Thm. (Sarkovskii 1964) Define an ordering on the real numbers as follows:

$$3 < 5 < 7 < 9 < 11 \ldots$$
$$< 3\cdot 2 < 5\cdot 2 < 7\cdot 2 < \ldots$$
$$< 3\cdot 2^2 < 5\cdot 2^2 < 7\cdot 2^2 < \ldots$$
$$\vdots$$
$$< \ldots 2^n < \ldots 2^4 < 2^3 < 2^2 < 1$$

If an interval map has a point of period p then it has a point of period q where $p < q$ with respect to the above ordering.

We can see a connection between the special role played by the number 2 and our period doubling operator on maps.

~~Horse~~

We want to consider diffeomorphisms ~~of~~ in dimensions greater than 1.

We want to find a family of diffeomorphisms that behave like the expanding maps of the circle in that we can code orbits by ~~a~~ symbolic sequences.

(Note that expanding diffeomorphisms do not exist in 1-dimension.)

Instead of starting with a definition we will start with an example, the linear ^(dimale) horseshoe. ~~map~~

Observation about itineraries. We constructed itineraries $\omega(x)$ based on the location of $f^j(x)$ for $j \geq 0$. If f is invertible we can consider sequences with positive <u>and negative</u> indexes

Based on this ^(idea) we would like to expand our collection of symbolic models.

Two sided shift.

Let $\Omega = \{0, 1, 2 \ldots N-1\}$ where $N \geq 2$.

Let $\Sigma = \Omega^{\mathbb{Z}} = \{(\omega_j)_{j=-\infty}^{\infty} : \omega_j \in \Omega\}$

Let $\lambda > 1$ and define a metric on Σ.

For $\omega, \omega' \in \Sigma$ let

$$d_\lambda(\omega, \omega') = \max_{n \in \mathbb{Z}} \frac{\varepsilon(\omega_j, \omega_j')}{\lambda^{|j|}}$$

where $\varepsilon(j,k) = \begin{cases} 1 & \text{if } j \neq k \\ 0 & \text{if } j = k. \end{cases}$

In particular $d_\lambda(\omega, \omega) = 0$ and the triangle inequality holds.

Prop. Σ is a compact metric space.

The ball of radius λ^{-n} around ω consists of ω' so that $\omega_j' = \omega_j$ for $|j| \leq n$.

The topology of Σ is independent of λ since the collection of balls is independent of λ.

The shift map $\sigma : \Sigma$ shifts a sequence one position to the left. That is if $\omega' = \sigma(\omega)$ then $\omega_n' = \omega_{n+1}$.

Unlike the case of the one-sided shift σ is invertible. The inverse is the right shift.

Prop. σ is a homeomorphism.

notation. If we write down a sequence in Σ using say 0's and 1's then we need a way to see the indices. Do this with a decimal pt.

$$\cdots \omega_3 \omega_2 \omega_1 . \omega_0 \omega_1 \omega_2 \omega_3 \cdots$$

$$\cdots 0\ 0\ 1\ ,\ 0\ 0\ 1\ 0 \cdots$$

Shifting left means moving the decimal point to the right.

A cylinder set is a subset of Σ obtained by specifying a finite number of coordinates, if $\omega = \omega_{-n} \omega_{-n+1} \ldots \omega_0 \ldots \omega_{n-1}$ is a finite word

$$C_\omega = \{ (\omega'_k)_{k=-\infty}^{\infty} : \omega'_k = \omega_k \text{ for } -n \le k \le n-1 \}.$$

C_ω $.**01$

$\omega : .**01$

A cylinder set is open: Given $\omega' \in C_\omega$ then for $\varepsilon = 2^{-(n+1)}$ any point within ε of ω' has the same entries as ω' for positions $-n$ through $n-1$. Thus the ball of radius ε around ω' is in C_ω.

Cylinder sets are closed: The complement of a cylinder set is a finite union of cylinder sets. So the complement of a cylinder set is open.

Def. If $f^t: X \circlearrowleft$ then $Y \subset X$ is an invariant set if $f(Y) = Y$.

Note that if we have an invariant set we can think of the restriction of f to Y as a dynamical system in its own right.

Prop. For the horseshoe $f: \mathbb{R}^2 \circlearrowleft$ the set $\Lambda = \bigcap_{n=-\infty}^{\infty} f^n(\Delta)$ is a closed invariant set.

(In fact it is the product of 2 middle-third Cantor sets.)