Let F be a field.

Definition. An affine variety over F is the set of solutions of a finite collection of polynomials

$$V_F = \{(x_1 \ldots x_m) : P_1(x_1 \ldots x_m) = \ldots = P_n(x_1 \ldots x_m) = 0\}.$$

Such varieties make sense over any field F.

ad hoc definition: The dimension is #variables − # of equations

(There are some issues with this definition which we ignore for now.)

0 equations: F^n has dim n and is called (affine) n-space. 1-space is the line. 2-space is the plane.

The second simplest case is 1 equation in 1 variable.

In this case the "varieties" have dimension 0 and are finite.

The second simplest case is 1 equation in 2 variables. Such varieties were first studied by the Greeks in the case of the real numbers.

Call these plane curves

Varieties defined by single polynomials in 2 complex variables are also called plane curves by algebraic geometers.

Part of the logic here is that when you are given an equation you might want to think about it as defining varieties over different fields. The corresponding varieties should be related.

We can think about them as revealing different aspects of the equation.

The geometric terminology should not change as the field changes. Thus a complex curve can be an example of a Riemann surface while a surface for an algebraic geometer may be an abstract of complex dimension 2 or real dimension 4.

Complex line is what we might call the complex plane.

Let us adopt the algebraic geometers language for today in an ad hoc way.

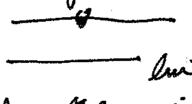
Say that the dimension of a variety is the number of variables minus the number of equations. This is not a good general definition but it will serve for this discussion.

The solution set of a single polynomial in two variables is a plane curve.

Consider some examples.

When f has degree 1 the corresponding "curve" is a line. In $\mathbb{C}^2$ this is just a copy of $\mathbb{C}$.

Individual lines are not so interesting but the nature of intersections leads to incidence geometry. 2 lines intersect in one pts. 0 pts the lines are parallel.



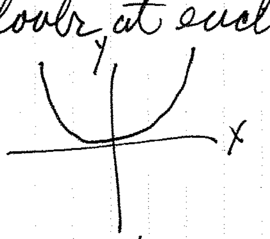
Parallel axiom holds.

What about f having degree 2?

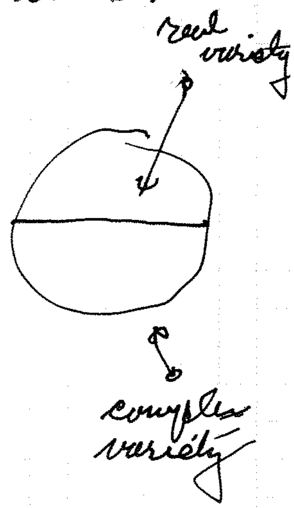
Examples: 1a $Y=X^2$, 1b $X^2+Y^2=1$, 1c $XY=1$, 1d $XY=0$.

1a We can look at each of these in $\mathbb{R}^2$ or $\mathbb{C}^2$

$Y=X^2$



parabola.

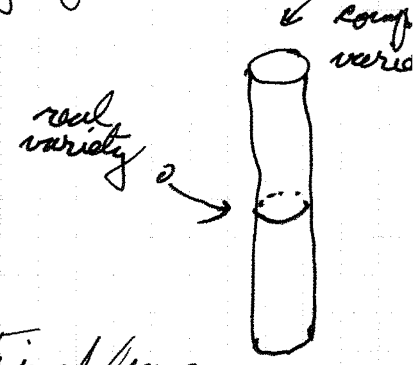


The curve is a graph.
We can parametrize the curve $\varphi(t) = (\underset{X}{t}, \underset{Y}{t^2})$

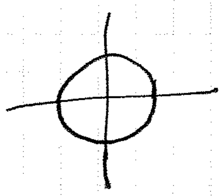
so the real curve is homeomorphic to $\mathbb{R}$.

Same thing works in $\mathbb{C}^2$.

Think of x, y ~~as com~~ t as complex variables.
The complex curve $y=x^2$ is conformally equivalent to $\mathbb{C}$.



1b $x^2+y^2=1$.



Real curve is homeomorphic to S^1. Parametrized by $t \mapsto (\cos t, \sin t)$. Map from $\mathbb{R}$ to S^1 is ~~a covering~~ the universal covering space. ~~&~~ Deck group is Γ, $2\pi\mathbb{Z}$ acting additively. $\pi_1(S^1)=\mathbb{Z}$.

Complex curve is parametrized by

$t \mapsto (\cos t, \sin t)$

$$\cos(t) = \frac{e^{it}+e^{-it}}{2} = \operatorname{Re} e^{it}$$

$$\sin(t) = i\frac{(e^{it}-e^{-it})}{2} = \operatorname{Im} e^{it}$$

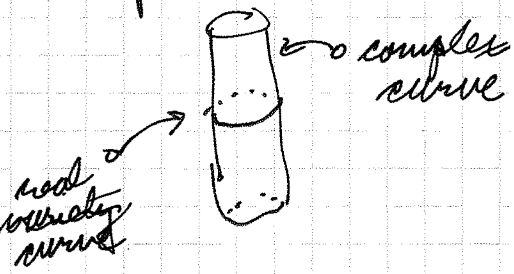
$$\cos^2 + \sin^2 = \frac{e^{2it}+e^{-2it}}{4} + \frac{1}{2} + \frac{-e^{2it}-e^{-2it}+2}{4} = 1.$$

V is conformally isomorphic to

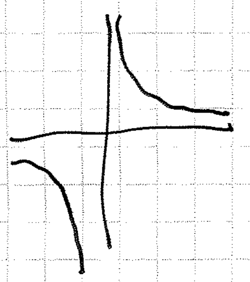
$\mathbb{C}/2\pi\mathbb{Z}$ ≈ cylinder
≈ $S^2 - 2$ pts.

Note that the complex variety is not compact but still $\pi_1 = \mathbb{Z}$; universal cover $= \mathbb{C}$.

"Real curve" is contained in the "complex curve".



1c. $XY = 1$



~~Reparametrization~~ Parametrization: Each non-zero x determines a Y. $t \mapsto \mathbb{C}-\{0\}$ maps to $(t, 1/t)$.

Curve is conformally isomorphic to $\mathbb{C}-\{0\}$.

Could also parametrize it as by $\mathbb{C}$.

$t \mapsto (e^t, e^{-t})$

Again $S^2 - 2$ pts.

p is a singular point if the partial derivatives

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \text{ at } p.$$

If p is not a singular point we call it a regular point.

$$\varphi(z) = (\varphi_1(z), \varphi_2(z))$$

If $\varphi: \mathbb{C} \to \mathbb{C}^2$ is a parametrization then

$p \in \mathbb{C}$ is a singular point of the parametrization

$$\frac{\partial \varphi_1}{\partial z} = \frac{\partial \varphi_2}{\partial z} = 0 \text{ at } p.$$

Example: $P(x,y) = xy$. $V = \{(x,y) : xy = 0\}$

$$\frac{\partial P}{\partial x} = y \qquad \frac{\partial P}{\partial y} = x$$

Both partials vanish at $(0,0)$. So this is a singular point.

We ~~rather~~ have discussed linear and quadratic curves. We will look carefully at cubic and quartic curves.

Newton classified real cubic curves.

He found 72 cases, missed 6.

Criticised by Euler for not having a guiding principle.

Two simplifying viewpoints. Consider projective and complex curves. We will see that non-singular cubics and quartics are all topologically equivalent and represent a 1-parameter family of conformal classes.

There are multiple ways to look at a

Riemann surfaces ~~sitting in~~ defined

$V = \{P(x,y)=0 : x,y \in \mathbb{C}\}$

by a polynomial in $\mathbb{C}^2$.

alg. geom – ~~to~~ [illegible] count some varieties
analytically – parametrizing by sin, cos
topologically – think of V as a 2 dim mfld in $\mathbb{R}^4$

One of these is by analogy with

curves defined by a poly. in $\mathbb{R}^2$.

To make your lines different

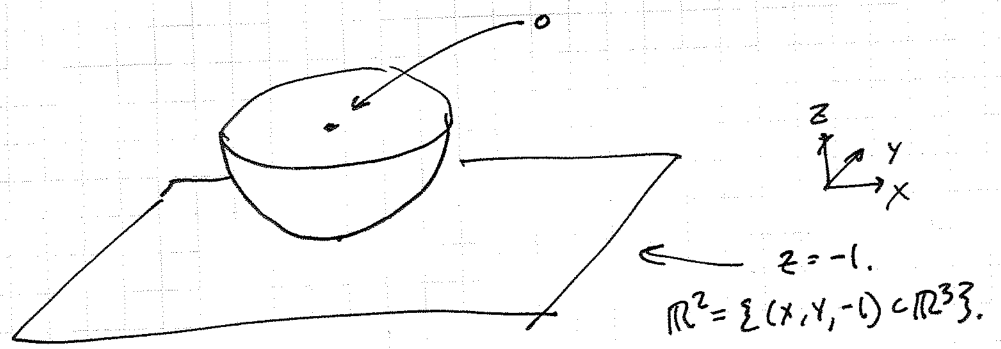
We observed that $\{(x,y)\in\mathbb{C}^2 : x^2+y^2=1\}$ is not compact.

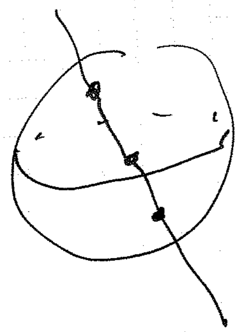
Return to this. Our objective is to add a "line at ∞" in $\mathbb{C}^2$ so that we can study the non-compact affine variety by adding a finite # of points at ∞ to complete it.

We start with a geometric construction for lines in the real plane $\mathbb{R}^2$. We will formalize this algebraically.

This construction has a certain formal ~~a~~ manipulation of equations and it makes sense over $\mathbb{R}$. Pictures are easier to draw so we look at this case first.

We will see that this formulation makes sense over any field in particular over $\mathbb{C}$.





A point in the plane determines a line in $\mathbb{R}^3$. A line in $\mathbb{R}^3$ determines a pair of points on the unit sphere.

$(x,y,-1)$ $\quad$ $(rx, ry, -r)$

We can identify the point in $\{z=-1\}$ with a unique point on the lower hemisphere.

We identify the lines in the x-y plane with "points at ∞". Note that each of these lines corresponds to 2 antipodal points on the equator.

We define $\mathbb{RP}^2$ to be the space of lines in $\mathbb{R}^3$. We can topologize this as

$S^2/p\sim-p$ or $D^2/\sim$ where $p\sim-p$ on $\partial D^2 = S^1$.

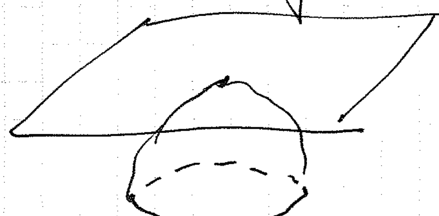
We can think of $\mathbb{R}^2$ as sitting inside ~~on~~ $\mathbb{RP}^2$.

Opportunity for confusion.

Previously we completed $\mathbb{C}$ to get $S^2 = \mathbb{C} \cup \{\infty\}$.
Think of this as a 1-dim complex object.

Now we complete $\mathbb{R}^2$ to get $\mathbb{RP}^2$ by adding a circle. Think of this as a 2 dim real object.

For the sake of making equations turn out nicely we redraw the picture



$\mathbb{R}^2 = \{(x, y, 1) \subset \mathbb{R}^3\}$ so $\mathbb{R}^2$ corresponds to $z = 1$.

$\mathbb{RP}^2 = \{(x, y, z) \in \mathbb{R}^3 : (x, y, z) \neq (0,0,0)\} \Big/ (x,y,z) \sim (\sigma x, \sigma y, \sigma z)$

Now the point (x, y) in $\mathbb{R}^2$ corresponds to the line $(x, y, 1)$ in $\mathbb{R}^3$.

What does it mean for a sequence of points (x_n, y_n) in $\mathbb{R}^2$ to converge to a certain point at ∞ in $\mathbb{RP}^2$?

$\boxed{\text{Prop. } p_n = (x_n, y_n) \text{ converges in } \mathbb{RP}^2 \text{ to a point on the line at } \infty \text{ if } \frac{x_n}{y_n} \to m \in \mathbb{R}_\infty}$

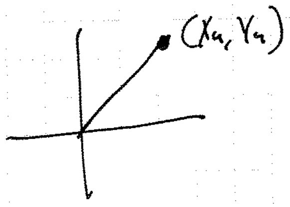
First (x_n, y_n) must go to ∞ in norm.

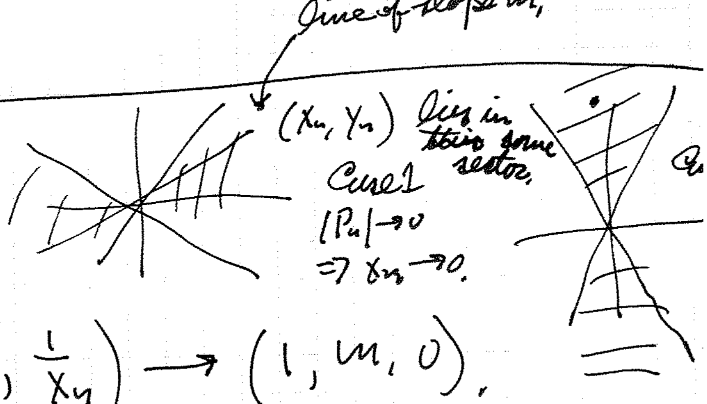
View this as $(x_n, y_n, 1)$ in $\mathbb{R}^3$. Equiv. to

or equ
$\{(x, y, z)$
$(x, y, z) \sim$
$(-x, -y,$

Prop. If $p_n = (x_n, y_n)$ satisfies $\|p_n\| \to \infty$ and $\frac{y_n}{x_n} \to m \in \mathbb{R} \cup \{\infty\}$

then p_n converges in $\mathbb{RP}^2$ to the pt.
corresponding to the line spanned by $(1, m, 0)$
if $m \neq \infty$ or $(0, 1, 0)$ if $m = \infty$.





Say $\frac{y_n}{x_n} \to m \neq \infty$.

$$(x_n, y_n, 1) \sim \left(1, \frac{y_n}{x_n}, \frac{1}{x_n}\right) \to (1, m, 0).$$

Say $\quad (x_n, y_n, 1) \sim \left(\frac{x_n}{y_n}, 1, \frac{1}{y_n}\right) \to (0, 1, 0).$

$\frac{y_n}{x_n} \to \infty \quad \frac{x_n}{y_n} \to 0$

We want to extend the notion of a line in $\mathbb{R}^2$ to a projective line in $\mathbb{RP}^2$ so that any two distinct projective lines intersect in a point.

A line in $\mathbb{R}^2$ corresponds to a linear equation $ax+by+c=0$

a line in $z=1$ plane determines a linear 2 dim space in $\mathbb{R}^3$.

A linear equation in the $z=1$ plane determines a 2-plane in $\mathbb{R}^3$ corresponding to ~~a~~ the equation $ax+by+cz=0$.

Note setting $z=1$ gives original equation. and eqn 2 is ~~linear~~ a homogeneous eqn. not an inhom

We define the corresponding projective line in $\mathbb{RP}^2$ to correspond to the collection of lines in $\mathbb{R}^3$ that satisfy this second linear equation.

Note that this includes line in $\mathbb{R}^3$

$$ax+by+cz=0 \qquad z=0 \qquad (= \ ax+by=0,\ z=0)$$

which corresponds to a point at ∞ in $\mathbb{RP}^2$.

In order to distinguish old lines from new lines we call lines in $\mathbb{R}^2$ affine lines.

Note that every affine line gives rise to a projective line which includes an extra point at ∞. The collection of all points at ∞ is in fact a line so the terminology "line at ∞" is justified.

Any two distinct projective lines intersect in one point. If two affine lines are parallel then the corresponding projective lines intersect in a point at ∞.

Projective lines are homeomorphic to circles.

in particular they are compact where affine lines were not compact.

This construction extends to all varieties in $\mathbb{R}^2$, An affine variety in $\mathbb{R}^2$ has an extension to a (compact) projective variety in $\mathbb{RP}^2$.

Any $V_{aff} = \{(X,Y) : P(X,Y) = 0\}$

Let $R(X,Y,Z) = P(\frac{X}{Z}, \frac{Y}{Z})$. Note that R is no longer a polynomial, it is a rational function and R has the property that $R(\lambda X, \lambda Y, \lambda Z) = R(X,Y,$ so it is well defined on $\mathbb{RP}^2$

If $P(X,Y) = \sum a_{nm} X^n Y^m$ Let $\text{degree}(P) = \max(n+m : a_{n,m} \neq$

$$R(X,Y,Z) = \sum a_{nm} \frac{X^n Y^m}{Z^{n+m}}$$

We can remove denominators by multiplying through by Z^d

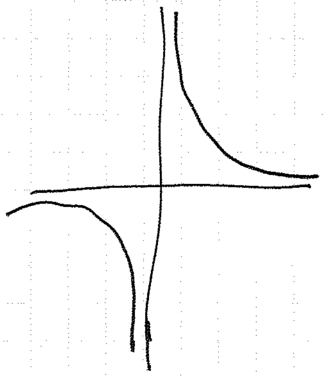
$$Q(X,Y,Z) = Z^d R(X,Y,Z) = \sum a_{nm} X^n Y^m Z^{d-(n+m)}.$$

Let $V_{proj} = \{ \underset{\neq (0,0,0)}{(X,Y,Z)} : Q(X,Y,Z) = 0 \} / {v \sim \lambda v}. \subset \mathbb{RP}^2.$

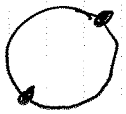
Q is an example of a homogeneous polynomial, all terms have the same degree. Q satisfies

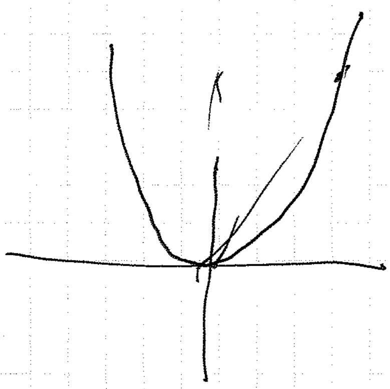
$$Q(\lambda X, \lambda Y, \lambda Z) = \lambda^d Q(X, Y, Z).$$

The value of Q is not well defined on lines in $\mathbb{R}^3$ so not on $\mathbb{RP}^2$ but the 0 set of Q is well defined on $\mathbb{RP}^2$.

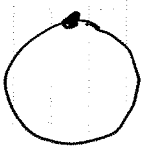


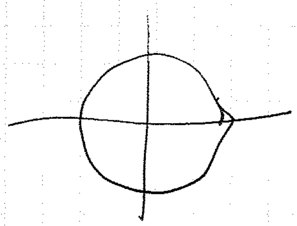
Add 2 pts. at ∞ ($m=0$, $m=\infty$), get a circle.





Add 1 pt. at ∞, get a circle



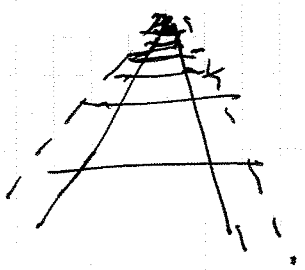


Add no pts. at ∞.

A plane curve of degree 1 is a line.
Lines are not so interesting as objects in themselves
but the collection of lines in the plane
leads to incidence geometry (affine geometry): 2 points
determine a line, 2 lines intersect in 0 or 1
point.

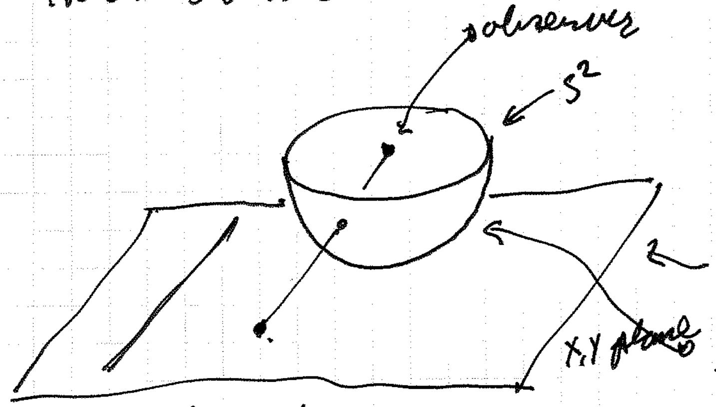
The problem of understanding how to
correctly render perspective in painting
led to the development of projective geometry.
In projective geometry it is useful to add
points at ∞ in such a way that parallel
lines in the affine plane intersect in a
point at ∞.

Consider a railroad track. The rails are (9)



parallel but when we draw it in perspective the lines appear to intersect at a "vanishing point".

How do we model this mathematically?



How do we construct the projective plane over $\mathbb{R}$?

$z=-1$.

visual sphere

affine plane sitting inside $\mathbb{R}^3$.

A point in the plane determines a line in $\mathbb{R}^3$ which determines a point on S^2.

Origin at the observer.

A line in the affine plane determines a plane passing through 0 (ie ~~a line~~ an affine line determines a linear plane). ~~The collection~~

A affine point lies on this line if the corresponding line lies in the corresponding plane.

The "line at ∞" corresponds to those lines in $\mathbb{R}^3$ which are parallel to this plane. Define $\mathbb{RP}^2$ to be the collection of lines in $\mathbb{R}^3$. Using this model we can think of $\mathbb{RP}^2$ as a completion of $\mathbb{R}^2$.

Each affine line in $\mathbb{R}^2$ corresponds to a projective line in $\mathbb{RP}^2$ which is the result of adding a single "point at ∞".

Two distinct projective lines intersect in a unique point in $\mathbb{RP}^2$ (perhaps "at ∞").

To algebraicize this we make one slight change. We put the affine plane at $z=1$. (No essential change in the picture.)

This means that the affine line $ax+by=c$ corresponds to the projective line defined by $ax+by=cz$. (Setting $z=1$ reduces the second equation to the first.) Note that the second equation is linear so that its solution set is a union of lines through the origin (or 1 dim subspaces).

Def. For F a field the F projective plane $F\mathbb{P}^2$ is the set $F^3 - \{0\}/\sim$ where

$v \sim w$ if $v = \lambda w$ for $\lambda \in F - \{0\}$.

F^2 is the affine F-plane and we think of F^2 as a subset of $F\mathbb{P}^2$ by mapping (X, Y) to the equivalence class of $(X, Y, 1)$.

Given a plane curve of any degree there is a corresponding projective curve obtained by adding an appropriate power of Z to each monomial.

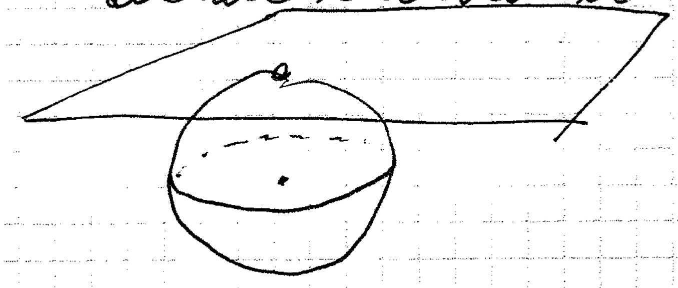
$$\begin{array}{lcl} P(X,Y) & & \overline{P}(X,Y,Z) \end{array}$$

Example: $X^2+Y^2-1 \longmapsto X^2+Y^2-Z^2$

Definition. A polynomial is homogeneous of degree d if each monomial has degree d. $\rightarrow \overline{P}(\lambda X, \lambda Y, \dots) = \lambda^d \overline{P}(X, \dots$ The procedure above associates to $P(X,Y)$ a homogeneous polynomial $\overline{P}(X,Y,Z)$ with $\overline{P}(X,Y,1) = P(X,Y)$ so that

its intersection with the affine plane is the
original variety.

Let me redraw the picture



Plane corresponds to $\{(X, Y, 1)\} \subset \mathbb{R}^3$.

Let $P(X,Y)$ be a polynomial equation.

We would like to find a polynomial equation

$Q(X,Y,Z)$ so that $\{P(X,Y)=0\} = \{(X,Y): Q(X,Y,1)=0\}$

and the zero set of Q is a union of lines through the origin.

Let $R(X,Y,Z) = P(\frac{X}{Z}, \frac{Y}{Z})$

$R(X,Y,Z) = R(\lambda X, \lambda Y, \lambda Z)$ so zero set of R is a union of lines.

If $P = \sum a_{nm} X^n Y^m$ then

$$R(X,Y,Z) = \sum a_{nm} \frac{X^n Y^m}{Z^{n+m}}$$

Note that the largest value of $n+m$ that occurs is $d = \deg(P)$. We can remove denominators by multiplying by Z^d.

$$Q(X,Y,Z) = Z^d R(X,Y,Z) = \sum a_{nm} X^n Y^m Z^{d-(n+m)}.$$

Q is an example of a homogeneous polynomial

all terms have the same degree.

$$Q(\lambda X, \lambda Y, \lambda Z) = \lambda^d Q(X,Y,Z)$$

Q is not constant on lines in general but the set where $Q=0$ is a union of lines.

Since this construction is purely algebraic we can perform it over any field (For consistency we should call the locus a "curve".)

Makes sense over $\mathbb{C}$ though it is not so easy to draw.

A complex projective curve corresponds to a complex affine curve where we have added points corresponding to " "asymptotic values".

Notation for homogeneous coordinates ①

$(X\ Y)$ vector in $\mathbb{F}^2$

$(X:Y)$ corresponding line in $\mathbb{F}^2$ or point in $\mathbb{FP}^1$

Only makes sense when X, Y not both zero.

$(\lambda X : \lambda Y) = (X : Y)$.

How do you find the points at ∞ for a projective curve?

Example. Consider the projective variety correspondi[ng] to $X^2+Y^2=1$ in $\mathbb{CP}^2$. What are the points at ∞?

Write $P(X,Y) = X^2+Y^2-1$. The corresponding homogene[ous] poly: $Q(X,Y,Z) = X^2+Y^2-Z^2$.

$V = \{Q=0 \text{ in } \mathbb{CP}^2\}$. $= \{(X:Y:Z) : Q(X:Y:Z)=0\}$

The complex line at ∞ in $\mathbb{CP}^2$ is the set of lines in $\mathbb{C}^3$ with $Z=0$.

The intersection $V \cap L$ is the set of lines $(X,Y,0)$ with $X^2+Y^2=0$.

We view this as a subset of the complex line at ∞. $\{(X:Y:0) : X^2+Y^2=0\}$. (Can identify with $\mathbb{CP}^1$: $(X:Y)$)

X^2+Y^2 factors as $(X+iY)(X-iY)$ so it determines 2 lines,

$X+iY=0$ and $X-iY=0$

We identify these lines with their slopes we get

$iY=-X$ $\quad Y=+Xi$ $\quad$ so $m=\frac{Y}{X} = +\frac{Xi}{X} = +i$.

$iY=X$ $\quad Y=-iY$ $\quad m=\frac{Y}{X}=-i$.

Compare this with the parametrization

$t \mapsto (\cos t, \sin t)$.

Line in $\mathbb{C}^2$ can be given 2 ways: generating vector or linear equation.

Equation for a line vs. vector generating a line.

Parametrize lines in $\mathbb{C}^2$ by their slope. in $\mathbb{C}\cup\infty$.

$aX+bY=0$

$bY=-aX$

$Y=-\frac{b}{a}X$

m if $a\neq$

$(1:-$

or $(a$

$bY=0$ slope ∞.

(Could also consider inv[erse] slope. Set the 2 stan[dard] charts for $\mathbb{CP}^1$.)

Lemma. If $P(X,Y)$ is a non-linear homogeneous polynomial of degree d in two variables with $\mathbb{C}$ coefficients then it factors as a product of linear polynomials

$$P(X,Y) = \prod_{i=1}^{d} (\alpha_i X + \beta_i Y)$$

Proof. $$P(X,Y) = \sum_{v=0}^{d} a_v X^v Y^{d-v} = Y^d \sum_{v=0}^{d} a_v \left(\frac{X}{Y}\right)^v$$

Let e be the largest index with $a_e \neq 0$.

$$= Y^d \sum_{v=0}^{e} a_v \left(\frac{X}{Y}\right)^v$$

$$= a_e Y^d \prod_{j=1}^{e} \left(\frac{X}{Y} - \gamma_j\right)$$

slope $= \gamma_i$

slope $= \infty$.

$$= a_e Y^{d-e} \prod_{j=1}^{e} (X - \gamma_j Y).$$

(Classification of 0-dim projective varieties in $\mathbb{CP}^1$ as points with multiplicity. Sum of multiplicities $= d$.)

These lines correspond to points added to the projective curve.

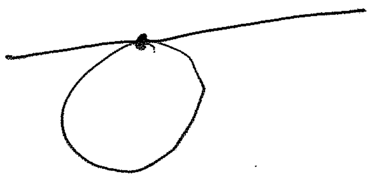
Remark. If we do the same construction with the parabola $Y = X^2$ we get

$P(X,Y) = X^2 - Y$

$Q(X,Y,Z) = X^2 - YZ$

$Q(X,Y,0) = X^2$.

The point at ∞ is the line of slope ∞ but it appears as a repeated factor. $X^2 = 0$.
This corresponds to the alg. geometric fact that the parabola is tangent to the line at ∞ whereas the ~~sphe~~ other quadrics are not.



Intersection pt with multiplicity 2.

(Tangency $\Leftrightarrow$ mult. > 1).

Let $P(z,w) = \sum a_{n,m} z^n w^m$ be a polynomial in 2 complex variables.

Is P a holomorphic function in the sense defined earlier?

$P : \mathbb{C}^2 \to \mathbb{C}$. Let's choose real coordinates for $\mathbb{C}^2$ and $\mathbb{C}$.

Write $z = x + iy$, $w = u + iv$ and

$P = R(z,w) + iQ(z,w) = R(x+iy, u+iv) + iQ(x+iy, u+iv)$.

In these coordinates the derivative of P is

$$DP = \begin{bmatrix} \frac{\partial R}{\partial x} & \frac{\partial R}{\partial y} & \frac{\partial R}{\partial u} & \frac{\partial R}{\partial v} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} & \frac{\partial Q}{\partial u} & \frac{\partial Q}{\partial v} \end{bmatrix}.$$

matrix for mult by i on $\mathbb{C}$ is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

" on $\mathbb{C}^2$ is $\left[\begin{array}{cc|cc} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$, Generalized Cauchy-Riemann equation is $J_1 \circ DP = DP \circ J_2$. This is equiv. to

$$\frac{\partial R}{\partial x} = \frac{\partial Q}{\partial y} \qquad \frac{\partial R}{\partial y} = -\frac{\partial Q}{\partial x} \quad \cdots$$

This is our notion of holomorphic.

Does DP satisfy this?

Note that P defines two 1-variable polynomial functions: (functions) $\phi_1(z)$: ~~$P(z_0$~~ $z \mapsto P(z, w_0)$ and $\phi_2(w)$: $w \mapsto P(z_0, w)$.

In $P = R, Q$, $z = x, y$ coordinates the first block is the derivative of the first function, ~~and~~ in $P = R, Q$, $w = u, v$ ~~the~~ coordinates the second block is the derivative. Thus each block satisfies the CR equations individually so DP is complex linear so P is holomorphic in our sense.

If we want to avoid introducing real coordinates (which we do) we write:

$$\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} + i\frac{\partial R}{\partial y}$$

and observe that the first block is just

real nos.

$$\frac{\partial R}{\partial x} \cdot I + \frac{\partial R}{\partial y} \cdot J.$$

2x2 matrices

$\frac{\partial P}{\partial z}$ standard complex analysis derivative of ϕ_1. Computed using standard rules.

This notation is compatible with the idea that we write the action of complex scalar multiplication on $\mathbb{C}$ as $\underbrace{a+bi}$ $\rightarrow$ $\underbrace{aI+bi}_{\substack{2\times2 \text{ matrix} \\ \text{wrt} \\ X,Y \text{ coord}}}$

complex no. (1×1 complex matrix)

of $z \mapsto P(z,w_0)$

$\frac{\partial P}{\partial z}$ is just the standard derivative in the sense of Complex Analysis so ~~sub~~ is computed following the classic rules:

$$\frac{\partial P}{\partial z} = \sum n a_{n,m} z^{n-1} w^m.$$

Recall $(z_0, w_0) \in \mathbb{C}^2$ is a regular point for V

if $DP = \begin{bmatrix} \frac{\partial P}{\partial z} & \frac{\partial P}{\partial w} \end{bmatrix} \neq 0$.

We interpret DP as a linear map from the tangent space at (z_0, w_0) to the tangent space to $\mathbb{C}$ at $P(z_0, w_0)$.

If (z_0, w_0) is regular the kernel of DP is 1 complex dimensional and we define it to be the tangent line to V.

If V has a local chart $\phi : U \overset{\subset \mathbb{C}}{\longrightarrow} V$ write $\phi(0) = (z_0, w_0)$ then $D\phi = \begin{bmatrix} \frac{\partial \phi_1}{\partial z} \\ \frac{\partial \phi_2}{\partial z} \end{bmatrix}$ is a vector which generates the tangent line. This follows from the chain rule $D(P \circ \phi) = DP \circ D\phi = 0$ since $P \circ \phi(z) = 0$.

Thm. A non-singular complex plane curve is a Riemann surface.

Last time introduced the real projective plane $\mathbb{RP}^2$ and the definition of projective varieties in $\mathbb{RP}^2$.

In formalizing this we will see that we can define a projective space of any dimension over any field $\mathbb{FP}^n$. When $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ we will show that $\mathbb{FP}^n$ is a ~~smooth~~ manifold or a complex manifold with charts in $\mathbb{R}^n$ or $\mathbb{C}^n$. In fact the construction works over any field $\mathbb{F}$ with "charts" in affine n-space $\mathbb{F}^n$ though we won't use these notions in this course. (Many constructions ~~requi~~ in alg. geom. require alg. closed fields, not relevant here.)

Projective n-space

In terms of the course this is the last section of examples ~~and~~ of mflds. We will also take the opportunity to introduce some ideas used later.

Let $\mathbb{F}^* = \mathbb{F} - \{0\}$.

Def. $\mathbb{FP}^n = \mathbb{F}^{n+1} - \{0\} \Big/ \mathbb{F}^*$ (space of $\mathbb{F}$ lines in $n+1$ space.)

$$\begin{tikzcd} & \mathbb{F}^{n+1} - \{0\} \arrow[d] \\ \mathbb{F}^n \arrow[ur, "\varphi"] \arrow[r, "\varphi"] & \mathbb{FP}^n \end{tikzcd}$$

Define. $\varphi : \mathbb{F}^n \longrightarrow \mathbb{FP}^n$ by

$\varphi(x_1 \ldots x_n) = (x_1 : \ldots x_n, 1)$.

We can call $\{(x_1 \ldots x_n\ 0)\} \subset \mathbb{F}^{n+1}$ where $(x_1 \ldots x_n) \neq 0$ the "hyperplane at ∞". ~~and~~

We can identify the hyperplane at ∞ with $\mathbb{F}^{n-1}$ by mapping

$$\begin{tikzcd} \mathbb{F}^n - \{0\} \arrow[r] \arrow[d] & \mathbb{F}^{n+1} - \{0\} \arrow[d] \\ \mathbb{FP}^{n-1} \arrow[r] & \mathbb{FP}^n \end{tikzcd}$$

$(x_1 \ldots x_n) \rightsquigarrow (x_1 \ldots x_n\ 0)$.

In this way we write $\mathbb{FP}^n$ as affine n-space $\mathbb{F}^n$ plus the hyperplane at ∞, $\mathbb{FP}^{n-1}$

Note that $\mathbb{FP}^0 = pt.$

$\mathbb{RP}^1 = \mathbb{R} \cup$ pt. at ∞

$\mathbb{RP}^2 = \mathbb{R}^2 \cup$ line at ∞ "$\mathbb{RP}^1$" ← show one

$\mathbb{CP}^1 = \mathbb{C} \cup$ pt at ∞

$\mathbb{CP}^2 = \mathbb{C}^2 \cup \mathbb{CP}^1$ at ∞.

$\mathbb{F}\mathbb{P}^n$ has an atlas with charts $\varphi_1 \ldots \varphi_{n+1}$, ~~where~~

Let $U_j \subset \mathbb{F}\mathbb{P}^n = \{(x_1 \ldots x_j \ldots x_{n+1}) : x_j \neq 0\}$. (invariant under scalar mult.

Let $\psi_j : U_j \longrightarrow \mathbb{F}^n$ be given by $\psi_j(x_1 \overset{x_j}{\ldots} x_{n+1}) =$

$$\left(\frac{x_1}{x_j} \ldots \widehat{\frac{x_j}{x_j}} \ldots \frac{x_{n+1}}{x_j}\right)$$

delete this coordinate

We are defining this on $\mathbb{F}^{n+1}$ but ψ_j is invariant under scalar multiplication so it is well defined on $U_j \subset \mathbb{F}\mathbb{P}^n$.

φ_j is given by $\varphi_j(y_1 \ldots y_{j-1}\ y_{j+1} \ldots y_{n+1}) = (y_1 \ldots y_{j-1}\ 1\ y_{j+1} \ldots y_{n+1})$

$$\varphi_j \circ \psi_j(x_1 \ldots x_j \ldots x_{n+1}) = \psi_j^{-1}\left(\frac{x_1}{x_j} \ldots \widehat{\frac{x_j}{x_j}} \ldots \frac{x_{n+1}}{x_j}\right)$$

$$= \left(\frac{x_1}{x_j} \ldots 1 \ldots \frac{x_{n+1}}{x_j}\right)$$

in j-th position

$$\psi_j \circ \varphi_j \underbrace{(y_1 \ldots y_{j-1}\ y_{j+1} \ldots y_{n+1})}_{\in \mathbb{F}^n} = \psi_j \underbrace{(y_1 \ldots y_{j-1}\ 1\ y_{j+1} \ldots y_{n+1})}_{\in \mathbb{F}\mathbb{P}^n}$$

$$= \underbrace{\left(\frac{y_1}{1} \ldots \frac{y_{j-1}}{1}\ \frac{y_{j+1}}{1} \ldots \frac{y_{n+1}}{1}\right)}_{\in \mathbb{F}^n}.$$

Overlap functions.

$\varphi_k : \mathbb{F}^n \to \mathbb{F}^n$

$$\varphi_{jk}(y_1 \dots y_n) = \varphi_j(y_1 \dots y_{k-1}\ 1\ y_k \dots y_n)$$

$$= \left(\frac{y_1}{y_j} \cdots \frac{y_{k-1}}{y_j}\ \frac{1}{y_j}\ \frac{y_k}{y_j} \cdots \frac{y_n}{y_j}\right).$$

Well defined on $\varphi_k^{-1} \cap U_j$ and smooth/holomorphic

Example: $\mathbb{FP}^1$ $\varphi_1(x_1 x_2) = \frac{x_2}{x_1}$ $\varphi_2^{-1}(y) = (y\ 1)$

$\varphi_{12}(y) = \varphi_1(y\ 1) = \frac{1}{y}$. $\boxed{\varphi_1(x\ y) = \frac{y}{x} \text{ gives slope coordinate,}}$

Conclude $S^2 = \mathbb{CP}^1$ as a Riemann surface.

since it has an atlas with 2 charts and the same overlap function.

Defined where the slope is not ∞.

$F=\mathbb{R}$ Function from $\mathbb{R}^n$ to $\mathbb{R}^n$ with smooth coordinate fcns. Clearly differentiable (continuous partials $\Rightarrow$ diff.)

$F=\mathbb{C}$ Function from $\mathbb{C}^n$ to $\mathbb{C}^n$ with hol. coord fcns.

Proposition. $\mathbb{CP}^n$ is compact as a topological space.

Proof.

Consider the unit sphere in $\mathbb{C}^{n+1}$. This is a $2n+1$ dimensional sphere in a real $2n+2$ dimensional space. It is closed and bounded, hence compact. We map a point in S^{2n+2} to the complex line that contains it. This is a surjective map from S^{2n+2} to $\mathbb{CP}^n$. The image of a compact space under a continuous map is compact.

Corollary. A projective variety V in $\mathbb{CP}^2$ is compact.

Proof. V is the zero set of a homogeneous poly in $\mathbb{C}^3$. The zero set of a homogeneous polynomial on $\mathbb{C}^3$ $\mathbb{C}^3 - \{0\}$ is closed. Its intersection with S^5 is closed and its image in $\mathbb{CP}^2$ is compact.