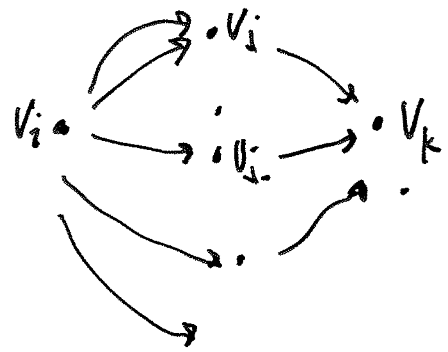
Counting words and counting periodic points.

Let G be a ~~graphs~~ directed graph. Assume the set of ~~edg~~ vertices is ~~$\{1 \dots n\}$~~ $\{v_1 \dots v_n\}$. Define a matrix ~~M_G by~~ $M = M_G$ by M_{ij} = # of paths from v_i to v_j.

Prop. The number of words of length n starting at v_i and ending at v_j is $(M^n)_{ij}$.

Proof. ~~The~~

Consider paths of length 2. The number



of paths of length 2 going through v_j is

$m_{ij} \cdot m_{jk}$. ← makes sen[se] if $m_{ij} = 0$ or

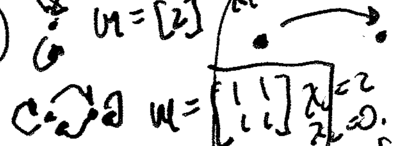
We get the total number of paths by summing over all possible middle vertices ie $\sum_j m_{ij} m_{jk}$. $= [M^2$

For a path of length n we sum over the possible intermediate vertices and get

$$\sum_{j_2 j_3 \dots j_{n-1}} m_{i j_2} m_{j_2 j_3} m_{j_3 j_4} \cdots m_{j_{n-1} k}.$$

Cor. The number of fixed points of σ^n is $\mathrm{Tr}(M^n)$.

Proof. Fixed points of σ^n correspond to sequences that repeat with period n. These correspond to paths of length n that start and end at the same point i.e. the diagonal entries of M^n. If we sum the diagonal entries then we get the total number of fixed points. This is the trace.



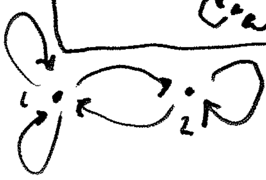
$M = [2]$ $\lambda = 2$

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$M = \boxed{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}$ $\lambda_1 = 2$, $\lambda_2 = 0$.

Example:

associated to f_A. $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.



$M = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

Think of M as a linear map from $\mathbb{C}^n \to \mathbb{C}^n$.
~~Diagonalize~~
Upper triangular
$\begin{bmatrix} \lambda & * \\ 0 & \lambda_n \end{bmatrix}$
$\mathrm{Tr}(M^n) = \lambda_1^n + \dots$

(Is it a coincidence that $M = A$? Yes and no. It is not a coincidence that they have the same largest eigenvalue. One matrix is constrained to be non-negative and the other is not.)

~~Eigenvalue~~ Eigenvalues of M are $\lambda = \frac{3+\sqrt{5}}{2}$ and $\mu = \frac{3-\sqrt{5}}{2}$.

$\mathrm{Tr}(M^n) = \lambda^n + \mu^n \sim \lambda^n$. ~~$\log \mathrm{Tr}(M^n)$~~ $\frac{\log \mathrm{Tr}(M^n)}{n} \to \log \lambda$.

Example:

$M = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

No periodic points of odd period.

$M^2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

Eigenvalues $\sqrt{2}$, $-\sqrt{2}$

$\mathrm{tr}\, M^n = 2^{n/2} + (-1)^n 2^{n/2}$.

How fast does the number of words grow?

Thm. Let $\rho = \rho(M) = \max_i |\lambda_i|$ where λ_i ranges over the real and complex eigenvalues.
$\rho(M)$ is the spectral radius of M.

Let $W_n(\mathcal{G})$ be the number of ~~words~~ paths in $\mathcal{G}$ of length n, ~~then~~

If $\Sigma_{\mathcal{G}} \neq \phi$ then

Prop. $\lim_{n\to\infty} \frac{\log W_n(\mathcal{G})}{n} = \log \rho(M)$.

Proof. Define a norm on $n \times n$ matrices by

$|A| = \sum_{i,j} |a_{ij}|$. Then $|M^n| = W_n(\mathcal{G})$ where $M = M(\mathcal{G}$

According to the spectral radius thm.

$$\lim_{n\to\infty} |M^n|^{1/n} = \rho(M).$$

If $\rho(M) \neq 0$ we can take logs of both sides to get
$\Updownarrow$
M not nilpotent

$$\lim_{n\to\infty} \frac{\log |M^n|}{n} = \log \rho.$$

This says that $W_n \sim \rho^n$.

Example: For $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ $f_{\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}}$ the order of growth of the # of words is the same as the order of growth of the # of fixed points of f^n.

Growth of the number of words of length n is ~~more~~ less robust than growth of number of fixed points of f^n.

On the other hand, while it is clear that the # of fixed points of f^n is a topological conjugacy invariant ~~the number of~~. it is not clear that the growth of the number of words is a top. conjugacy invariant.

~~In the~~

~~We claim that the growth rate~~

The # of words of length n is not a topological invariant since it depends on the particular A we choose but the growth rate of the # of words of length n is a topological invariant.

We will see this by identifying it with ~~the~~ an invariant defined for all discrete dynamical systems on compact spaces which is called the entropy.

(One approach would be to count periodic points but this is not very robust. Product with an irrational rotation destroys them.)

Our next topic is topological entropy.

This will be a real valued invariant of a dynamical system which measures the "amount of chaos".

For dynamical systems with Markov partitions the topological entropy should be the exponential growth rate of the number of words of length n.

$$f : X \circlearrowleft \qquad h(f) \stackrel{?}{=} \lim_{n\to\infty} \frac{\log \#\text{ of words of length } n}{n}$$

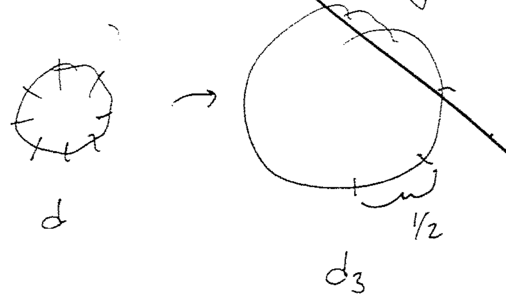
$$\text{so } \#\text{ of words of length } n \approx \exp(n \cdot h) \approx \exp(h)^n.$$

Example: If ~~we~~ our system is the shift map on 2 symbols $\sigma : \Sigma_2 \circlearrowleft$ then $\#$ of words of length $n = 2^n$ so we want $h(\sigma) = \lim_{n\to\infty} \frac{\log 2^n}{n} = \frac{n \log 2}{n} = \log 2$.

Similarly ~~if~~ the entropy of the doubling map on the circle should be $\log 2$.

On the other hand we would like the entropy to be defined whether or not our dynamical system has a Markov partition.

~~We will proceed by defining a family of metrics d_n on X which reflect the dynamical behavior of f. In the case of the doubling map of the circle the metric d_n has the effect of stretching the circle so that the pieces of the Markov partition into words of length n have size $1/2$.~~



~~The number of words of length n will then correspond to the size of the metric space measured appropriately.~~

Could think of the supremum of word growth over various partitions. We adopt a different approach.

~~Now consider~~
To get an idea of how to define these metrics d_n consider the case of the 1 sided shift map on Σ_2.

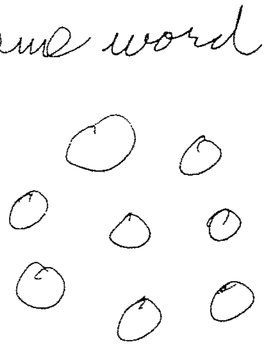
Consider words of length n 2 shift

$.w_0 \ldots w_{n-1}$

Say w and w' are sequences ~~where~~ which correspond to different words of length n.
Then $d(w,w') \geq \max \frac{\delta(w_i,w_i')}{2^i}$ depends on the position of the symbol which is different.
We would like d_n to treat each of these coordinates equally.

Define $d_n(w,w') = \max\limits_{0 \le l \le n-1} d(f^l(w), f^l(w'))$.

With respect to this metric d_n $d_n(w,w') = 1$ if w and w' correspond to different words and $d_n(w,w') \le \frac{1}{2}$ if the correspond to the same word. ~~Σ_2 is covered by 2~~ Σ_2 is covered by 2^n balls of radius $\frac{1}{2}$.



Let (X, d) be a compact metric space and $f : X \circlearrowleft$ a continuous map (possibly invertible). We use f to define a family of new metrics on X.

Let
$$d_n(p, q) = \max_{0 \le \ell < n} d(f^\ell(p), f^\ell(q)).$$

There are several ways of measuring a metric space. Here is one.

Let $B(p, n, \varepsilon) = \{q \in X : d_n(p, q) < \varepsilon\}$ be the ball of radius ε. A set E is an <u>(n, ε) spanning set</u> if $X \subset \bigcup_{p \in E} B(p, n, \varepsilon)$.

Let $S(n, \varepsilon)$ be the least cardinality of an (n, ε) spanning set.

Finite (n, ε) spanning sets exist since X is compact so $S(n, \varepsilon) < \infty$.

Define $h(f, \varepsilon) = \limsup_{n \to \infty} \frac{1}{n} \log S(n, \varepsilon)$.

The function $h(f,\varepsilon)$ is monotone in the sense that if ~~[illegible]~~ $0<\varepsilon'<\varepsilon$ then $h(f,\varepsilon')\geq h(f,\varepsilon)$.

This follows from $S(n,\varepsilon')\geq S(n,\varepsilon)$ as every (n,ε') spanning set is also an (n,ε) spanning set.

Define $h(f)=\lim_{\varepsilon\to 0} h(f,\varepsilon)$.

Monotonicity implies that the limit exists but it can be infinite. Since $S(n,\varepsilon)\geq 1$, $h(f,\varepsilon)\geq 0$ and $h(f)\geq 0$.

Since $S(n,\varepsilon)$ is defined as ~~an~~ the cardinality of a minimal cover it is easy to get an upperbound but can be hard to get a lower bound.
We give an alternate approach to entropy where it is easier to get lower bounds.

Let $\varepsilon > 0$. A set $E \subset X$ is (n,ε) separated if $d_n(x,y) \geq \varepsilon$ for all pairs of distinct elements $x, y \in E$.
Since X is compact any (n,ε) separated set must be finite.
Let $N(n,\varepsilon)$ be the maximum number of elements in an (n,ε)-separated set. suffices to consider $N(n,\varepsilon)$ to be the number of elements in a maximal (n,ε)-separated set.

Prop. Let (X,d) be a compact metric space and $f: X \circlearrowleft$ continuous. Then

$$h(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N(n,\varepsilon).$$

Proof. First we show $N(n,\varepsilon) \geq S(n,\varepsilon)$.
Let E be an (n,ε)-separated set of cardinality $N(n,\varepsilon)$ (maximum possible). Then we claim that $\bigcup_{p \in E} B(p,n,\varepsilon) = X$. If not there is a ~~pt~~ $q \in X$ not in any set $B(p,n,\varepsilon)$, so $d_n(p,q) > \varepsilon$ for all $p \in E$.

In this case we can add q to E to create a larger (n,ε) separated set contradicting our assumption.

Thus an (n,ε) separated set of max cardinality is an (n,ε) spanning set and, $\cancel{N(n,\varepsilon) \geq S(n,\varepsilon)}$ its cardinality is at least $S(n,\varepsilon)$, ~~Next we show that~~ the cardinality of the minimal spanning set.

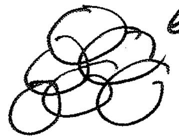
So $N(n,\varepsilon) \geq S(n,\varepsilon)$.

Remark $\boxed{\text{Suffices to use maximal } (n,\varepsilon) \text{ separated sets.}}$

Next we claim that $N(n,\varepsilon) \leq S(n,\varepsilon/2)$.

Let E' be an $(n,\frac{\varepsilon}{2})$ spanning set of cardinality $S(n,\frac{\varepsilon}{2})$.

Let E be an (n,ε) separated set.



$(n,\varepsilon/2)$ balls cover.

At most one point of E is in any ball $B(q,n,\varepsilon/2)$ for $q \in E'$ since if $p,p' \in E$ are in $B(q,n,\varepsilon/2)$ we have $d_n(p,p') \leq \varepsilon$ by the triangle inequality. So $\# E \leq S(n,\varepsilon/2)$.

Putting these together we have

$$S(n,\varepsilon) \leq N(n,\varepsilon) \leq S(n,\varepsilon/2)$$

Since log is monotone increasing

$$\frac{1}{n}\log S(n,\varepsilon) \leq \frac{1}{n}\log N(n,\varepsilon) \leq \frac{1}{n}\log S(n,\varepsilon/2)$$

Taking lim sup

$$h(f,\varepsilon) \leq \limsup \frac{1}{n}\log N(n,\varepsilon) \leq h(f,\varepsilon/2)$$

Letting $\varepsilon \to 0$

$$h(f) \leq \lim_{\varepsilon\to 0}\limsup_{n\to\infty} \frac{1}{n}\log N(n,\varepsilon) \leq h(f)$$

~~Scholium. Take any sequence of (n,ε) separated sets~~ ~~$\sigma(n,\varepsilon)$~~ $\sigma(n,\varepsilon)$ ~~and any seq. of card of (n,ε) spanning sets than~~ of cardinalities of $n(n,\varepsilon)$ Then $\lim_{\varepsilon\to 0} \lim_{n\to\infty} \frac{1}{n} \log$ [illegible]

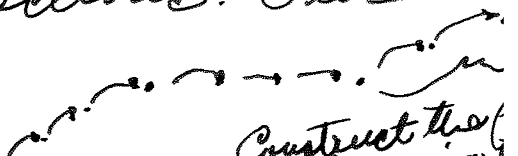
Proposition. ~~Let~~ Assume that every node of $\mathcal{G}$ [illegible] has at least one entering and one exiting edge [illegible] (Rule out $\bullet \to \bullet$ for example) ~~Then~~ Consider the shift $(\Sigma_{\mathcal{G}}, \sigma)$ [illegible]

$$h(\sigma) = \rho(M_{\mathcal{G}}).$$

Proof. The ball of radius ~~2^{-j}~~ λ^{-j} around ω is the set $\{\omega' : \omega'_k = \omega_k \text{ for } |k| \le j\}$. ~~Call these~~

For each ~~word~~ finite ~~word~~ path in $\mathcal{G}$ $\alpha = \alpha_{-j} \dots \alpha_{-1}\, \alpha_0 \dots \alpha_{k-1}$ we can define $C_\alpha = \{\omega : $ ~~$\omega_k = \alpha_k$ for $|k| \le j\}$~~ $\omega_\ell = \alpha_\ell$ for $-j \le \ell$ [illegible]

These are examples of cylinder sets. We want to know that cylinder sets C_α are non-empty. This means that the finite path α can be extended infinitely in both directions. Our hypothesis takes care of this.



Construct the [illegible] extension using hypothesis of outg[illegible]

~~We construct an (n~~

~~The balls~~ The λ^{-j} balls around ω for the d^n metric correspond to cylinder sets

$$C_\alpha = \{\omega : \omega_k = \alpha_k \text{ for } -j \le k \le j+n\}.$$

We can construct an (n, λ^{-j})-separated set by picking one ω from each set C_α. This is a maximal (n, λ^{-j}) separated set.

Add [illegible] 1 more [illegible] would [illegible] not to be [illegible] separated [illegible]

The number of such sets is the number of words of length $n+k+1$.

~~So~~ $d(\omega,\omega') \le \frac{1}{\lambda^{\ell}}$ means that

$$\omega_j = \omega'_j \text{ for } |j| \le \ell.$$

$$\underbrace{\omega_{-\ell} \cdot \omega_2\,\omega_1 . \omega_0 \cdots\cdots \omega_\ell}$$

ω and ω' agree on a "window" of length $2\ell+1$.

~~$d(f(\omega)$~~

$d(\sigma(\omega), \sigma(\omega')) \le \frac{1}{\lambda^{\ell}}$ means that $\omega_j = \omega'_j$ for $-\ell+1 \le j \le \ell+1$. They agree on a "shifted window".

$d_n(\omega,\omega') \le \frac{1}{\lambda^{\ell}}$ means that ~~ω and ω' agree on n the~~ $d(\sigma^m(\omega), \sigma^m(\omega')) \le \frac{1}{\lambda^{\ell}}$ for $m = 0 \ldots n$. So ω and ω' agree on the $n+1$ shifted windows:

$$\omega_{-\ell} \quad \cdots \qquad \omega_{\ell+n}.$$

These windows fit together to give one large window going from position $-\ell$ to $\ell+n$.

$$\limsup_{n \to \infty} \frac{\log\left(N(n, \lambda^{-k})\right)}{n} = \limsup_{n \to \infty} \frac{\log\left(\# W_{n+k}(\mathcal{A})\right)}{n}$$

$$= \limsup_{n \to \infty} \frac{n+k}{n} \cdot \frac{\log\left(\# W_{n+k}(\mathcal{A})\right)}{n+k}$$

$$= \log \rho .$$

$$h(\sigma) = \lim_{k \to \infty} \limsup_{n \to \infty} \frac{\log\left(N(n, \lambda^{-k})\right)}{n} = \log \rho .$$

Can understand the dynamics of f in terms of 2 maps from words of length $n+1$ to words of length n.

Process of refining our analysis of f.

Define Σ^v_G

~~Refinements of graphs.~~

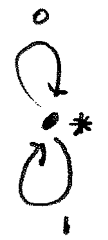
Observation:

$\boxed{\text{Can think of } \Sigma^e_G \text{ as an } \Sigma^v_{G'} \text{ for a new graph } G'.}$

How is G' defined?

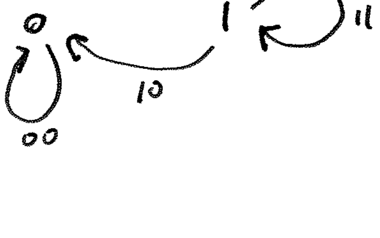
The vertices of G' are the edges of G and the edges of G' corresponds to paths of length 2 in G.

Example:



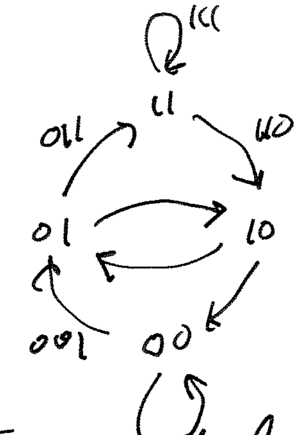
G.

G'



~~Pt~~ We can repeat this procedure: 8

$\mathcal{G}''$



000 ✓
001 ✓
010 ✓
011 ✓
100 ✓
101 ✓
110 ✓
111 ✓

Define $\mathcal{G}^{(i)}$ to be the graph whose vertices consist of paths of length n. Edges consist of paths of length

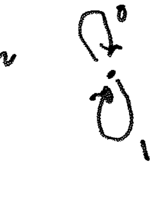
This is a combinatorial analogue of what we were doing ~~on~~ with the torus.

$$\boxed{\begin{aligned} h(w_0\dots w_n) &= w_1\dots w_n \\ t(w_0\dots w_n) &= (w_0\dots w_{n-1}) \end{aligned}}$$

Cor. We can have two different graphs $\mathcal{G}_0, \mathcal{G}_1$ for which $\Sigma_{\mathcal{G}_0}$ and $\Sigma_{\mathcal{G}_1}$ are topologically equivalent. In fact ~~to~~ all $\mathcal{G}^{(i)}$ produce top. conjugate systems. $(\Sigma_{\mathcal{G}^{(i)}}, \sigma)$.

~~What~~ How do we get topological conjugacy invariants ~~from the~~ of $(\Sigma_{\mathcal{G}}, \sigma)$ from $\mathcal{G}$?

~~Ans~~ ~~The~~ Subshifts of finite type ~~correspond to subgra~~ are given by saying that a certain collection of words of length n cannot occur. These correspond to ~~subgra~~ sets $\Sigma_{\mathcal{H}}$ where $\mathcal{H}$ is a subgraph of $\mathcal{G}^{(n)}$ and $\mathcal{G}$ is the graph



We have been looking at 3 types of maps that have chaotic behaviour: expanding maps of the circle, the horseshoe ~~un. linear hyper~~ and hyperbolic linear maps of the torus. We have constructed Markov partitions. Markov partitions allow us to answer many questions but for ~~some questions we d~~ answering some questions we don't need the full force of this machinery. We can get by ~~u~~ using a lighter touch. (Nevertheless we will see ~~w~~ Markov partition ideas reappearing.)

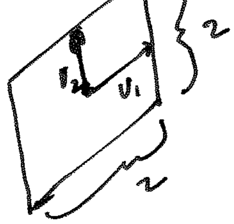
Stat

Thm. Let A be a hyperbolic integral matrix with $\det A = \pm 1$. Let $f_A: \mathbb{T}^2 \circlearrowleft$ be the corresponding linear hyperbolic diffeomorphism. Say the eigenvalues are λ and μ with $|\lambda| > 1 > |\mu|$. Then

$$h(f_A) = \log(|\lambda|).$$

Proof. We have seen that the calculation of entropy is independent of the metric used. We exploit this to choose a metric adapted to the problem. Start by defining $\tilde{d}$ in $\mathbb{R}^2$.

Say $v = a v_1 + b v_2$ where v_1 and v_2 are ~~a~~ unit length eigenvectors corresponding to λ and μ. Define $|v|_{max} = \max\{|a|, |b|\}$. The unit ball for this norm is a rhombus with side length 2.



~~$h(\tilde{p})$~~

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "A"] \arrow[d, "\pi"'] & \mathbb{R}^2 \arrow[d, "\pi"] \\ \mathbb{T}^2 \arrow[r, "f"] & \mathbb{T}^2 \end{tikzcd}$$

$\pi A(\tilde{p}) = f(\pi \tilde{p})$

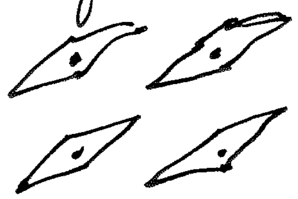
~~Define $\tilde{d}(p,$~~ For $\tilde{p}, \tilde{q} \in \mathbb{R}^2$ define $\tilde{d}(\tilde{p}, \tilde{q}) = |p - q|_{max}$.

Let $\pi: \mathbb{R}^2 \to \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Define a metric d_{max} on $\mathbb{T}^2$ by

$$d_{max}(p, q) = \min_{\substack{\tilde{p}, \tilde{q}: \\ \pi(\tilde{p}) = p, \\ \pi(\tilde{q}) = q}} \tilde{d}_{max}(\tilde{p}, \tilde{q})$$

The map π is a local (but not global) isometry. If $\tilde{d}(\tilde{p},\tilde{q}) < \frac{1}{4}$ then $d(\pi(\tilde{p}), \pi(\tilde{q})) = \tilde{d}(\tilde{p},\tilde{q})$.

This follows from the fact that if $\tilde{d}(\tilde{p},\tilde{q}) < \frac{1}{4}$ then no point $\tilde{q} + \mathbb{Z}^2$ is closer to $\tilde{p}$ than $\tilde{q}$ itself. Equivalent to the disjointness of integral translates of the $\frac{1}{4}$ ball (rhombus).



~~The~~ Claim:

~~Now~~ $\tilde{d}(A\tilde{p}, A\tilde{q}) \le \lambda \, \tilde{d}(\tilde{p},\tilde{q})$

Say $\tilde{p} - \tilde{q} = \begin{pmatrix} \Delta X \\ \Delta Y \end{pmatrix}$ then $|A(\tilde{p}) - A(\tilde{q})| = \left| \begin{pmatrix} \lambda \cdot \Delta X \\ \mu \cdot \Delta Y \end{pmatrix} \right|$ ~~$\le \lambda \max$~~

$\tilde{d}(\tilde{p},\tilde{q}) = \max\{\Delta X, \Delta Y\}$

$\tilde{d}(A(\tilde{p}), A(\tilde{q})) = \max\{\lambda \cdot \Delta X, \mu \cdot \Delta Y\}$
$\le \max\{\lambda \cdot \Delta X, \lambda \Delta Y\} \le \lambda \cdot \max\{\Delta X, \Delta Y\}$.

Want to get an explicit description of the ~~B(...)~~ $B(p,n,\varepsilon)$ ball in $\mathbb{T}^2$ when ε is small.

Assume $\varepsilon < \frac{1}{4\lambda}$. Say $q \in B(p,n,\varepsilon)$. ie $\max d(f^{l}(p), f^{l}(q))$ for $l = 0 \dots n$

Pick $\tilde{p}, \tilde{q}$ ~~mapping~~ in $\mathbb{R}^2$ mapping to p, q ~~...~~ with $\tilde{d}(\tilde{p},\tilde{q}) = d(p,q)$.

Now $\tilde{d}(A\tilde{p}, A\tilde{q}) \le \lambda \cdot \frac{1}{4\lambda} = \frac{1}{4}$ so ~~...~~ $d(\pi A\tilde{p}, \pi A\tilde{q}) = d(f(p), f(q))$. ~~By induction~~ ~~Now~~ By assumption since $q \in B(p,n,\varepsilon)$ $d_n = \max\{f^{l}(p), f^{l}(q)\} = d_n < \varepsilon$, $d(f(p), f(q)) < \frac{1}{4\lambda}$ so we can repeat the argument to get $d(f^{l}(p), f^{l}(q)) = \tilde{d}(A^{l}\tilde{p}, A^{l}\tilde{q})$ for $l = 0 \dots n$.

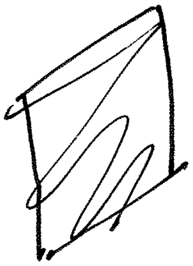
$A^\ell \tilde{q}$

$$\tilde{d}(A^\ell \tilde{p}, \tilde{A} \tilde{q}) = |A^\ell(\tilde{p} - \tilde{q})| = \max\{|\lambda|^\ell |\Delta x|, |\mu|^\ell |\Delta y|\}$$

$$d_n(p,q) = \max_{\ell = 0 \dots n} \max\{|\lambda|^\ell |\Delta x|, |\mu|^\ell |\Delta y|\} = \max\{|\lambda|^n |\Delta x|, |\Delta y|$$

$B(p, n, \varepsilon) =$

$= \{q : d_n(p,q) \le \varepsilon\}$



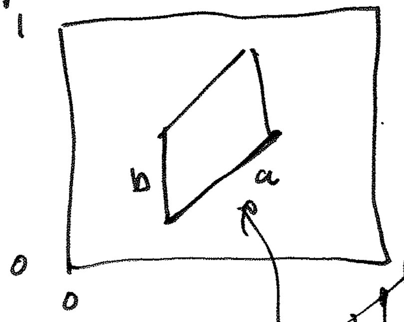
is a rhombus

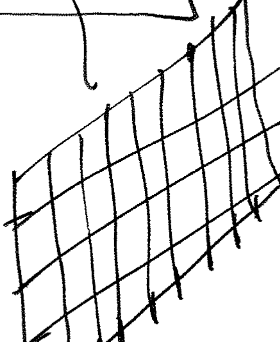
$|\lambda^n| |\Delta x| \le \varepsilon$

$|\Delta y| \le \varepsilon$

We can get upper and lower bounds on the entropy by constructing (n,ε) spanning sets and (n,ε) ~~separated~~ separated sets.

separated sets:





Construct a rhombus that maps isometrically to $\mathbb{T}^2$. Divide the v_1 side into $\left[\frac{|\lambda^n|}{a \cdot 2\varepsilon}\right]$ $\cancel{\left[\frac{2\varepsilon}{|\lambda^n|}\right]}$ ~~sides and the v_2 sides~~ intervals.

Divide the v_2 side into $\left[\frac{1}{b \cdot 2\varepsilon}\right]$ intervals.

Let $E_{n,\varepsilon}$ consist of points at the centers of these rhombi

$$\# E_{n,\varepsilon} = \cancel{\left[\frac{2\varepsilon}{|\lambda^n|}\right]} \cancel{[2\varepsilon]} \cdot \left[\frac{|\lambda^n|}{a \cdot 2\varepsilon}\right] \left[\frac{1}{b\, 2\varepsilon}\right].$$

Divide the sides into intervals of length $< \frac{2\varepsilon}{|\lambda^n|}$ and $< 2\varepsilon$;

# of such intervals is

$$\lim_{n\to\infty} \frac{1}{n} \log \# E_{n,\varepsilon} = \frac{1}{n} \log \cancel{\left[\frac{2\varepsilon}{|\lambda^n|}\right]\left[2\varepsilon\right]} \left[\frac{\lambda^n}{2\varepsilon}\right]\left[\frac{1}{2\varepsilon}\right]$$

$$= \cancel{\frac{1}{n} \log|\lambda|^n + \frac{1}{n} \log |\lambda|^n \left[\frac{2\varepsilon}{|\lambda^n|}\right] \cdot \left[2\varepsilon\right]}$$

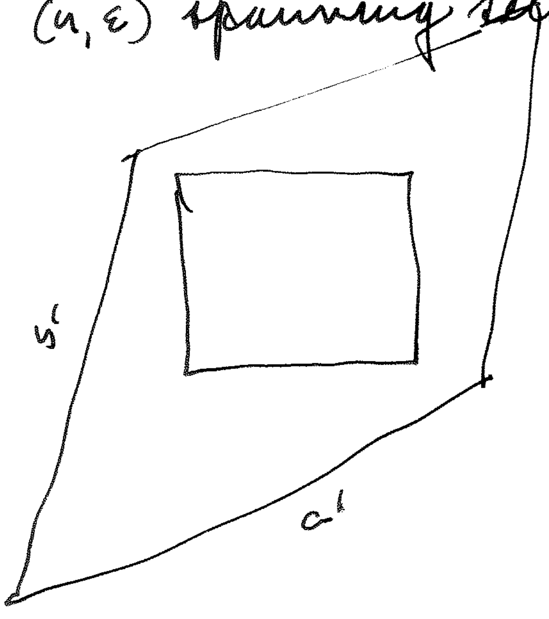
bounded above

$$= \frac{1}{n} \log|\lambda^n| + \frac{1}{n} \log \quad \cancel{\log 2\varepsilon}.$$

$$\underbrace{\frac{1}{|\lambda|^n}\left[\frac{|\lambda|^n}{2\varepsilon}\right]\left[\frac{1}{2\varepsilon}\right]}_{n} \le \left[\frac{1}{2\varepsilon}\right]^2$$

$$= \log|\lambda|. \quad \text{So } h(f) \ge \log|\lambda|.$$

To get an upper bound for entropy we construct (n, ε) spanning sets. Find ~~an~~ a rhombus that covers $[0,1]\times[0,1]$. ~~Divide the side into~~ Cover the intervals side of the rhombus by intervals of length greater than $\cancel{\frac{|\lambda|^n}{2\varepsilon}}$ ~~and~~ $\frac{2\varepsilon}{|\lambda|^n}$ and 2ε. # of such



intervals is $\left[a' \cdot \frac{|\lambda|^n}{2\varepsilon}\right]+1$ and $\left[\frac{b'}{2\varepsilon}\right]+1$.

~~Estimate~~ Evaluate the growth to get $h(f) \le \log|\lambda|$.

Remark. Similar arguments work for hyperbolic $n \times n$ matrices. Answer is no longer the log of the largest eigenvalue but the log of the absolute value of the products of eigenvalues with abs. value greater than 1.

Definition. Let (X,d) be a metric space and let $f: X \circlearrowleft$ be a map or homeomorphism. Let (p_i) be a sequence of points in X where $i \in \mathbb{N}$ (map) or $i \in \mathbb{Z}$ (homeo).
We say that (p_i) is an ε-pseudo-orbit if

$$d(p_{i+1}, f(p_i)) \leq \varepsilon.$$

~~If $\varepsilon = 0$~~
(When $\varepsilon = 0$ a pseudo-orbit is an orbit. In general we can think about it as an orbit computed by a computer with ~~some error~~) finite precision.)

Def. f has the shadowing property if for any $\varepsilon > 0$ there is a $\delta > 0$ so that for any δ-pseudo-orbit there is an orbit q_i with

$$d(p_i, q_i) \leq \varepsilon \text{ for all } i. \quad (1)$$

We further require that if ε is sufficiently small then any two orbits (q_i) and (q_i') with

$$d(q_i, q_i') \leq \varepsilon \text{ are equal.} \quad (2)$$

In particular (q_i) in (1) is unique.

~~Exam~~ Can you believe what you see when you simulate a dynamical system on a computer? Every time you iterate the computer returns an approxima[tion] to $f(p)$. The difference is on the order of 10^{-60} with double precision. After 60 iterations the location of the point depends more on the error than the map f? On the other hand if you are considering a map with the shadowing property then there is <u>some</u> initial condition which corresponds to the orbit you see (but probably not the one you started with.).

Theorem. Any (non-linear) expanding map of the circle ~~or torus~~ or a linear expanding map of the torus has the ~~shado~~ shadowing property. $\boxed{m^2+n^2 \text{ to } 1 = |\det A|.}$

Example: Let $A = \begin{pmatrix} m & n \\ -n & m \end{pmatrix}$ with $m^2+n^2 > 1$, $m, n \in \mathbb{Z}$.
A induces a map from $\mathbb{T}^2$ ↺ (which is not invertible).
~~This~~ Can describe the linear transformation A as sending $(x+iy) \mapsto (m+in)(x+iy)$.
This map multiplies distance by exactly $\sqrt{m^2+n^2} > 1$.

Remark: This map has a Markov partition but the pieces of the partition have fractal bound[aries]

Proof. In the circle case choose λ so that $|f'(p)| \geq \lambda > 1$ for all $p \in \mathbb{R}/\mathbb{Z}$. ~~Given Given~~ Let λ' be such that $|f'(p)| \leq \lambda'$ for all p. ~~Say we are given ε with~~ If $\varepsilon < \frac{1}{2\lambda'}$ and $d(p,q) < \varepsilon$ then

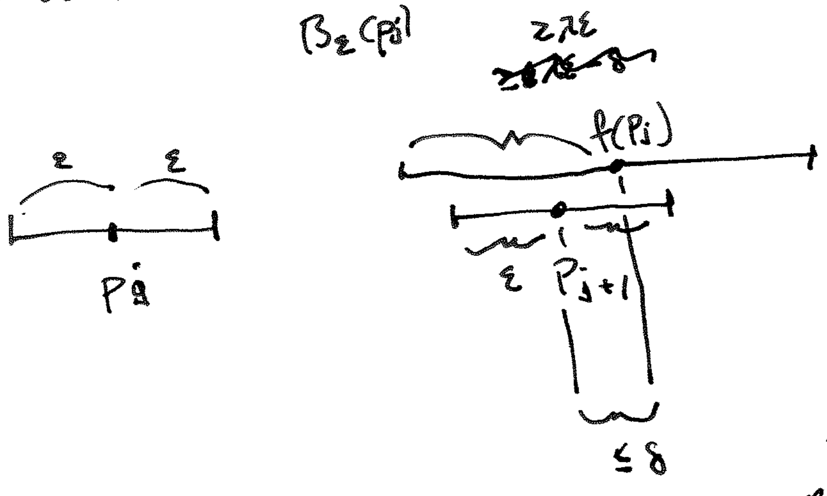
$d(f(p), f(q)) < \frac{1}{2}$ and $d(f(p), f(q)) \geq \lambda \cdot d(p,q)$.

In the torus case choose $\varepsilon < \frac{1}{2\lambda}$.

Now let $\varepsilon < \frac{1}{2\lambda'}$ be given. ~~Consider a sequence of balls (intervals) $B_\varepsilon(p_j)$.~~

Choose $\delta < (\lambda - 1)\varepsilon$.

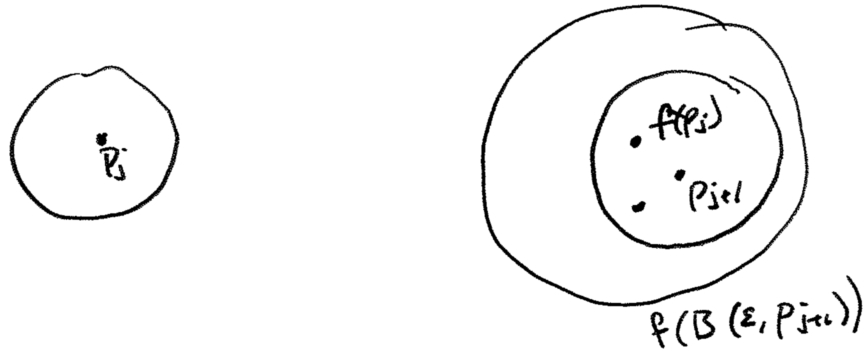
Now construct intervals (disks) around ~~p_i~~ points p_i



~~Say~~ Want to know that the ~~interval of length~~ ball of radius $\lambda\varepsilon$ around $f(p_j)$ contains the ball of radius ε around p_{j+1}

Want $\lambda\varepsilon - \delta \geq \varepsilon$

$$\lambda\varepsilon - \delta \geq \lambda\varepsilon - (\lambda - 1)\varepsilon$$
$$= \lambda\varepsilon - \lambda\varepsilon + \varepsilon.$$



Want

$|q - p_{j+1}| \le \varepsilon$

$\Rightarrow |q - f(p_j)| \le \lambda\varepsilon$

Triangle inequality:

$$d(q, f(p_j)) \le \underbrace{d(q, p_{j+1})}_{\le \cancel{\delta}} + \underbrace{d(p_{j+1}, f(p_j))}_{\cancel{\lambda\varepsilon}}$$

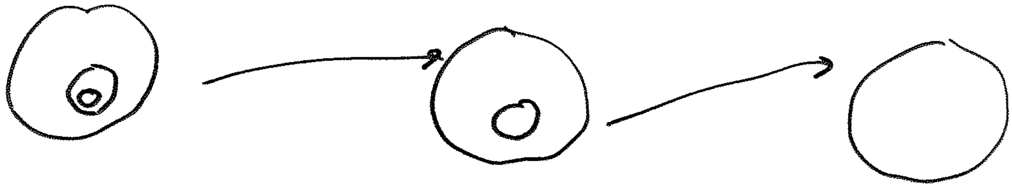
$$\cancel{\le (\lambda+1)\varepsilon + \lambda\varepsilon}$$

$$\le$$

$$\le \varepsilon + \delta$$

$$\le \varepsilon + (\lambda - 1)\varepsilon$$

$$= \lambda\varepsilon.$$



· Let $N_j = \{q : d(f^{\ell}(q), p_\ell) \le \varepsilon \text{ for } \ell = 0 \ldots j\}$

$f^{\ell}(N_\ell) = B(\varepsilon, p_\ell)$.

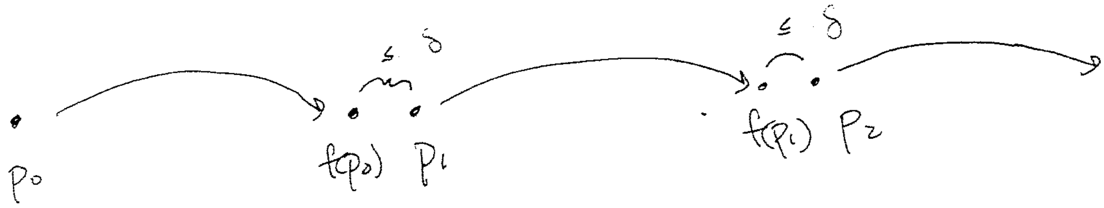
radius $N_j \le \frac{1}{\lambda^j}$

Let $q = \bigcap_{j=0}^{\infty} N_j$.

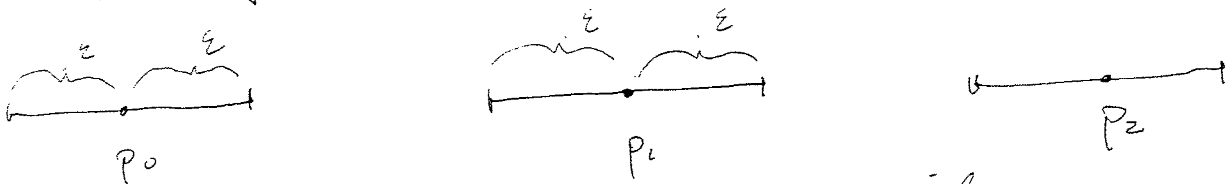
Proof. Assume $|f'(p)| \ge \lambda > 1$ for all $p \in \mathbb{R}/\mathbb{Z}$.

~~Let~~ Given ε choose $\delta < (\lambda-1)\cdot\varepsilon$

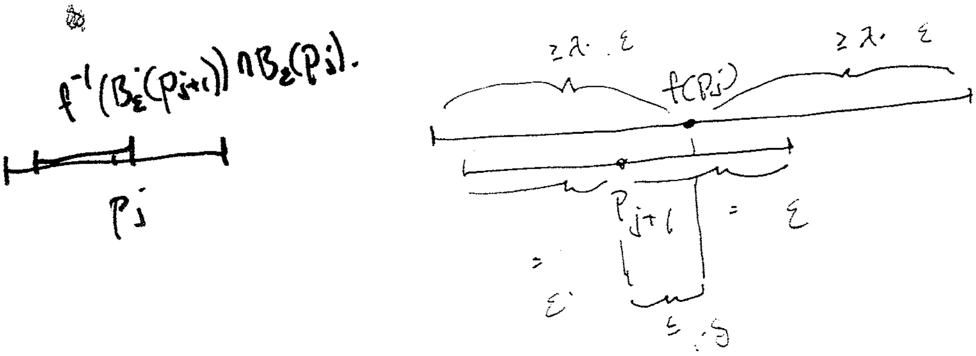
Consider an ε pseudo orbit



Now draw ~~circular~~ balls $B_\varepsilon(p_j)$. These are intervals of length 2ε centered at p_j.



We want to find an actual orbit (q_j) with $q_j \in B_\varepsilon(p_j)$. Claim that $f(B_\varepsilon(p_j))$ contains $B_\varepsilon(p_{j+1})$.



Need ~~ε~~ $\lambda\cdot\varepsilon \ge \varepsilon + \delta$ or $(\lambda-1)\varepsilon \ge \delta$.

Now, as in the proof of the Sarkovskii thm., there is a sequence of intervals $I_0 = B_\varepsilon(p_j) \supset I_1 \supset I_2 \dots$ where $f^\ell(I_j) \subset B_\varepsilon(p_\ell)$ ~~and~~ for $\ell < j$ and $f^\ell(I_\ell) = B_\varepsilon(p_\ell)$.

In addition we have $|I_j| \le \frac{\varepsilon}{\lambda^j}$ since $f^j(I_j) = B(p_j, \delta)$. $|f'| \ge \lambda$

Let $q_0 = \bigcap_{j=0}^{\infty} I_j$. Then by construction $f^j(q_0) \in I_j$ for all j and q_0 is the unique point with this property.

Remarks.

Quantitative uniqueness:

If $d(f^\ell(q_i), f^\ell(q_i')) \le \delta$ for $0 \le \ell \le n$ then $d(q_i, q_i') \le \frac{8\varepsilon}{\lambda^n}$.

In this case q_i, q_i' are both in I_n.

Let $I_n = \cancel{\{q : d(f^n(}$

$= \{q : d(f^e(q), p_e) \le \varepsilon \text{ for } 0 \le e \le n\}$

I_n form a nested sequence of non-empty intervals and

$\mathrm{diam}(I_0) = 2\varepsilon$

~~diam~~

$f^n(I_n) \supseteq B_\varepsilon(p_n)$ (in particular I_n is non-empty).

f multiplies distance by at least λ.

~~$x, y \in I_0$ then~~

$f | B_\varepsilon(p_e)$ multiplies distance by at least λ

$x, y \in I_n \Rightarrow d(f^n(x), f^n(y)) \ge \lambda^n d(x, y)$.

~~$\Rightarrow$ diam~~

$\Rightarrow \lambda^n d(x,y) \le 2\varepsilon$

$\Rightarrow d(x,y) \le \frac{2\varepsilon}{\lambda^n}$.

$\mathrm{diam}(I_n) \le \frac{2\varepsilon}{\lambda^n}$.

~~We also~~

We also see that any two orbits $(f^e(p))$ and $(f^e(q))$ which remain ε-close for all forward time are equal. Apply same analysis replacing p_i by $f^i(p)$ above.

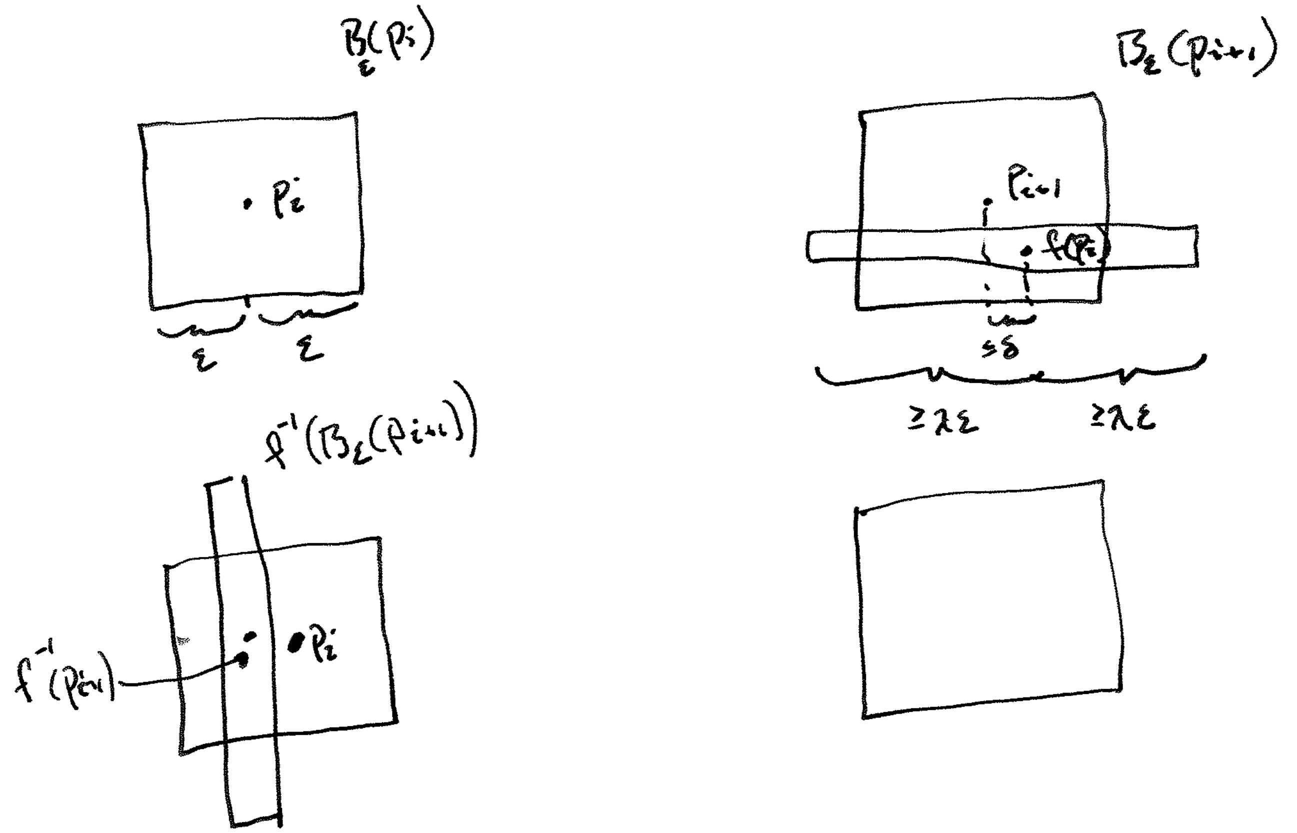
Thm. Let f_A be a linear hyperbolic diffeomorphism of the torus then f_A has the shadowing property

Proof. If f_A has the shadowing property with respect to one metric metric then it has the shadowing property with respect to any (with different ε, δ). equivalent metric. Let us then show that f has the shadowing property with respect to the distance that comes from the max norm

$|av_1 + bv_2|_{max} = \max\{|a|, |b|\}$ where v_1 and v_2 are eigenvectors of unit length, λ_1, λ_2 are the eigenvalues $|\lambda_1| > 1 > |\lambda_2|$ and $\lambda = |\lambda_1|$.

Given $\varepsilon > 0$ choose $\delta < \underbrace{(\lambda - 1)}_{\text{positive}}\varepsilon$.

Let p_i be an δ-pseudo-orbit. Let $B(p_i, \varepsilon)$ be the unit ball around p_i with respect to the max norm. We claim that $f(B(p_i, \varepsilon))$ crossed $B(p_{i+1}, \varepsilon)$ in a "Markov fashion".



~~Choose ε~~

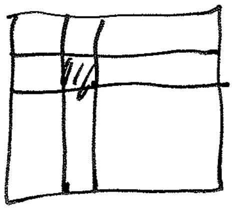
Use the fact that f^{-1} is uniformly continuous.

Choose ε small enough that $d(f^{-1}(p_{i+1}), p_i) \le \varepsilon$.

Let I_n^+ be the set of points q so that $f^\ell(q) \in B(p_\ell, \varepsilon)$ for $0 \le \ell \le n$ then arguing by induction I_n^+ is a full height rectangle in $B(p_0, \varepsilon)$ and the width of I_n^+ is $\frac{\varepsilon}{\lambda^n}$.

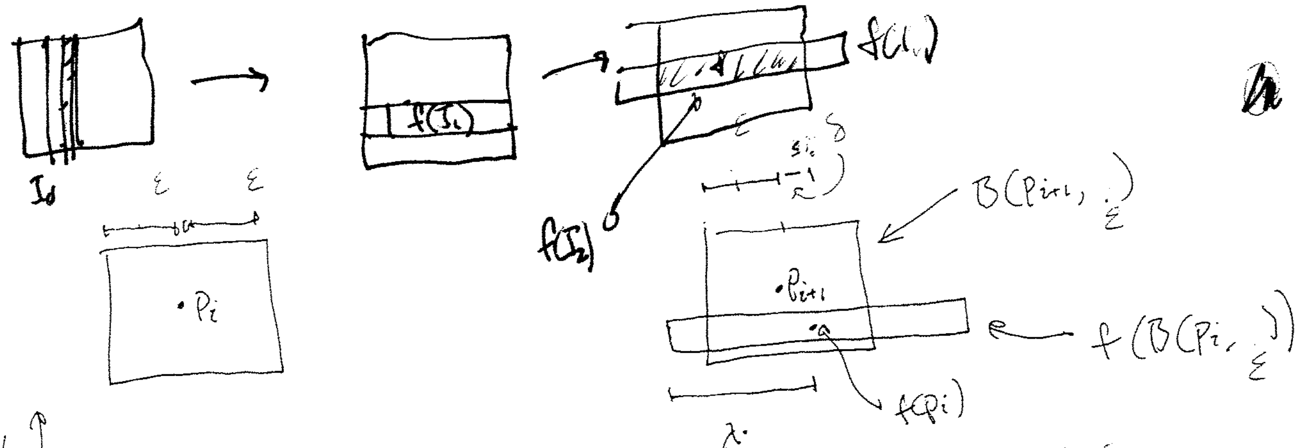
Let I_n^- be the set of points q in $B(p_0, \varepsilon)$ so that $f^\ell(q) \in B(p_\ell, \varepsilon)$ for $-n \le \ell \le 0$. Then I_n^- is a full width rectangle and the height is $\frac{\varepsilon}{\lambda^n}$.

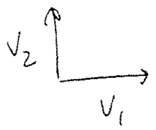
The sets $I_n^+ \cap I_n^-$ are intersections of full height and full width rectangles. ~~so are~~ They are non-empty rectangles of height and width



$\frac{2\varepsilon}{\lambda^n}$. Furthermore they are nested.

Let $q = \bigcap_{n=1}^{\infty} I_n^+ \cap I_n^-$. Then ~~the~~ by construction the orbit of q ε-shad the pseudo-orbit p_i.

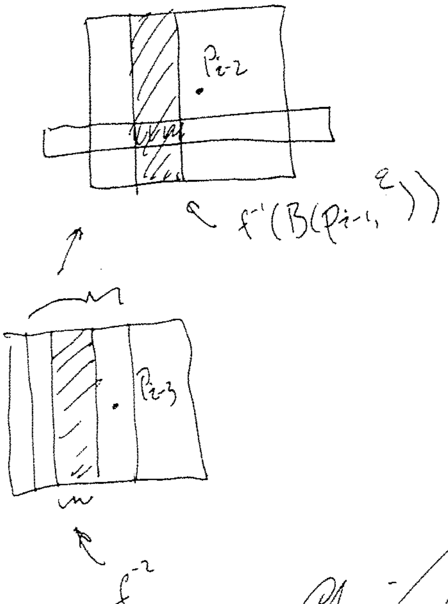




Need to check $\lambda \cdot \epsilon > \epsilon + \delta$

or $(\lambda - 1)\epsilon > \delta$.

Now the set of points in $B(p_i, \epsilon)$ that map to $B(p_i, \epsilon)$ is a vertical (full height) rectangle inside $B(p_i, \epsilon)$



Let I_j for $j \geq 0$ be the set of points in ^{v} $B(p_0, \delta)$ so $f^\ell(v) \in B(p_\ell, \delta)$ for $0 \leq \ell \leq j$.

Let $I_{j,k}$ be the set of points ^{v} for which $f^\ell(v) \in B(p_\ell, \delta)$ for $j \leq \ell \leq k$.

Claim that $I_{j,k}$ is a full width rectangle in $B(p_k, \delta)$ and a full height rectangle in $B(p_j, \delta)$. Proof by induction following the Markov partition proof.

~~Furthermore the width of~~

Let I_j for $j \geq 0$ be the set of points $r \in B(p_0, \varepsilon)$ so that $f^l(r) \in B(p_l, \varepsilon)$ for $0 \leq l \leq j$.

Claim that I_j is a full height rectangle in $B(p_0, \varepsilon)$ and $f^j(I_j)$ is a full width rectangle in $B(p_j, \varepsilon)$. Furthermore width of I_j is $\frac{\varepsilon}{\lambda^j}$.

Proof by induction on j.

For $j \leq 0$ let I_j'' be the set of points $r \in B(p_0, \varepsilon)$ so that $f^l(r) \in B(p_l, \varepsilon)$ for $j \leq l \leq 0$. The height of I_j'' is $\frac{\varepsilon}{\lambda^{|j|}}$.

Conclude that there is a unique point z in $\bigcap_{j=-\infty}^{\infty} I_j$ and this is the unique point that ε-shadows the pseudo-orbit p_j.

Quantitative uniqueness:

If we have two orbits p_i, p_i' and

$d(p_i, p_i') \leq \varepsilon$ for $-N \leq i \leq N$ then $d(p_0, p_0') \leq \ldots \frac{\varepsilon}{\lambda^N}$.

Structural stability.

How does the topological conjugacy type of a dynamical system vary as we vary the parameters?

Can it be locally constant?

Definition. We say that a dynamical system $f: M \circlearrowleft$ defined on a manifold (circle, torus, $\mathbb{R}^2$) is structurally stable if

C^1 topology.
We say f, g are C^1

~~Two examples~~

Examples: $R_\theta : \mathbb{R}/\mathbb{Z} \circlearrowleft$.

If θ is close to θ' $\theta' \neq \pm\theta$ then R_θ is <u>not</u> topologically conjugate to $R_{\theta'}$ since $\cancel{\rho(R_\theta) = \theta \text{ is a topological}}$ $\rho(R_\theta) = \theta$ is a topological invariant (up to sign.)

Example: $f_{\alpha,\beta}(x) = \alpha + \beta \sin(2\pi x) \bmod \mathbb{Z}$.

~~[illegible]~~ Property might hold on an Arnold tongue where $\rho(f_{\alpha\beta})$ is rational.
Seems to fail ~~on~~ outside of interiors of Arnold tongues

Expanding map: $f(x) = mx + \alpha \sin(2\pi x) \bmod 1$.

α small.

For an expanding map the topological conjugacy type is determined by the degree since nearly maps are topologically conjugate.

~~We say~~

What about hyperbolic maps of the torus?

~~We say that $f, g: T^2 \to$~~

Let f, g be C^1 maps with C^1 inverses. We say that f and g are C^1 close if $d(f(p), g(p)) \le \varepsilon$, $d(f^{-1}(p), g^{-1}(p)) \le \varepsilon$ and Df_p and Dg_p are close.

Def. f is structurally stable if any C^1 close diffeo. is topologically conjugate to f.

Thm. If f is a hyperbolic linear diffeo. of the torus then f is structurally stable.

Do not have the tools that we need to prove this but shadowing plays an important role.

If g is C^1 close to f then g has shadowing.

How do we construct a top. conjugacy from f to g? If $p \in T^2$ then set $p_i = f^i(p)$. p_i is an orbit for f and a pseudo-orbit for g. Find a q ~~that~~ so that the g orbit of q is δ close to the set p_i.

Set $h(p) = q$.

Where do we go from here?

~~Ideally~~

~~If we understand $f:X^{\circlearrowleft}$ can we understand maps (diffeos)~~
~~close to f?~~

① Extend the notion of hyperbolicity to diffeos. f where Df can vary ~~f~~ from point to point.

We did this for expanding maps of the circle. Want to be able to speak of expanding directions and contracting directions at each point but these can vary with the point.

② As in the case of the horseshoe we can focus on some set Λ and make sense of "hyperbolicity on Λ".

④ Show that hyperbolicity ~~is an of~~ persists when we perturb our map provided that the derivatives of our maps do not change too much. (C^1 closeness)

⑤ Show hyperbolic maps are structurally stable using shadowing.

If $f:M^{\circlearrowleft}$ is hyperbolic and $g:M^{\circlearrowleft}$ is C^1 close then g is hyperbolic. ~~Both maps have~~

Hyperbolic maps have the shadowing property.

Construct the conjugacy $h:M\to M$. Let $p\in M$.

$$\begin{array}{ccc} M & \xrightarrow{f} & M \\ \downarrow & & \downarrow \\ M & \xrightarrow{g} & M \end{array}$$

Let $p_j = f^j(p)$. p_j is an orbit for f and an ε-pseudo-orbit for g. So there is a $q_j = g^j(q)$ so that (q_j) is ε close to (p_j). ~~Send~~ Let $h(p)=q$.