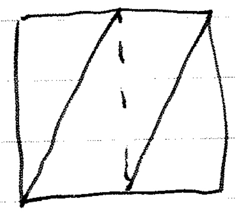
Our next topic is expanding maps of the circle. We will see different dynamical behavior from circle homeos. in this case. In particular we will see "chaotic" behavior for the first time.

Definition. A continuously differentiable map $f: \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is called expanding if $|f'(x)| > 1$ for all $x \in \mathbb{R}/\mathbb{Z}$.

Examples: (1) ~~f(x)~~ $f_m: \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ $\quad f_m(x) = mx \bmod 1$ for $m \neq -1, 0, 1$.



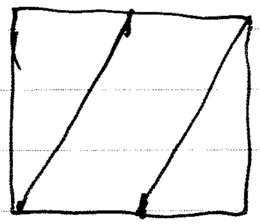
(2) $g_m: \{z \in \mathbb{C} : |z| = 1\}$ $\quad g(z) = z^m$.

Note that f_m and g_m are topologically conjugate via the conjugacy $h: \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ $\quad h(v) = e^{2\pi i v}$.

Recall that any continuous map f from $\mathbb{R}/\mathbb{Z} \to$ has a continuous lift $F: \mathbb{R} \to$ where $f(x) = F(x) \bmod 1$.

Example. The lift of f_m is $F(x) = mx$.

Picture:



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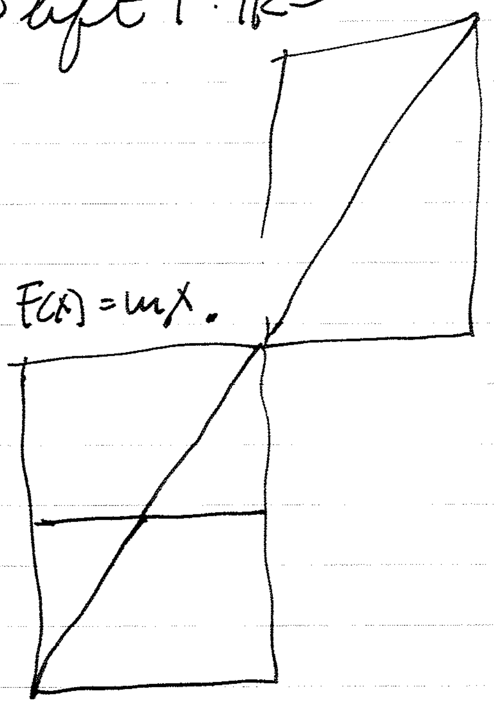


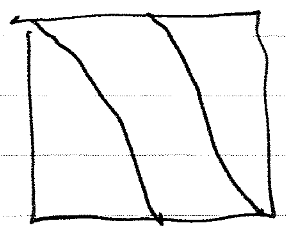
$F|[0,1]$

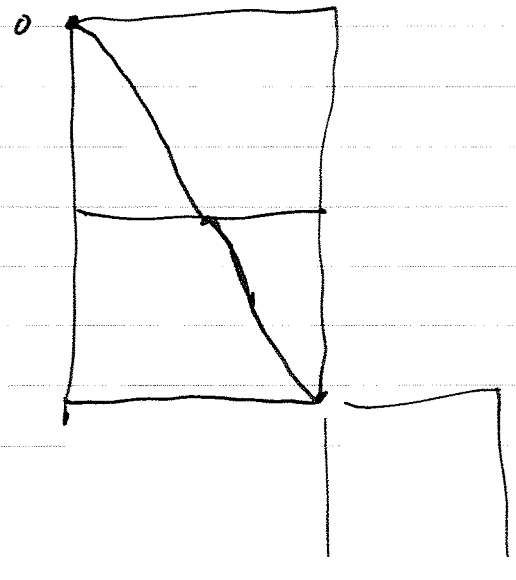
Any two lifts differ by a constant which is an integer.

The <u>degree of f</u> is $F(1) - F(0)$ and F satisfies (This depends only on f.)

$F(x+1) = F(x) + \deg f$.

$d = -2$





Recall that $T(x) = x+1$. The equation

$F(x+1) - F(x) = d$ gives $F(x+1) = F(x) + d$ or

$F \circ T = T^d \circ F$.

If f is a homeomorphism then $d = \pm 1$ depending on whether f preserves or reverses orientation.

~~In general~~

Lemma. If f is expanding then $|d| > 1$.

Proof. Let F be a lift of f then the mean value theorem gives

$$\frac{F(1) - F(0)}{1 - 0} = F'(x_0) \text{ for some } x_0 \in (0,1)$$

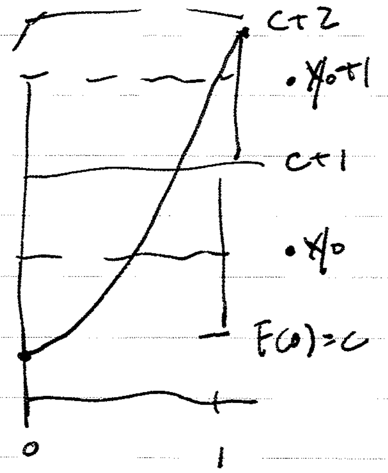
So $F'(x_0) = d$ and the expanding condition gives $|d| = |F'(x_0)| > 1$.

Remark. Any $d \neq 0, 1, -1$ is possible for an expanding map.

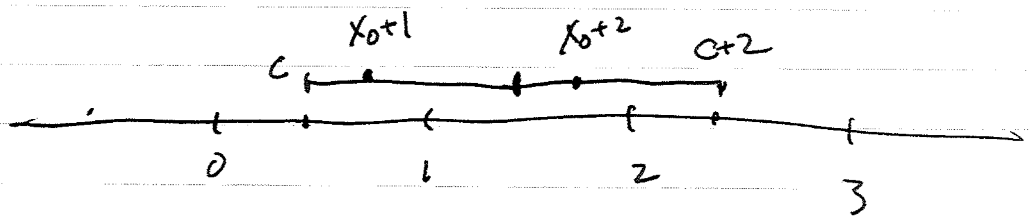
Lemma. Say f is expanding, $\deg(f)=d$ and $x_0 \in \mathbb{R}/\mathbb{Z}$ then $\#\{f^{-1}(x_0)\} = |d|$.

Proof. Let F be a lift of f. $f(x)=x_0$ if and only if $F(x) = x_0 + \mathbb{Z} = x_0 + n$ for $n \in \mathbb{Z}$.

Now



$F([0,1]) \supset [F(0), F(1)] = [F(0), F(0)+d] = [c, c+d]$ by the intermediate value thm. and $F|[0,1]$ is injective since if $F(a)=F(b)$ we get $F'(x)=0$ by the intermediate value thm. So $F([0,1]) = [F(0), F(0)+d] = [c, c+d]$



$[c; c+d]$ can be divided into d intervals of length 1. Each of these intervals contains exactly one point of the form x_0+n. Since F is an bijection for each x_0+n in $[c, c+d]$ there is an x_n ~~with~~ $\in [0,1]$ with $F(x_n) = y_n$. These x_n are distinct points in $\mathbb{R}/\mathbb{Z}$.

Cor. An expanding map of $\mathbb{R}/\mathbb{Z}$ is not invertible.

Lemma. If f is expanding and $\deg(f) = d$ Then f has $|d-1|$ fixed points.

Proof. Let F be a lift of f. A fixed point of f is a solution of $f(x) = x \bmod 1$ or

$F(x) - x \in \mathbb{Z}$. Let $G(x) = F(x) - x$.

$G([0,1]) \supset [G(0), G(1)] \cancel{= [F(0)-0, F(1)-1]}$

and has length $|G(1) - G(0)| = |(F(1)-1) - (F(0)-0)|$

$= |F(1) - F(0) - 1| = |d-1|$.

$G|[0,1]$ is injective since $G(a) = G(b)$

$\Rightarrow G'(x_0) = 0$ for some x_0 so $F'(x_0) - 1 = 0$ or

$F'(x_0) = 1$. This violates expansion.

Now we argue as before that the number of solutions is equal to the length of the interval which is $|d-1|$.

Cor. Expanding maps are not minimal.

Proof. They have fixed points which are points whose orbits are not dense.

Prop. If f and g are maps of the circle then $\deg(f \circ g) = \deg(f) \cdot \deg(g)$.

Proof. Let F and G be lifts of f and g. Let d_f and d_g be the respective degrees of f and g then

$$F \circ T = T^{d_f} \circ F \quad , \quad G \circ T = T^{d_g} \circ G$$

so

$$F \circ G \circ T = F \circ T^{d_g} \circ G = F \circ \underbrace{T \circ \ldots T}_{d_g} \circ G$$

$$= T^{d_f} \circ F \circ \underbrace{T \circ \ldots T}_{d_g - 1} \circ G$$

$$= T^{d_f \cdot d_g} \circ F \circ G.$$

Cor. f^n has $|(\deg f)^n - 1|$ fixed points.

In particular f has infinitely many periodic points of infinitely many periods unlike circle homeomorphisms.

Definition. A dynamical system $f^t : X \circlearrowleft$ is topologically transitive if it has a dense orbit.

Thm. The doubling map f_2 is topologically transitive.

Proof. Consider an ∞ sequence $a_1, a_2, a_3 \ldots$ of 0's and 1's. This sequence corresponds to an $x \in [0,1]$ where

$$x = \frac{a_1}{2} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \ldots$$