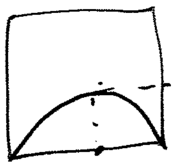
Interval maps.

Unimodal maps

Quadratic maps

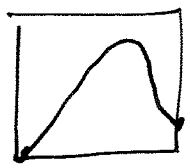
$f_\lambda : x \mapsto \lambda x(1-x)$

$f_\lambda : [0,1] \to [0,1]$

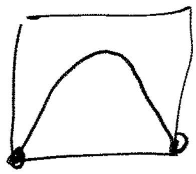


$f(\tfrac{1}{2}) = \tfrac{\lambda}{4}$.

The role of the ordering.



Symmetric unimodal:



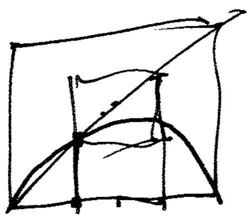
~~Role of~~

What happens as λ goes from 0 to 4?

Note $f_\lambda = \lambda x - \lambda x^2$

$f'_\lambda(0) = \lambda$. If $\lambda < 1$ then there is a unique periodic point which is fixed.

If $\lambda > 1$ there is a second fixed point.



~~If~~ $\lambda x - \lambda x^2 = x$

$\lambda - \lambda x = 1 \qquad \lambda = 1 + \lambda x$

$\cancel{\lambda(1-x) = 1} \qquad \lambda - 1 = \lambda x$

$x = \frac{\lambda - 1}{\lambda} = 1 - \frac{1}{\lambda}$

$f'(x) = \lambda - 2\lambda x$

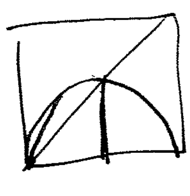
$$f'\left(\frac{\lambda-1}{\lambda}\right) = \lambda - 2\lambda\left(\frac{\lambda-1}{\lambda}\right)$$

$$= \lambda - 2(\lambda - 1)$$

$$= \lambda - 2\lambda + 2$$

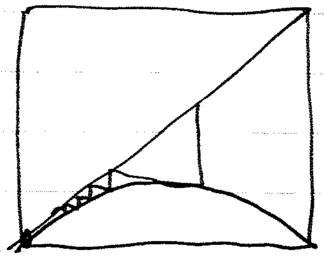
$$= 2 - \lambda$$

$\lambda < 2$

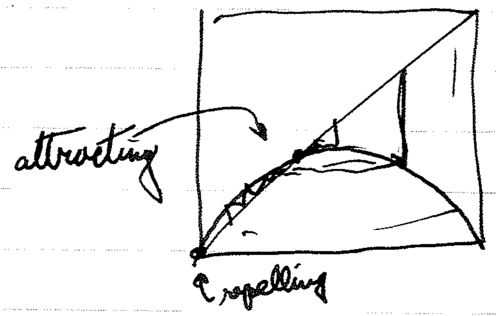


$\lambda = 2$

If $\lambda < 1$ ~~there~~ then $f'(0) = \lambda < 1$. f has a unique fixed point which is attracting

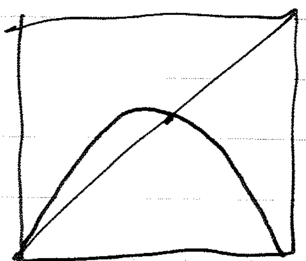


When $\lambda > 1$ f develops a second fixed point. First pt. becomes repelling.

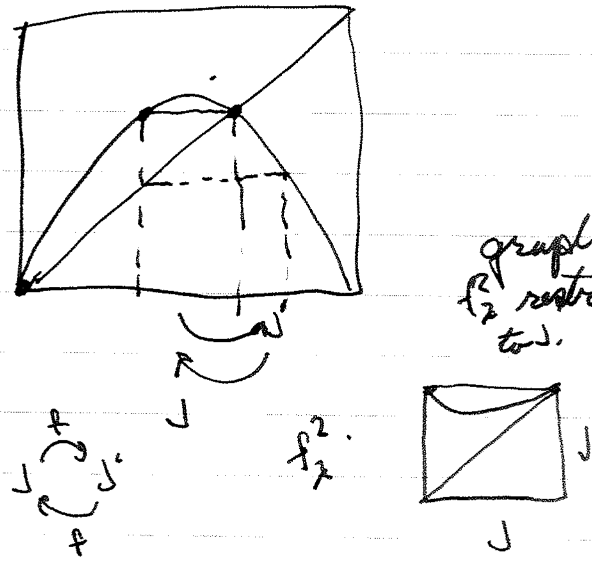


When $\lambda = 2$ $f'(\frac{1}{2}) = 0$

$\frac{1}{2}$ is super-attracting



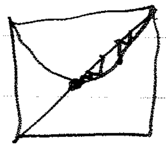
$\lambda > 2$



graph of f_λ^2 restricted to J.

$\lambda = 3$ $f'(p) = -1$ at fixed pt.

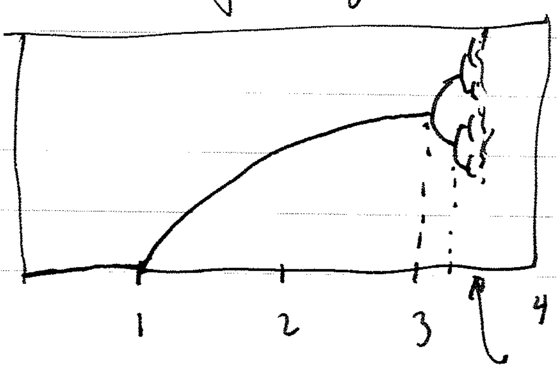
$\lambda = 3$.



$\lambda > 3$ f develops a point of period 2.

As we increase λ f ~~does~~ goes through a sequence of "period doubling" bifurcations.

$\lambda > 3$ creates ~~a saddle~~ an attracting pt. of period 2.



$:[0,1]\circlearrowleft$

f is unimodal and symmetric if $f(x)=f(1-x)$,
f is monotone increasing on $[0,1/2]$ and monotone
decreasing on $[1/2,1]$.
$f(0)=f(1)=0$.

Periods of periodic points. x has period n if

$f^n(x)=x$ but $f^m(x)\neq x$ for $0<m<n$.

(sometimes called "least period" or "prime period".)

What effect does the unimodal condition have on the collections of possible periods of periodic points?

Proposition. Let f be a symmetric unimodal map with periodic points of periods $^{\text{exactly}}$ $p_1, p_2 \ldots$ Then there is a symmetric unimodal map with periodic points of periods $^{\text{exactly}}$ $1,\ 2p_1,\ 2p_2 \ldots$

Remark. Every map of the interval has points of period 1. ($f(0)=0$)

The list $^{\text{of periods}}$ $p_1, p_2 \ldots$ can be finite or infinite.

Proof. We will locate the periodic points of g.

Where do we see fixed points for g?

$p = 0$ and $q = 0.6$

How do points move?

$$\begin{tikzcd}
I_1 \arrow[loop above] \arrow[r, bend left] & I_2 \arrow[r, bend left] & I_3 \arrow[r, bend left] & I_4 \arrow[l, bend left] & I_5 \arrow[llll, bend left] \arrow[lll, bend left]
\end{tikzcd}$$

Where are other periodic points located?

There are no periodic points in I_5.

There are no periodic points in I_1 other than p.
since any periodic point in I_1 remains in I_1 for all time.
Any point other than p leaves I_1 eventually and never returns.

There are no periodic points in I_2.

Every periodic point in $I_3 \cup I_4$ other than q moves from I_3 to I_4 and from I_4 to I_3.

In particular if $f^n(x) = x$ then n is even.

$g|I_3 = L_4 \circ f \circ L_3^{-1}$ $\qquad$ $g|I_4 = L_3 \circ s \circ L_4^{-1}$

$g^2|I_3 = L_3 \circ s \circ L_4^{-1} \circ L_4 \circ f \circ L_3^{-1} = L_3 \circ s \circ f \circ L_3^{-1}$

$g^2|I_4 = L_4 \circ f \circ L_3^{-1} \circ L_3 \circ s \circ L_4^{-1} = L_4 \circ f \circ s \circ L_4^{-1}$.

$g^{2n}|I_4 = L_4 \circ (f \circ s)^n \circ L_4^{-1}$

~~If $g^{2n}(x) = x$ (we can assume~~

$= L_4 \circ f^n \circ L_4^{-1}$

If $f^n(x) = x$ then $g^{2n}(L_4(x)) = L_4 \circ f^n \circ L_4^{-1} \circ L_4(x) = L_4 f^n(x) = L_4(x$

($f^m(x) \neq x$ for $1 < m < n$)

If $g^m(L_4(x)) = L_4(x)$ then (i) m is even and $f^{m/2}(x) = x$.

So a periodic point of period ~~n~~ n for f has period 2n for g. Conversely a periodic pt. point of period 2n for g has period n for f.

Corollary. There are symmetric unimodal maps with periods of periodic points equal to $1, 2, 4, 8 \ldots 2^n$ for any $n \geq 0$.

Corollary. There is a symmetric unimodal map with periods of periodic points equal to ~~1, 2, 4, 8, … 2^n~~ all powers of 2.

Thm. (Period 3 implies chaos.) If f is a continuous map of the interval (not assumed to be unimodal) with a point of period 3 then f has points of all periods.

Remark: p has period n if $f^n(p) = p$ but $f^m(p) \neq p$ for $m < n$.