Two basic questions about circle homeomorphisms with irrational rotation numbers

Q. ① Are all orbits dense?

② ~~Can they have orbits which are not dense~~?

② Are they topologically ~~equivalent to~~ conjugate to rotations?

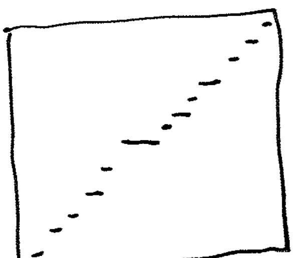
~~A~~ According to Poincaré's theorem on conjugacies, these two questions are equivalent.

Our next project is to construct "Denjoy's example" which shows that the answer to ① and ② is "not necessarily". [To some extent the structure of this counter-example is determined by what we already know.

Poincaré's theorem on semi-conjugacies tells us that such a counter example is semi-conjugate to an irrational rotation. Thus there is an h from $\mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ which is increasing but not strictly increasing and which conjugates f to R_θ.

We assume $h(0)=0$. Can think of h: ~~$\mathbb{R} \to$~~ $[0,1] \to [0,1]$.

graph of h.



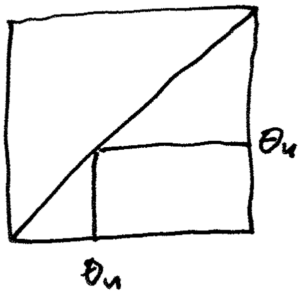
$$\begin{tikzcd} \mathbb{R}/\mathbb{Z} \arrow[r, "f"] \arrow[d, "h"] & \mathbb{R}/\mathbb{Z} \arrow[d, "h"] \\ \mathbb{R}/\mathbb{Z} \arrow[r, "R_\theta"] & \mathbb{R}/\mathbb{Z} \end{tikzcd}$$

What can we say about the image of the horizontal plateaus? If $h(x)=h(x')$ then $h(f(x))=h(f(x'))$ so if $h(I)=y$ then $h(f(I)) = R_\theta(y)$. and $h(f^n(I)) = R_\theta^n(y)$. Let $I_n = f^n(I)$ then the image of $\bigcup_n I_n = \mathcal{O}(y)$ is dense in $\mathbb{R}/\mathbb{Z}$.

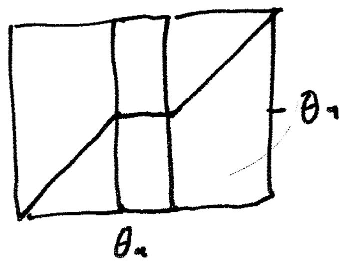
It follows that h is constant on a dense subset of the circle. Such a function is sometimes called a "Devil's staircase".

~~It~~ In order to construct f we start by constructing h. Let's assume that the image of the intervals of constancy I_n is the orbit of 0 so $h(I_n) = R_\theta^n(0) = n\theta \bmod 1$. Let us call this point θ_n and think of $\theta_n \in [0,1)$. Let l_n be the length of I_n. Assume $\sum l_n = L < \infty$.

Here is a geometric construction of ~~h~~ the graph of h. Start with the diagonal in the square. For each θ_n cut the square vertically and insert a rectangle of height 1 and width l_n. Make the graph ~~h~~ constant on this rectangle taking the value θ_n.



The challenging part is that we need to do this operation countably many times.



In the end we wind up with a function from $[0, 1+L]$ to $[0,1]$.

For future use we want an analytic formula for h. In fact it is easier to get a formula for h^{-1}.

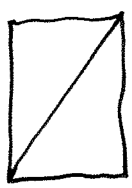
For ~~x~~ $y \notin O(0)$ we get $h^{-1}(y) = y + \sum_{\theta_j \in [0,y)} l_j$.

If $y \in O(0)$ then h^{-1} has a jump type discontinuity at y where the left and right hand limits are $y + \sum_{\theta_j \in [0,y)} l_j$ and $y + \sum_{\theta_j \in [0,y]} l_j$.

It is not hard to show that h^{-1} is continuous at the remaining points.

Knowing h determines f on the complement of $\cup I_n$. If $x \notin \cup I_n$ then $R_\theta h(x) \notin O(0)$ so $h^{-1} R_\theta h(x)$ is a unique point which is the only possible value for $f(x)$.

It remains to define f on $\cup I_n$. Arguing as above we see that $f(I_n)$ must be equal to I_{n+1}. If we take f to be linear on each I_n then we get a continuous homeomorphism semi-conjugate to R_θ.


I_{n+1}

I_n

We have countably many rectangles on which f is not defined.

If $x \notin \bigcup I_n$ then $f(x)$ is uniquely determined by $f(x) = h^{-1}(R_\theta(h(x)))$.
If $x \in I_n$ then $f(x) \in h^{-1}(R_\theta(h(x))) = I_{n+1}$.

We can fill in the graph in these rectangles with straight lines to produce a homeomorphism f.

Question. How ~~can~~ differentiable can we make f?

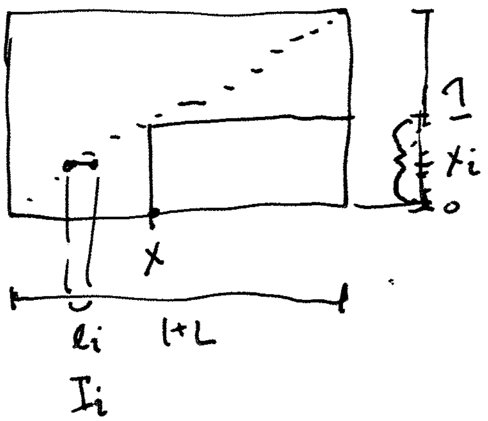
Recall that if f arises from the time one map of an ODE then the differentiability of f is the same as the differentiability of the coefficients of the ODE. Something we can get our hands on.
Does this differentiability influence the dynamics that can occur

Theorem. (Denjoy) There is a C^1 diffeomorphism of the circle which ~~is~~ has irrational rotation number but which is not minimal.

Proof. The things we can control are the lengths of gaps l_n and graph of f when it maps I_n to I_{n+1}.

Assume $\frac{l_n}{l_{n+1}} \to 1$ as $n \to \pm\infty$. For example take $l_n = \frac{1}{n^2+25}$ $\quad \frac{1}{2} < \frac{l_{n+1}}{l_n} < 2$.

Say ~~$l_n = [a_n, b_n]$~~ or $I_n = [a_n, b_n]$ where $b_n - a_n = l_n$.



Recall that there is a semi-conjugacy from f to the rotation R_θ.

The ~~disolex~~ interval of length 1+L is obtained from the interval of length 1 by replacing the ~~point~~ $x_i = R_\theta^i(0)$ by an interval of length l_i.

~~x maps to~~ If $x \notin O(0)$ then x maps to $h(x)$ where

$$x = h(x) + \sum_{x_i \in [0, h(x)]} l_i .$$

This is really a formula for h^{-1} at least when $x \notin O(x)$.

Write $y = h(x)$. ~~Then~~ and $h^{-1}(y) = x$

$$h^{-1}(y) = y + \sum_{x_i \in [0, y]} l_i .$$

h^{-1} is a function with jump type discontinuities at the points x_i.

skip

Want a formula for H the lift of h to $\mathbb{R}$.
Write $x_{i,j}$ for $T^j R_\theta^i(0)$. We called the set of $x_{i,j}$'s Λ_2 in the context of the Poincaré thm.
Define $H(x)$ implicitly by

$$x = H(x) + \sum_{x_{i,j} \in [0, H(x)]} l_i .$$

Claim: ① $H(x) = h(x)$ for $x \in [0,1)$

② $H(x+L+1) = H(x)+1$

③ H is monotone.

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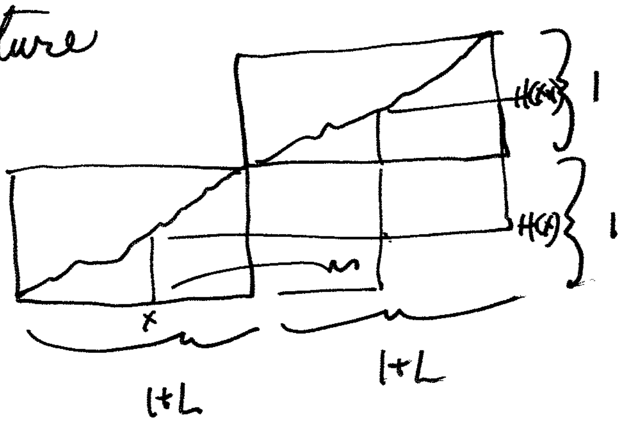
① ~~follows from~~

Picture

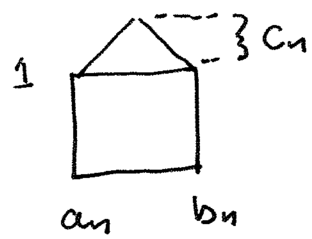


② follows from $\sum_{X_{i_N} \in [H(x), H(x+1))} l_i = L$.

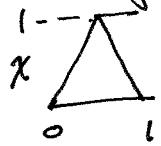
① follows from every $X_i \bmod 1$ has a unique representative as $X_{i,j}$.

We want $f|I_n$ to have a continuous first derivative and we will want f to have derivative 1 at either end of I_n. Say $I_n = [a_n, b_n]$.

Take f' on I_n to have the following graph.



Let $\chi(x) = 1 - |1-2x|$.



Then $f'(x) = 1 + c_n \chi\left(\frac{x-a_n}{b_n-a_n}\right)$.

Since f maps $[a_n, b_n]$ to $[a_{n+1}, b_{n+1}]$

we need $$\int_{a_n}^{b_n} f'(x)\,dx = \underbrace{(b_n - a_n)\left(1 + \frac{c_n}{2}\right)}_{\text{from the picture}} = b_{n+1} - a_{n+1}$$

This gives $l_n\left(1+\frac{c_n}{2}\right) = l_{n+1}$ or $c_n = 2\left(\frac{l_{n+1}}{l_n} - 1\right)$.

Differentiability argument starts in lecture 9 page 5.

Why is f differentiable with a continuous derivative?

$$\begin{tikzcd}
x \quad y \arrow[r, "f"] \arrow[d, "h"] & f(x) \quad f(y) \arrow[d, leftrightarrow, "H"] \\
H(x) \quad H(y) \arrow[r, "R_\alpha"] & H(x)+\alpha \quad H(y)+\alpha
\end{tikzcd}$$

If $x \in \cup I_n$ this is true by construction.

Assume $x, y \notin \cup I_n$.

Want to show that $f'(x) = 1$ or

$$\lim_{y \to x} \frac{f(y) - f(x)}{y - x} = 1 \quad \text{or} \quad \lim_{y \to x} \frac{f(y) - f(x) - (y-x)}{y-x} = 0.$$

Assume $x < y$

Now $x = H(x) + \sum_{i : x_i \in [0, h(x)]} l_i \qquad y = H(y) + \sum_{i : x_i \in [0, h(y)]} l_i$

$$y - x = H(y) - H(x) + \sum_{i : x_i \in (h(x), h(y)]} l_i$$

$$f(x) = H(f(x)) + \sum_{i:\, x_i \in [0,\, H(f(x))]} l_i$$

But $H(f(x)) = R_\alpha(H(x)) = H(x)+\alpha$

$$f(x) = H(x) + \alpha + \sum_{i:\, x_i \in [0,\, H(x)+\alpha]} l_i$$

Åi

$$f(y) - f(x) = (H(y)+\alpha) - (H(x)+\alpha) + \sum_{i:\, x_i \in [H(x)+\alpha,\, H(y)+\alpha]} l_i$$

$$= H(y) - H(x) + \sum_{i:\, x_i \in [H(x),\, H(y)]} l_{i+1}$$

$$f(y) - f(x) = (H(y) + \alpha) - (H(x) + \alpha) + \sum_{i:\, x_i \in [h(x)+\alpha,\, h(y)+\alpha]} l_i$$

$$= H(y) - H(x) + \sum_{i:\, x_i \in [h(x),\, h(y)]} l_{i+1}$$

So

$$\frac{f(y) - f(x) - (y-x)}{y-x} = \frac{\sum l_{i+1} - \sum l_i}{\underbrace{H(y) - H(x)}_{\text{positive}} + \sum l_i}$$

$x_i \in (h(x), h($

$$\le \frac{M \cdot \sum l_i - \sum l_i}{\sum l_i}$$

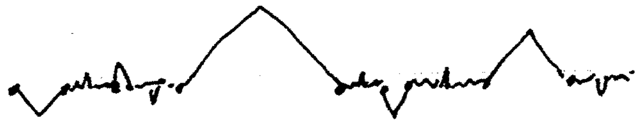
where $M = \max\limits_{i:\, x_i \in [H(x), H(y)]} \frac{l_{i+1}}{l_i}$ $\qquad \le M - 1.$

Now as $y \to x$, $H(y) \to H(x)$ and $M \to 1$.

since $\frac{l_{i+1}}{l_i} > \varepsilon$ for only finitely many indices

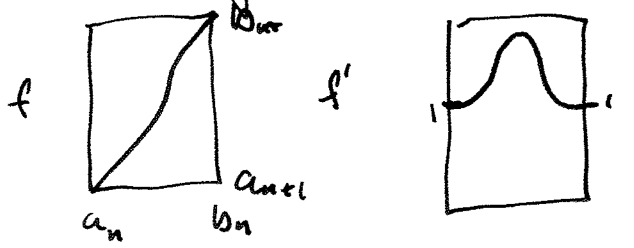
To get the lower bound argue with $\min \frac{l_{i+1}}{l_i}$.

Picture of f'



For any $\varepsilon > 0$ only finitely many peaks are above $1+\varepsilon$ or below $1-\varepsilon$.
Thus f' is continuous.
At any $x \notin \bigcup I_n$ we have heights of wedges converging to 0 but the sum of the heights of all wedges is infinite.

Denjoy's example has a pretty wild derivative. Is it possible that this example could be made C^2 rather than just C^1? We could replace our maps from interval to interval by something smoother, but we will still have the structure of "peaks and valleys." Not so clear how to deal with this

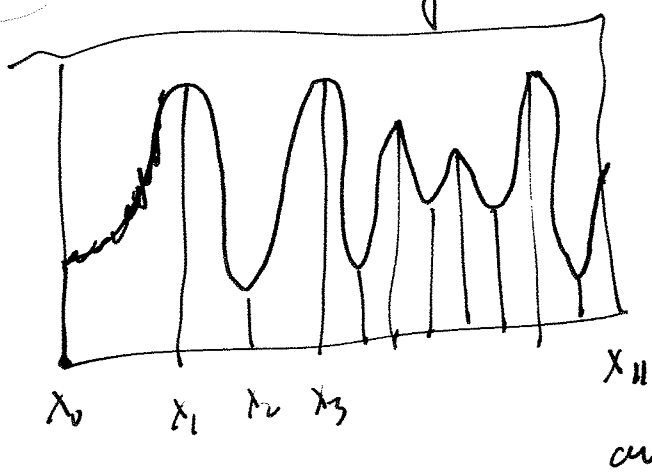


Denjoy shows that a C^2 diffeo. of the circle with irrational rotation number must be minimal. We will give his proof. We start with a definition.

Definition. We define the variation of a continuous function $g:[0,1]\to\mathbb{R}$ to be

$$\mathrm{var}(g) = \sup\left\{ \sum_{i=0}^{n-1} |g(x_{i+1})-g(x_i)| : 0 = x_0 < x_1 < \dots x_n \right.$$

where n can be arbitrary



We are summing the sizes of the jumps. When n is large we are looking at the behavior of smaller jumps. Roughness on a smaller scale.

Theorem (Denjoy) Let f be a circle homeomorphism with irrational rotation number ρ. If f is C^1 and $\log|f'|$ has bounded variation then f is topologically conjugate to R_ρ.

Remarks. You will show that if f is C^2 then $\log|f'|$ has bounded variation. Thus this result says there are no C^2 Denjoy counterexamples.

If f is not minimal then this reflected in the growth of the distortion of f^n as measured by the ~~[illegible]~~ size of ~~[illegible]~~ $(f^n)'$.

Lemma. Let f be a C^1 diffeo of the circle with irrational rotation number. If f is not minimal then ~~there is~~ there is an x_0 so that
depending on n.

$$|(f^n)'(x_0)| \cdot |(f^{-n})'(x_0)| \to 0 \text{ as } n \to \infty.$$

Proof. If f is not minimal there is an interval $I_0 \subset \mathbb{R}/\mathbb{Z}$ mapped to a point by h. ~~Assume that~~ Let $I_j = f^j(I_0)$. These intervals are disjoint (since they map to different pts on an orbit under R_ρ)

Since $\sum |I_n| \leq 1$ we have $|I_n| \to 0$ as $n \to \pm\infty$.

$$\frac{|I_n| + |I_{-n}|}{2} = \frac{1}{2}\left(\int_{I_0} (f^n)'(z)\,dz + \int_{I_0} (f^{-n})'(z)\,dz \right)$$

$$= \int_{I_0} \frac{(f^n)'(z) + (f^{-n})'(z)}{2}\,dz$$

$$\geq \int_{I_0} \sqrt{|(f^n)'(z)\,(f^{-n})'(z)|}\,dz$$

$$\geq \sqrt{m} \cdot |I_0|$$

where $m = \min_{I_0} (f^n)'(z)(f^{-n})'(z)$. ~~Since~~ Pick x realizing the minimum then $m \to 0$ as $n \to \infty$.

Let's try to understand this result.

By the ~~inverse~~ rule for differentiating an inverse function

$$(f^{-n})'(x_0) = \frac{1}{(f^n)'(f^{-n}(x_0))} = \frac{1}{(f^n)'(x_{-n})}$$

So we have

$$\frac{|(f^n)'(x_0)|}{|(f^n)'(x_{-n})|} \to 0 \quad \text{as } n \to \infty.$$

~~Taking logs turns the chain rule from a~~

Can analyze $(f^n)'(x_0)$ in terms of the chain rule.

$$(f^n)'(x_0) = f'(x_0) \cdot f'(x_1) \dots f'(x_{n-1}) = \prod_{i=0}^{n-1} f'(x_i).$$

Taking logs makes this a sum along an orbit:

$$\log (f^n)'(x_0) = \sum_{i=0}^{n-1} \log f'(x_i).$$

We have $\log \frac{|(f^n)'(x_0)|}{|(f^n)'(x_{-n})|} \to -\infty$ or

$$\sum_{i=0}^{n-1} \log f'(x_i) - \sum_{i=0}^{n-1} \log f'(x_{i-n}) \to -\infty.$$

$$= \sum_{i=0}^{n-1} \log f'(x_i) - \log f'(x_{i-n}) \to 0$$

~~Usefully~~ ~~At~~ approaching this naively not surprising that ~~we~~ have an orbit ~~along~~ segment along which f ~~grows~~ is larger and one along which f ~~decreas~~ is smaller. The trickiness in the lemma allows us to locate these two points as x_0 and x_n.

x_0 . . . . . x_n

x_n . . . . . . x_0

Lemma. Let f be a circle homeomorphism with an irrational rotation number. Then there is a sequence $n_k \to \infty$ such that for any $x \in \mathbb{R}/\mathbb{Z}$ the intervals (x_i, x_{i+n_k}) (where $x_i = f^i(x)$) for $0 \le i \le n_k$ are disjoint.

Proof. Let R_ρ be the rotation with same rotation number as f. Let $y_k = R^k_\rho(x)$. for some x.

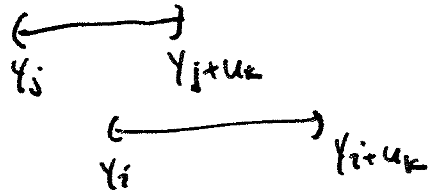
Let $n_0 = 1$ and define n_k recursively by

$$n_k = \min \{ i \in \mathbb{N} : \mathrm{dist}(y_0, y_i) < \mathrm{dist}(y_0, y_{n_{k-1}}) \}$$

$R^{n_k}(x)$ gives the sequence of closest approaches to 0. ~~A~~ Since $R^{n_k}(x)$ is close to x, $|n_k \rho - m_k|$ is small so ~~ρ~~ $\rho - \frac{m_k}{n_k}$ is very small. Gives "best rational" approximations to ρ.

Say intervals (y_i, y_{i+n_k}) and (y_j, y_{j+n_k}) overlap with $i, j \le n_k$

~~fuzy~~



Say $i > j$ then

$$\underset{\mathrm{dist}(y_0,\, y_{i-j})}{\underset{\parallel}{\mathrm{dist}(y_j, y_i)}} < \underset{\mathrm{dist}(y_0,\, y_{n_k})}{\underset{\parallel}{\mathrm{dist}(y_i, y_{i+n_k})}}$$

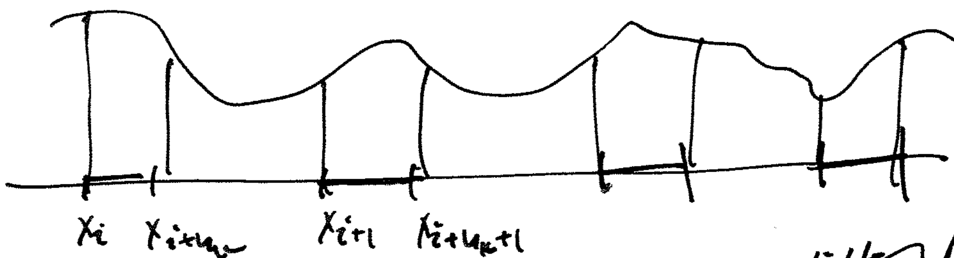
This violates the assumption that n_k is a closest return. (Holds if $|i-j| \le n_k$).

~~(21)~~ (6)

Now let f be any homeomorphism with rotation number ρ. We have a semiconjugacy ~~H~~ h from f to R_ρ. The monotonicity of H implies that $f^i(x) \in [f^j(x), f^k(x)]$ if and only if

$$R_\rho^i(h(x)) \in [R_\rho^j(h(x)), R_\rho^k(h(x))].$$

We will use this to compare averages of function at nearby points. ~~Any~~ g is continuous then $\left|\sum_{i=0}^{n_k} g(x_i) - \sum_{i=0}^{n_k} g(x_{i+n_k})\right| \le Var(g)$ since

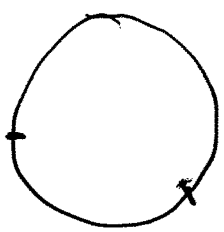


~~shift indices~~ take $y = x - n$

Apply this to $g = \log|f'|$ and we get

$$\left|\sum_{i=0}^{n_k} \log\{f'(x_i)\} - \log f'(x_{i-n})\right| \le var(\log|f'|)$$

This ~~contradicts~~ shows that f must be minimal



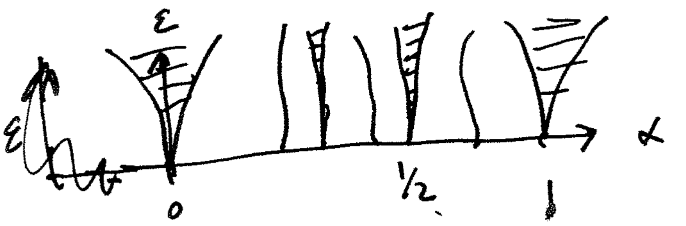
What happens in a generic family of homeomorphisms of the circle?

Weakly coupled oscillators.
α is ratio of periods.
ε is strength of coupling.

Say we have a family like $f(x) = x + \alpha + \varepsilon \sin(2\pi x)$.

α corresponds to a rotation parameter and ε represents the "coupling" or "non-linearity". ~~This~~ We can plot the rotation number in the (ε, α) plane.

Arnold tongues.



Baire category:

large in a top. sense:
- generic - contains a dense ...
- comeagre - contains a countable intersection of dense open sets.

If we increase α and fix ε ρ is monotone increasing but not strictly monotone. At each irrational value f is conjugate to a rotation and ρ is strictly increasing as a function of α. So we have lines of irrational rotation #

Out of the ~~...~~ rational points however we get "tongues". Each of these tongues gives a plateau. The result is a "Devil's staircase". In fact ρ is constant on a dense set. ρ is non-~~...~~ constant on a set of positive measure.

Phase locking is generic but not of full measure. Thus non-linear case is substantially different from the linear case.

Points in the interior of an Arnold tongue are top. conjugate to one another.

Def. f is structurally stable if ~~a~~ C^1 ~~when~~ all sufficiently C^1 close maps are topologically conjugate.

If $\rho(f)$ is irrational then f is not structurally stable.

~~A~~ Structural stability is dense.

Rings of Saturn show a similar pattern of gaps.