~ ~~$c_2 : \mathbb{R}^+ \to$~~ $c_2 : \mathbb{R}^+ \to \mathbb{C}$ is given by $c_2(t) = t+1$.

We are removing branch points and adding them back in.



We know that small circles around the punctures in U lift to single circles around punctures in V'. Since each puncture in V' is a non-singular point these small circles surround disks in V.

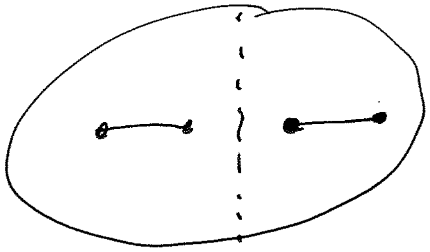
Consider $\text{wind}(\gamma, c_2(t)) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z - c_2(t)}$.

As $t \to \infty$ the integrand tends to 0 since $\frac{1}{z - c_2(t)} \to 0$. It follows that $\lim_{t\to\infty} \text{win}(\gamma, c_2(t)) = 0$.

Since γ does not intersect ~~with~~ the image of c_2 the integrand is continuous.

On the other hand $\text{wind}(\gamma, c_2(t)) \in \mathbb{Z}$

so in fact it is constant and equal to 0.



Gluing $\hat{u}_1$ and $\hat{u}_2$ together we get a disk with 2 slits. Gluing the slits together is equivalent to adding a handle. We get a torus with one pt. removed.

Prop. If $d = 2k+1$ with $k \geq 1$ then a non singular hyperelliptic curve is homeomorphic to a surface of genus k with one point removed.

$U_0 = U - \bigcup D_j$ where D_j is a small disk around x_j.

$V_0 = \pi_X^{-1}(U_0)$.

Prop. If V is non-singular then the inverse image of each disk D_j is a disk $\tilde{D}_j$ so that

$V = V_0 \cup \tilde{D}_j$.

Motivation. Let us back up and explain the motivation for what we were trying to do on Wednesday
an interesting
(1)

We are trying to build a family of examples of Riemann surfaces which are also plane curves.

Let $P(x)$ be a polynomial in one variable.

Let $V = \{(x,y) : y^2 = P(x)\}$.

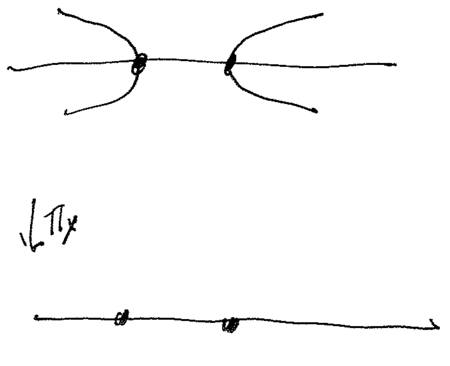
Let $\pi_x : V \to \mathbb{C}$ send (x,y) to x. using the implicit fcn. thm.

We have shown that π_x is a local homeo. at points (x,y) where $y \neq 0$.

Let's throw out these pts $y=0$, $P(x)=0$. Pts in V with $y=0$ satisfy ~~eq~~ $P(x)=0$.

Let $V' = V - \{(x_j, 0) : j = 1 \dots k\}$ where $x_1 \dots x_k$ are the roots of P.

Let $U = \mathbb{C} - \{x_j : j = 1 \dots k\}$. $\pi_x : V' \to U$ is a local homeomorphism.



We would like to do 2 things.
We want to show that
$\pi_x : V' \to U$ is a covering map.

We would like to identify which covering space V' corresponds to in the Galois correspondence.

Our technique is to actually construct a parametrization of V' using integration the integration construction we have described.

Hyper-elliptic surfaces are connected to multivalued functions.

A defining property of hyper-elliptic curves is that
we can make sense of the function $\sqrt{P}$ on V by taking $\sqrt{P} = Y$ since both Y and $\sqrt{P}$ satisfy $Y^2 = \sqrt{P}^2 = P$.

Our construction using integration produces a surface on which $\log P$ is defined. We want to modify this using $\sqrt{P} = \exp(\frac{1}{2} \underbrace{\log P})$.

We generalize by taking n-th roots instead of just square roots.

On Wednesday I stated ~~as as a~~ as a proposition that "G" is a well defined function on U_{Γ}.

I want to readjust some of my definitions to make the proof obvious.

Recall that given a polynomial P we defined

$$h : \pi_1(U) \to \mathbb{C} \text{ by } h(\gamma) = \int_\gamma \frac{P'}{P}\,dz,$$

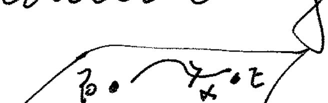
Let me call this h_0

We observed that h was a homomorphism:

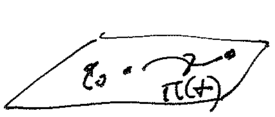
$h_0(\alpha\cdot\beta) = h_0(\alpha) + h_0(\beta)$.

Key obvious remark: This is true for any pair of paths not just for loops.

In constructing $F(z) = \int_{\pi(\alpha)} \frac{P'}{P}\,dz = h_0(\pi(\alpha))$



I used this fact without observing it.



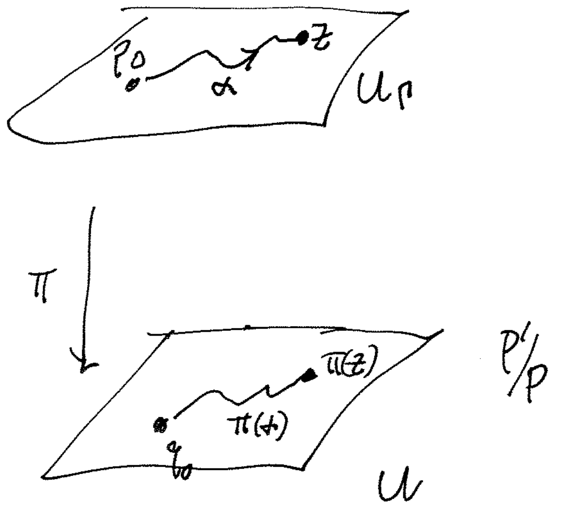
and showing that F is well defined on U_Γ where $\Gamma = \ker h_0 = h_0^{-1}(0)$ in $\pi_1(U; q_0)$.

(= "log P" or $F' = \frac{P'}{P}$.)

$(F\circ\pi)' = \frac{P'}{P}$, F has the cor derivative,

F+c have same property.

Now h_0 defines a function F upstairs.



$F(z) = h_0(\pi(\alpha))$.

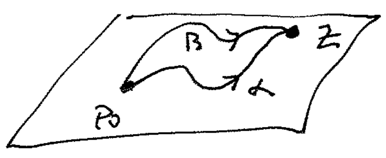
(Note that we are using the fact that h_0 is defined on paths.)

$h_0 : \pi_1(U, q_0) \to \mathbb{C}$ (here we are just using h_0 on loops)

$\Gamma = \ker h_0 = h_0^{-1}\{0\}$.

The fact that F is well defined upstairs is super easy now,

If β is a second path joining p_0 to z then



we want to show that

$h_0(\pi(\beta)) = h_0(\pi(\alpha))$.

Use the homomorphism property (for paths and loops).

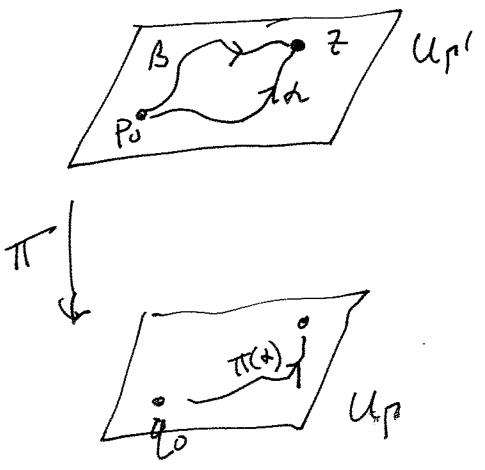
$$h_0(\pi(\beta)) - h_0(\pi(\alpha)) = h_0(\pi(\beta)\cdot\pi(\alpha)^{-1}) = h_0(\pi(\beta\cdot\alpha^{-1}))$$

but the loop $\beta\cdot\alpha^{-1}$ lies in Γ by the construction of U_Γ, so $h_0(\pi(\beta\cdot\alpha^{-1})) = 0$,

Now define $h_1(\gamma) = \exp\left(\frac{1}{n}\int_\gamma P'/P\, dz\right) =$
$\exp\left(\frac{1}{n} h_0(\gamma)\right)$

h_1 is a homomorphism from the groupoid of paths to the multiplicative group $\mathbb{C}^*$.

$h_1(\gamma \cdot \rho) = h_1(\gamma) \cdot h_1(\rho)$.
Define $\Gamma' = \ker h_1 = h_1^{-1}(1)$.
Define $G(z) = h_1(\pi(\alpha))$



$G(z)$ is well defined.
Want to show $G(\alpha) = G(\beta)$

$G(\alpha) \cdot G^{-1}(\beta) = h_1(\pi\alpha) \cdot h_1^{-1}(\pi\beta) = h_1(\pi\alpha \cdot \pi\beta^{-1}) = h_1(\pi(\alpha \cdot \beta^{-1}))$

$\pi(\alpha \cdot \beta^{-1}) \in \Gamma'$ since it is the image of a loop in $U_{P'}$ so $h_1(\pi(\alpha \cdot \beta^{-1})) = 1$ as was to be shown.

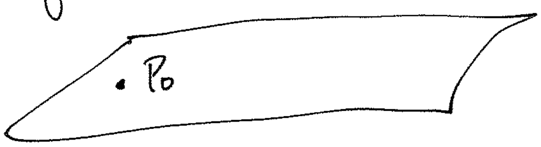
If F is well defined on U_p then $F + \text{const.}$ is also well defined. Changing F by an additive constant corresponds to changing G by a multiplicative constant.

Proposition

For an appropriate constant c:

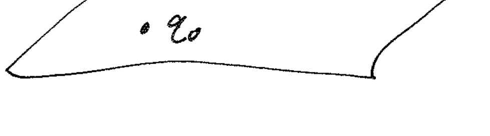
$$cG(z) = \sqrt[n]{P(z)} \quad \text{or} \quad (cG)^n(z) = P(\pi(z)).$$

Proof. By construction $G(p_0) = 1$. It need not be the case that $P(\pi(p_0)) = P(q_0) = 1$. We choose c so that



$$(cG)^n(p_0) = P(q_0),$$

ie $c^n = P(q_0)$.



Now we want to use the fact that $U_{p'}$ is connected as a topological space.

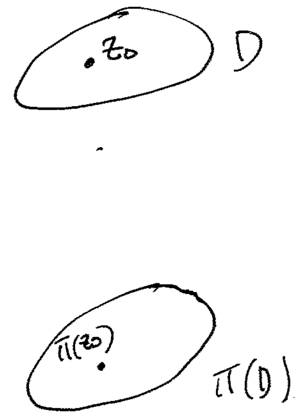
Consider the set Σ of z in $U_{p'}$ for which

$\Sigma = \{ z \in U_{p'} : (cG)^n(z) = P(\pi(z)) \}$. This set is closed since $(cG)^n$ and $P \circ \pi$ are continuous.

The set is non-empty since $p_0 \in \Sigma$.

Let us show that the set is open.

Let $z_0 \in \Sigma$ let D be a disk containing z_0.



Given a holomorphic function f on D we can choose a branch of the $\log f$ over D.
Choose branches of $\log((cG)^n)$ and $\log P(\pi(z))$ that agree at z_0. Let's differentiate them,

$$\log((cG)^n) = n \log(cG) = n(\log c + \log G)$$

where $G = \exp(\frac{1}{n} F)$

$$= n(\log c + \log(\exp \tfrac{1}{n} F))$$

$$= n \log c + n \cdot \tfrac{1}{n} F = n \log c + F.$$

Der. $= \frac{d}{dz} F = \frac{P'}{P}$.

We have constructed a covering space of degree n (we think). Is it connected?

8

$$\frac{d}{dz} \log P(\pi(z)) = \frac{P'(\pi|z))}{P(\pi(z))}$$

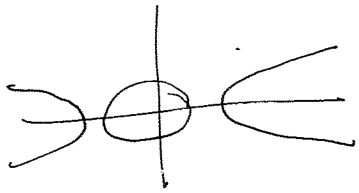
Want to specialize to hyper elliptic curves.

Let $P(x)$ be a polynomial.

$$V = \{(x,y) \in \mathbb{C}^2 : y^2 = P(x)\}$$ (Note $n=2$).

(~~Plane~~ Affine curve not projective)

Say $P = \prod_j (x - x_j)^{m_j}$.



$$V' = V - \{(x_j, 0)\}.$$

$$U = \mathbb{C} - \{x_j\}.$$

Recall that $h_0(\gamma) = \int_\gamma \frac{p'}{p} dz$. ②

$h_0 : \pi_1(U, p) \to \mathbb{C}$.

$$h_0(\gamma) = 2\pi i \sum_j m_j \cdot \mathrm{wind}(\gamma, x_j) \in 2\pi i \mathbb{Z},$$

$$h_1(\gamma) = \exp\left(\tfrac{1}{2} h_0(\gamma)\right) \in \exp\left(\frac{2\pi i \mathbb{Z}}{2}\right) = \{\pm 1\}.$$

$h_1(\gamma) = +1$ if $h_0(\gamma)$ is even

-1 if $h_0(\gamma)$ is odd.

$\bar{h}_1(\gamma) = h_0(\gamma)/2\pi i \mod 2 \in \{0, 1\}$.

Informally $h_1(\gamma) = (-1)^{\bar{h}_1(\gamma)}$.

$$\bar{h}_1(\gamma) = \sum m_j \cdot \mathrm{wind}(\gamma, x_j) \mod 2.$$

$\Gamma = \ker h_1 = \ker \pi_1$.

U_Γ is the covering space associated to Γ.

We defined a function $G : U_\Gamma \to \mathbb{C}$ and showed that $(cG)^2 = P(\pi(z))$. (for some c.)

This means that the map

$z \in U_\Gamma$ maps to

$\Phi(z) = (\underset{X}{\pi(z)}, \underset{Y}{cG(z)})$ is contained in $V = \{(X,Y) : Y^2 = P(X)\}$

It is not hard to see that this map is a bijection when Γ has index 2

in $\pi_1(U, p)$ since for every x in U the image of the map contains two solutions of $Y^2 = P(X)$.

$$\begin{tikzcd} U_\Gamma \arrow[r, "\Phi"] \arrow[d, "\pi"', "\deg 2"] & V \arrow[d, "\pi_X"', "2 \text{ to } 1."] \\ U \arrow[r] & \mathbb{C} \end{tikzcd}$$

Cor. $\pi_X : V' \to U$ is a covering map.

If $\Gamma = \ker h_1$ has index 2 then the covering

$\pi: U_p \to U$ is a normal covering and the deck group has order 2. Unraveling the definitions of G shows that:
The deck group on U_p corresponds to the automorphism $(X,Y) \mapsto (X,-Y)$ of V.
Note that $(X,Y) \mapsto (X,-Y)$ preserves the equation $Y^2 = P(X)$.

(Simple) example.

$Y^2 = X \qquad P(X) = X$

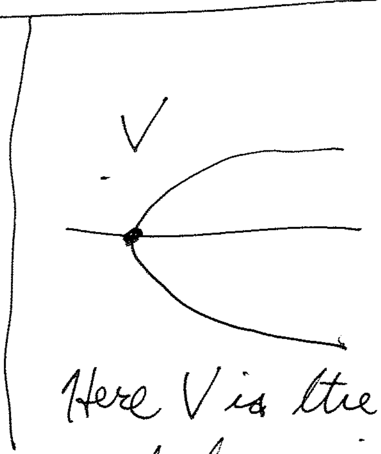
$V' = V - \{(0,0)\}$.

P has 1 root at 0 of multiplicity 2.

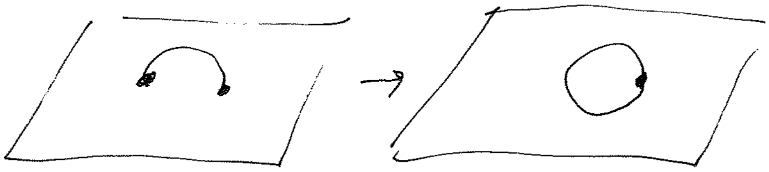
$U = \mathbb{C} - \{0\}$

$\bar{\pi}_1(\alpha) = \mathrm{wind}(\alpha, 0) \bmod 2$.

$\Gamma = \ker \bar{\pi}_1$ is generated by 2·generator of $\pi_1(\mathbb{C} - 0)$ and has index 2.



Here V is the parabola on its side.



Example 2. $P(X) = X^2$. V is singular but our method still works

$V = \{Y^2 - X^2 = 0\}$. $\quad V' = V - \{0,0\}$.

P has 1 root of multiplicity 2 at 0.

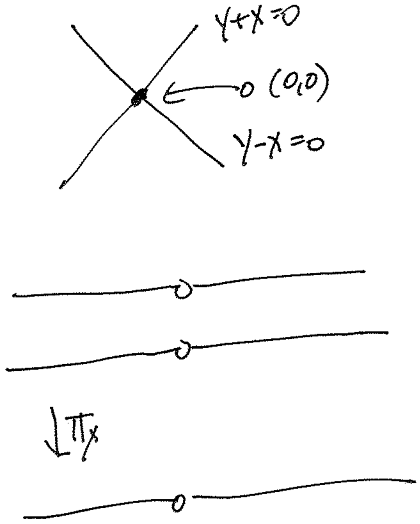
$U = \mathbb{C} - \{0\}$.

$\bar{\pi}_1(\alpha) = 2 \cdot \mathrm{wind}(\alpha, 0) \bmod 2 = 0$.

Γ is the whole group. It does not have index 2.

In this case $V' = V - (0,0)$ is disconnected.

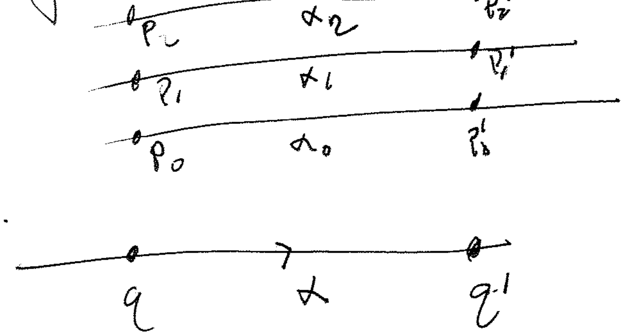
$V = \{Y^2 = X^2\} = \{ \quad Y^2 - X^2 = 0\} = \{(Y-X)(Y+X) = 0\}$



V consists of 2 lines that intersect at $(0,0)$. V' consists of 2 components each of which projects isomorphically to U, $\pi_x|V'$ is still a covering space.

There is an alternative view of covering spaces which works even when the covering space is not connected. A covering space gives rise to a homomorphism $\mu : \pi_1(U, q) \to \mathrm{Perm}(\pi^{-1}(q))$. This is called the monodromy representation and we define it using path lifting:

Any α is a path



α defines a map from $\pi^{-1}(q)$ to $\pi^{-1}(q')$ by considering the endpoint of α_j the lift of α starting at p_j

$\mu_\alpha : \pi^{-1}(q) \to \pi^{-1}(q')$.

Now take $q = q'$,

Example 3.

$x^2+y^2=1$

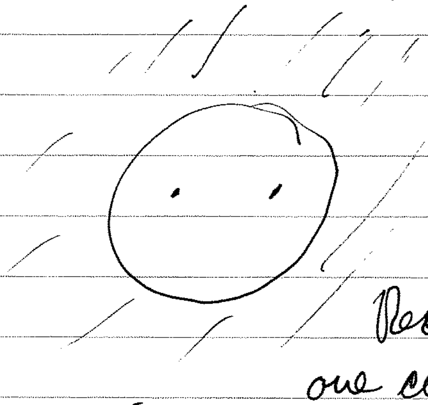
$y^2=1-x^2$

$y^2=(1+x)(1-x)$ $\qquad U=\mathbb{C}-\{\pm 1\}$. Has 2 simple roots,

$-1 \qquad +1$

$\frac{h_1(\gamma)}{2\pi i} = \mathrm{wind}(\gamma,-1)+\mathrm{wind}(\gamma,+1) \mod 2.$

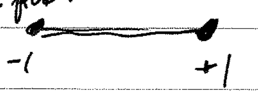
Note that the loop that surrounds both pts. maps trivially.



In a nbd. of ∞ the covering space has 2 distinct sheets.
Recall that there are 2 points at ∞ one corresponding to each sheet. (Curve has 2 asymptotes with slope $\pm i$.)

Slit picture:

In the corresponding projective curve. These sheets give punctured nbhs of these 2 pts.

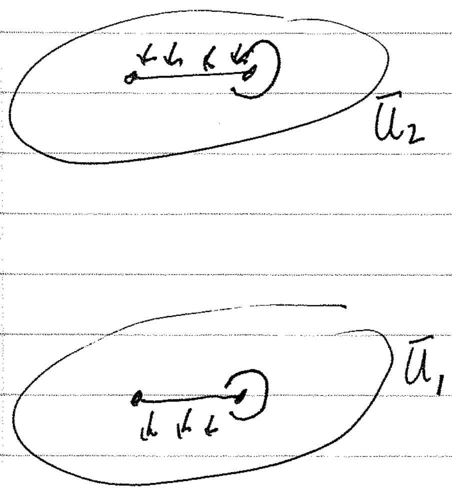


Cut $\mathbb{C}$ along the slit between -1 and 1. Call this $\tilde{U}$. Note that all loops in $\tilde{U}$ correspond to the trivial permutation.

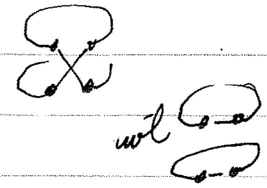
This means that $\pi_x^{-1}(\tilde{U})$ is disconnected. In fact it consists of two sheets.

Each sheet looks like a slit plane

V can be reconstructed by gluing these two sheets together along the slit.

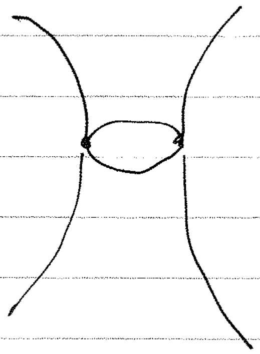


Since any loop which crosses the slit has non-trivial monodromy we see that the top slit must be glued to the bottom slit.



Loop has non-trivial monodromy if it does not lift to a loop upstairs.

If we cut the loop it lifts to 2 pieces, one in $\bar{U}_1$ and one in $\bar{U}_2$.



Reconstruction of V (the cylinder).

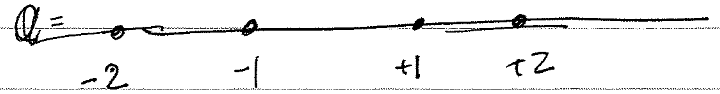
(Projective curve is the sphere.)

Now consider

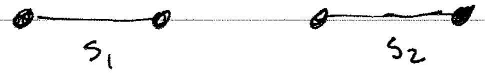
$$y^2 = (x^2-1)(x^2-4).$$

V is non-singular since P has no multiple roots.

$U = \mathbb{C} - \{\pm 1, \pm 2\}$



lets do the slit analysis again.



If we use our slits to pair up the roots of P, then any loop γ which avoids the slits has the property that

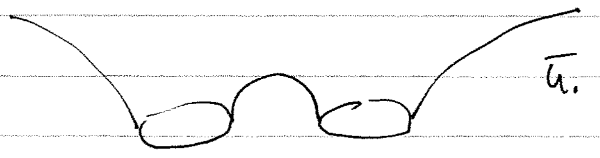
$$\mathrm{wind}(\gamma, -2) = \mathrm{wind}(\gamma, -1) \quad \text{and}$$

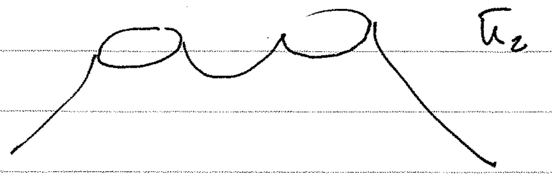
$$\mathrm{wind}(\gamma, 1) = \mathrm{wind}(\gamma, 2) \quad \text{so}$$

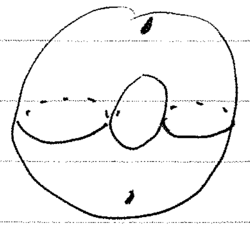
$$\frac{h_1}{2\pi i}(\gamma) = \sum \mathrm{wind}(\gamma, z_j) = 2 \cdot \mathrm{wind}(\gamma, -2) + 2 \cdot \mathrm{wind}(\gamma, 1) = 0 \mod 2$$

Furthermore any γ that crosses one slit exactly once satisfies $\frac{h_1}{2\pi i}(\gamma) = 1 \mod 2.$

Consider $\bar{U}_1$, $\bar{U}_2$ as before





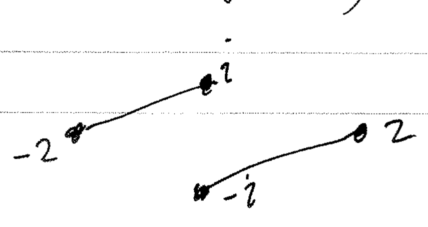


Conformally V is equivalent to a torus with two punctures

If we have 4 distinct roots at any location in $\mathbb{C}$ we can perform the same construction to obtain a topological torus. Typically these tori will have different <u>conformal</u> structures. Interesting to study these.

Consider the polynomial:

$(x^2-4)(x^2+1)$

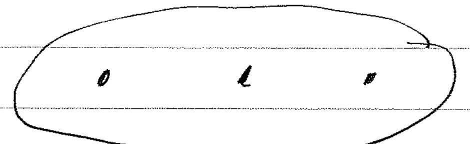


Choice of slits is not unique, but the slits are only a tool.

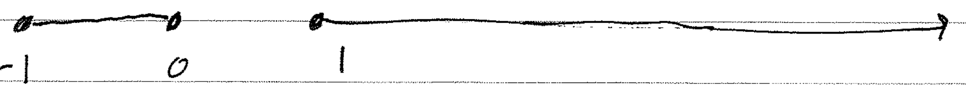
Note that when the # of roots is odd a large loop around all the roots has non-trivial monodromy.

What about a cubic?

$Y^2 = X(X^2-1)$.

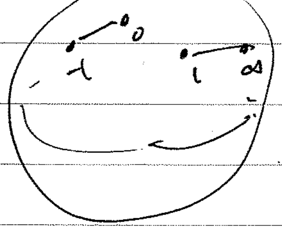


Consistent with having a unique pt. at ∞. ~~which~~



We can construct a second slit by connecting 1 and ∞.

This picture makes more sense in $\mathbb{CP}^1$



Still get a topological torus.

More ~~than~~ roots? Surface of higher genus.