MA424 Example Sheet 1

2 October 2014

1. Consider the differential equations for the harmonic oscillator:

$$\dot{x} = v$$
$$\dot{v} = -kx$$

For a given $t \in \mathbb{R}$ let $f^t : \mathbb{R}^2 \to \mathbb{R}^2$ be the "time advance map" defined by the equation $f^t(x_0, v_0) = (x(t), v(t))$ where $s \mapsto (x(s), v(s))$ is a solution to the above equations satisfying the initial condition $(x(0), y(0)) = (x_0, y_0)$. Write the matrix that represents f^t and show that it satisfies the defining properties of a dynamical system:

(a) $f^s \circ f^t = f^{s+t}$

(b) $f^0 = Id$

2. A natural metric on the circle $\mathbb{R}/\mathbb{Z}$ is given by

$$d(x, y) = \min\{|b - a| : a \in x + \mathbb{Z}, b \in y + \mathbb{Z}\}.$$

Show that R_α is an isometry with respect to this metric i.e. show that $d(R_\alpha(x), R_\alpha(y)) = d(x, y)$.

3. It is not easy to find an n so that the initial digit of 2^n is 7 (without using a calculator or computer).

(a) Show that the initial digit of 2^n depends on the location of $R_\theta^n(0)$ in the circle where $\theta = \log_{10} 2$.

(b) Prove that $\log_{10} 2$ is irrational. Note that it is very close to the rational number $3/10$.

(c) We have shown that if θ is irrational then orbits of R_θ are dense in the circle $\mathbb{R}/\mathbb{Z}$ but, for a given $\epsilon > 0$, we may have to choose a very large n in order that the set of points $\{R_\theta^j(0) : j = 0 \ldots n\}$ be ϵ dense in the circle.

(d) Plot the orbit $R_\alpha^n(0)$ where $\alpha = 3/10$. Using the geometry of this orbit and the relation between R_α and R_θ of estimate the value of n for which 9 first appears as an initial digit of 2^n. Estimate the value of n for which 7 first appears.

4. Are the following maps lifts of circle homeomorphisms?

   (a) $F(x) = x + \frac{1}{2}\sin(x)$
   (b) $F(x) = x + \frac{1}{4\pi}\sin(2\pi x)$
   (c) $F(x) = x + \frac{1}{\pi}\sin(2\pi x)$
   (d) $F(x) = -x + \frac{1}{4\pi}\sin(2\pi x)$

5. Show that if F is a homeomorphism of $\mathbb{R}$ such that $F(x+1) = F(x)+1$ then F is a lift of a circle homeomorphism.

6. Show that if F is a lift of f then F^n is a lift of f^n for any integer n.