Cor. f^n has $|(\deg f)^n - 1|$ fixed points.

In particular f has infinitely many periodic points of infinitely many periods unlike circle homeomorphisms.

— ~~Cor. Also~~ Minimality of a system is an "irreducibility" property. Here is a weaker irreducibility property:

Definition. A dynamical system $f^t : X^{\circlearrowleft}$ is topologically transitive if it has a dense orbit.

Thm. The doubling map f_2 is topologically transitive.

Proof. Consider an ∞ sequence $a_1, a_2, a_3 \ldots$ of 0's and 1's. This sequence corresponds to an $x \in [0,1]$ where

$$x = \frac{a_1}{2} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \ldots$$

The correspondence between sequences and points is not injective but it is surjective since every x in $[0,1]$ has a binary expansion.

Now if x corresponds to $(a_1, a_2, a_3 \ldots)$ then $f_2(x)$ corresponds to $(a_2, a_3, a_4 \ldots)$.

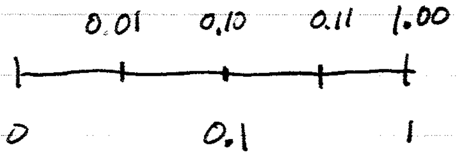
$$2x = a_1 + \frac{a_2}{2} + \frac{a_3}{2^2} + \ldots \quad \text{mod } 1$$

$$= \frac{a_2}{2} + \frac{a_3}{2^2} + \ldots$$

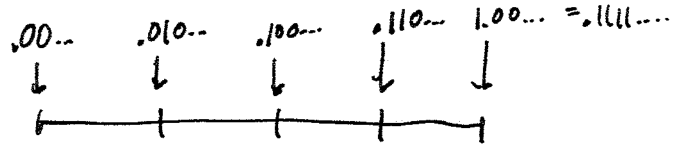
$.0111111\ldots$ and $.10000\ldots$ map to the same pt.

Thus $f_2^n(x)$ corresponds to the sequence obtained by shifting left n places.

What does it mean for $\{f_2^n(x)\}$ to be dense in terms of sequences?



Now let all finite words in 0,1



0
1
00
01
10
11
000
001
...

$$x = \frac{a_1}{2} + \underbrace{\frac{a_2}{4} + \frac{a_3}{8} + \dots}_{\text{tail}}$$

$x =$ Let $a_1, a_2 \dots$ be formed by concatenating them

0100011011000001.... $f^n(x)$ is dense.

Thm. For the doubling map periodic points are dense.

Proof. A point x is periodic if the binary expansion

$a_1, a_2, a_3, \dots$ is periodic.

To find a periodic point in the interval $.a_1 a_2 \dots a_n$ take $a_1, a_2, a_3 \dots a_n a_1 a_2 a_3 \dots a_n \dots$

One possible definition of chaotic behavior:

A dynamical system is chaotic if it is topologically transitive and periodic points are dense.

Example: t_2 is chaotic in this sense.

Other sense: Contains "a coin toss".

Remark. Homeomorphisms of S^1 are never chaotic.

Proof. Let f be an or. preserving circle homeomorphism. We

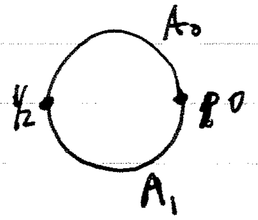
(9)

If f has any periodic points then all periodic points have the same ~~number~~ period, say n. So a periodic point ~~x~~ satisfies $f^n(x)=x$. Since f^n is continuous the set of points satisfying this equation is closed. If this set is ~~also~~ also dense then it is all of $\mathbb{R}/\mathbb{Z}$. Thus ~~...~~ every point is periodic and ~~no point has a dense orbit~~ has a finite orbit. In particular no point has a dense orbit.

If f is or. reversing then ~~apple~~ we can apply this argument to f^2.

Conclude for a circle homeomorphism

What is going on geometrically?



We have a decomposition of the circle into 2 pieces.

$f(\partial A_0) = f(\partial A_1) \subset \partial A_0 \cup \partial A_1$. (Markov property)

Given ~~p~~ x we can construct an itinerary $\omega_0, \omega_1, \omega \dots$ where $\omega_j = 0$ or 1

~~$f^j(p)$ $\omega_j(p)$~~

$$f^j(x) \in A_{\omega_j}.$$

This itinerary is the binary expansion. We run into problems if $f^j(x) = 0$ or $\frac{1}{2}$ in which case it is not clear how to define $\omega_j(x)$.

This idea of a "Markov partition" and the corresponding coding is useful in many settings.

~~The map σ:~~

Let $\Sigma_2 = \{ (\omega_k)_{k=0}^{\infty} : \omega_k \in \{0,1\} \}$

Let d be the distance defined by

$$d(\omega, \omega') = 2^{-\min\{k : \omega_k \neq \omega'_k\}} \quad \text{if } \omega \neq \omega'$$

and $d(\omega,\omega) = 0$.

Let $\sigma : \Sigma_2 \circlearrowleft$ be defined by

$$\sigma(\omega_0, \omega_1, \omega_2 \dots) = (\omega_1, \omega_2 \dots).$$

In the course of this proof we have shown that the map $h(\omega) = \frac{\omega_0}{2} + \frac{\omega_1}{2^2} + \dots$ from Σ to $\mathbb{R}/\mathbb{Z}$ is a semi-conjugacy between $\cancel{f_2}$ σ and f_2.