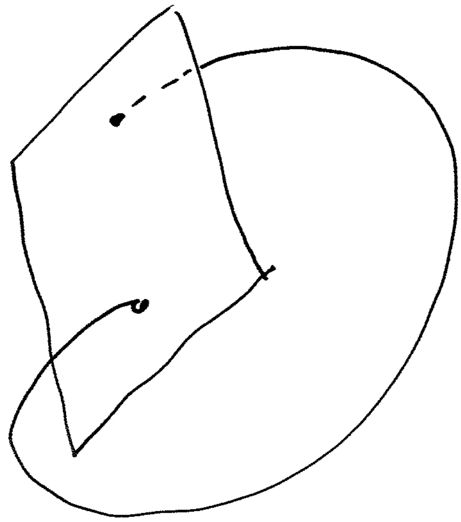
In 1938 Cartwright and Littlewood began working on a differential equation which arose from radio and radar work. from the Radio Research Board.

In 1945 they proved that this equation displayed chaotic behavior

In ~~1960's~~ 1966 Smale constructed a simple geometric model of a diffeomorphism which demonstrated this behavior.

There is a connection between ~~dif~~ autonomous differential equations in 3-variables and diffeomorphisms in two variables.

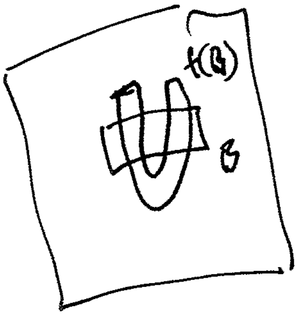
Choose a surface $S \subset \mathbb{R}^3$ transverse to the flow. $f^t: \mathbb{R}^3 \circlearrowleft$.



We define a map from $F: S \to S$ by sending $p \in S$ to $f^{t_0}(p)$ where t_0 is the smallest $t > 0$ for which $f^t(p) \in S$.

F will be a 1-1 differeo from some set $U \subset S$ to some set $V \subset S$. ~~A~~ Typically U and V have a large overlap.

The picture of the flow the Smale imagined was



What does chaos mean in this context?

It means that the behavior of solutions looks random. Solutions look like the past behavior is not determining the future behavior. A classic model of such random behavior is the tossing of a coin. Knowing which ~~how~~ previous tosses were heads or tails does not allow you to calculate whether the next toss will be a head or a tail.

A topological model of the coin tossing experiment is the 2-sided 2-shift ~~Σ_2~~ $\sigma : \Sigma_2 \circlearrowleft$ where the outcome of the k-th toss corresponds to the value of ω_k.

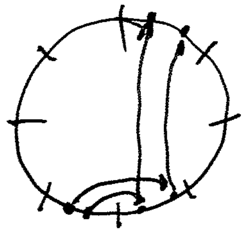
Thus Σ_2 corresponds to the set of all possible outcomes, ~~and so~~ ~~is~~ σ corresponds to doing the next toss.

We will show that ~~we can map Σ_2~~ there is an injective semi-conjugacy from $\sigma : \Sigma_2 \circlearrowleft$ to the horseshoe $f : \mathbb{R}^2 \circlearrowleft$. It is in this sense that the horseshoe demonstrates chaos.

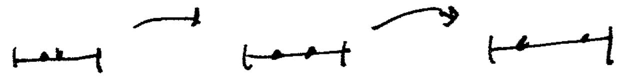
We have already seen chaotic behavior in expanding maps of the circle.

Why is expansion important for maps?

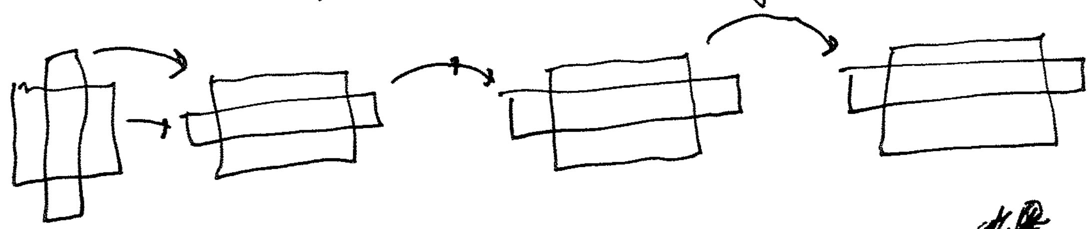
Expansion is used to show that distinct points correspond to distinct symbolic sequences



If p and q are sufficiently close then $d(f(p), f(q)) \geq \lambda \cdot d(p,q)$. so eventually the points must land in different intervals.



Now let's consider the same question for diffeomorphisms. Again we want to show that if p and q are close then $f^n(p)$ and $f^n(q)$ cannot continue to be close but in this case n can be positive <u>or</u> negative.



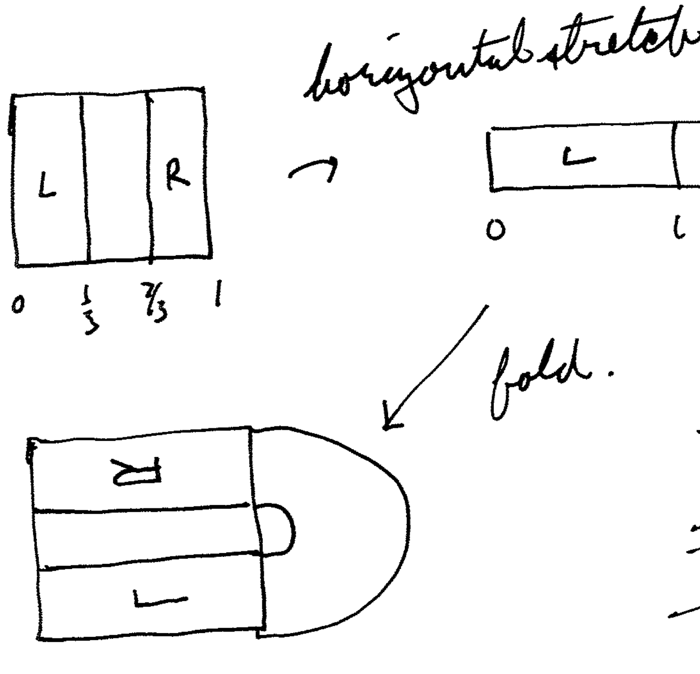
We can acheive this with a diffeo. that of a rectangle that stretches the x-direction but shrinks the y-direction.

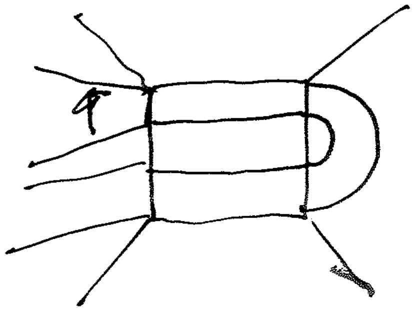
If p and q have distinct x-coordinates then ~~even~~ the will eventually pushed apart in forward time. If they have the same x-coordinate they will be pushed together in positive time. If they have distinct y-coordinates they will be pushed apart in <u>backward</u> time. So if $p \neq q$ they will be pushed apart in forward or backward ~~an~~ time.

This is what happens in Smale's horseshoe.

Our "model ~~no~~ diffeomorphism" $f:\mathbb{R}^2 \circlearrowleft$ is constructed in order to clearly show the important dynamical behavior. In fact it is quite artificial but we will worry about that later.

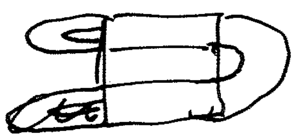
Let $\Delta = [0,1]\times[0,1]$. We will focus on ~~the~~ $f|\Delta$ and $f^{-1}|\Delta$.





$$f(x,y) = \begin{cases} (3x, \tfrac{1}{3}y) & (x,y)\in L \\ (3-3x, 1-\tfrac{1}{3}y) & (x,y)\in R \end{cases}$$

We do not specify f outside of Δ but we assume that any point in Δ which leaves Δ at some ~~time~~ future time never returns ~~into~~ to Δ in the future. We assume that any point leaving Δ at some past time never returns in the past.



Thm. Let $\Omega = \bigcap_{n=-\infty}^{\infty} f^n(\Delta)$. Then Ω is an invariant set and $f|\Omega$ is topologically conjugate to the shift on Σ on two symbols.

The proof will follow the outline of the construction of the semi-conjugacy for an expanding map of the circle.
(Version in the notes improves on my lecture.)

Let $p \in \Delta$. If $f^n(p) \in \Delta$ for all n then

~~Let A(~~ $f^n(p) \in L \cup R$ for all n and we

can define an itinerary $\omega(p)$ to be a bi-infinite sequence where

$$\omega_j(p) = \begin{cases} 0 & \text{if } f^j(p) \in L \\ 1 & \text{if } f^j(p) \in R. \end{cases}$$

$$\omega_j(f(p)) = \begin{cases} 0 & \text{if } f^{j+1}(p) \in L \\ 1 & f^{j+1}(p) \in R \end{cases}$$ so $\omega_j(f(p)) = \omega_{j+1}$ and

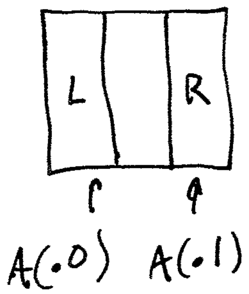
$$\omega(f(p)) = \sigma(\omega(p)).$$

As in our analysis of expanding maps it is useful to consider finite itineraries.

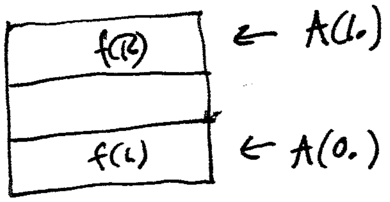
Let ω be a finite word $\underbrace{\omega_j \ldots \omega_1}_{|j|\text{-indices}} . \underbrace{\omega_0 \ldots \omega_{k-1}}_{k\text{-indices}}$

Let $A^{j,k}(\omega) = \{ p : f^\ell(p) \in \omega_\ell \text{ for } j \le p \le k-1. \}$

Example.



$A(.0) = \{p : p \in L\}$.

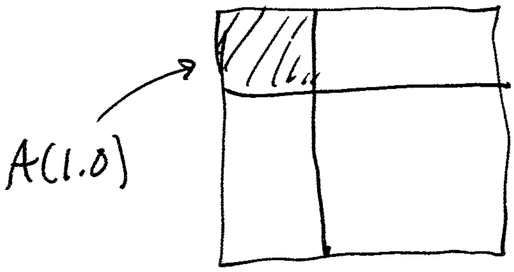


$\ldots -2\ -1\ .\ 0\ 1\ 2 \ldots$

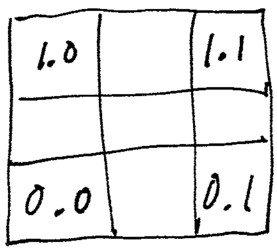
$A(1.) = \{p : f^{-1}(p) \in R\}$

or $\{p : p \in f(R)\}$.

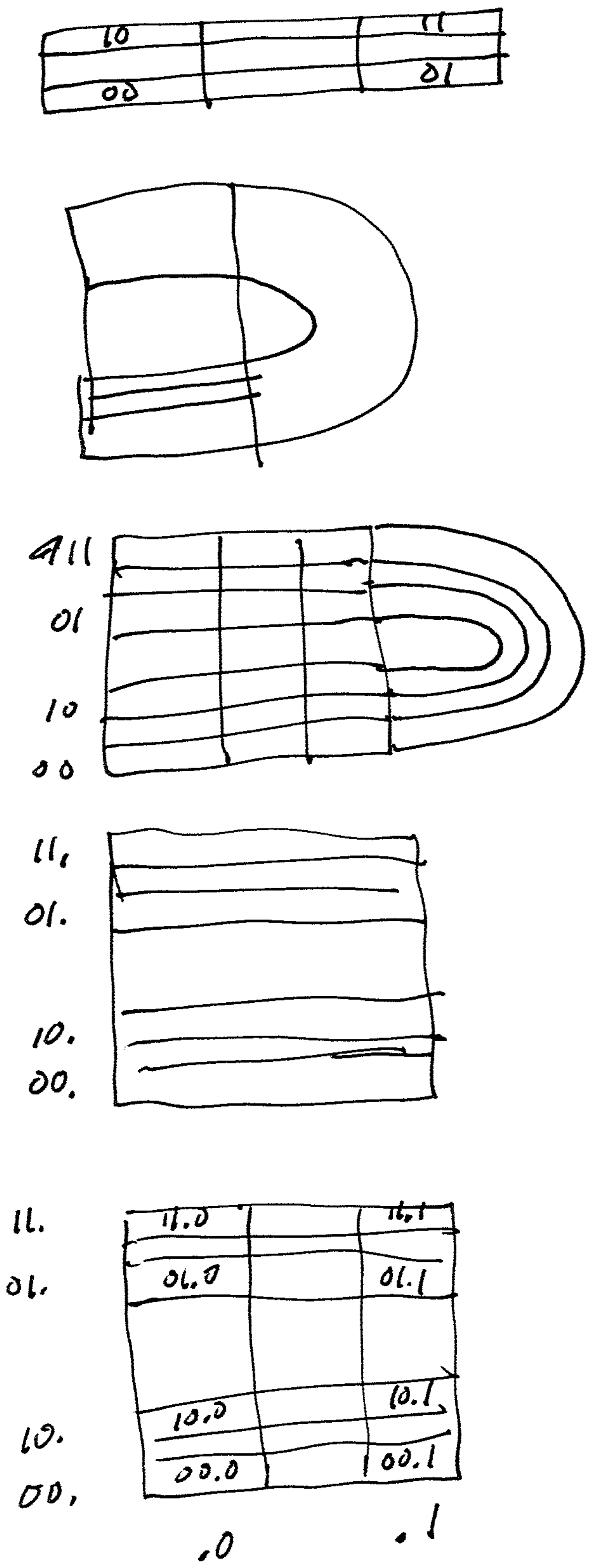
By definition $A(1.0) = A(.0) \cap A(1.)$



We get



If we map this decomposition by f we shift the decimal pt. to the right:



For any word $w_j \dots w_{k-1}$ with $j \geq 0$, $k \geq 0$

Claim. $A^{j,k}(w)$ is a rectangle in Δ ~~of~~ with horizontal and vertical sides. The height is $\frac{1}{3^{|j|}}$ and the width is $\frac{1}{3^{|k|}}$.

.0 $j=0$, $k=1$

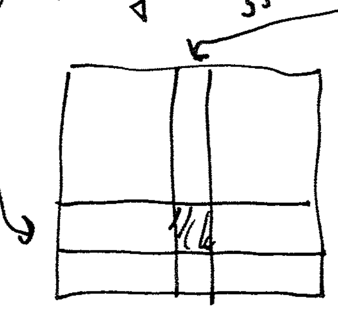
height$(A(.0))=1$
width$(A(.0))=\frac{1}{3}$

Proof. By induction. It is true for words of length 1: $0.$, $1.$, $.0$, $.1$.

Assume it is true for words of length ~~over~~ $<n$.

Consider a word of length n: $w_j \dots w_1 . w_0 \dots w_{k-1}$ with $j \geq 1$ and $k \geq 1$. By induction hypothesis

$A(w_j \dots w_1.)$ is a rectangle in Δ of width 1 and height $\frac{1}{3^j}$



$A(.w_0 \dots w_{k-1})$ is a rectangle in Δ of height 1 and width $\frac{1}{3^k}$.

Their intersection has width $\frac{1}{3^k}$ and height $\frac{1}{3^j}$. Thus the claim holds.

Now consider a word $.w_0 \dots w_{n-1}$ of length n.

$j=1$
$k=n-1$

shift left

$A(w_0.w_1 \dots w_{n-1})$ ~~satisfies~~ is a rectangle of width $\frac{1}{3^{n-1}}$ and height $\frac{1}{3}$. It is contained in L or R (depending on w_0). $f|L$ or R is a linear map which multiplies ~~width~~ height by $\frac{1}{3}$ and ~~height~~ width by 3 so $f(A(w_0.w_1 \dots w_{n-1}))$ $= A(.w_0 \dots w_{n-1})$ has height 1 and width $\frac{1}{3^n}$ as was to be shown.

Construction of conjugacy:

Let $\omega \in \Sigma_2$. The sequence of sets

$A(\omega_{-n} \ldots \omega_{-1} \omega_0 \ldots \omega_{n-1})$ are nested and have decreasing diameter.

Let $h(\omega) = \bigcap_{n=1}^{\infty} A(\omega_{-n} \ldots \omega_{n-1})$.

$d = d_\lambda$

h is continuous, Fix $\lambda > 1$. ~~Following~~ ~~Let $\varepsilon > 0$. Choose n so that $\lambda^n < \varepsilon$.~~ Given $\nu > 0$ and $\omega' \in \Sigma$ s.t. $d(\omega, \omega') < \lambda^{-\nu}$. Then $\omega'_n = \omega_n$ for all $|n| < \nu$.

So $h(\omega)$ and $h(\omega') \in A(\omega_{-\nu} \ldots \omega_{\nu-1})$. In particular

$\|h(\omega) - h(\omega')\| \le 2 \cdot 3^{-\nu}$.

h is injective: If $\omega \ne \omega'$ then there is some n, k with $\omega_k \ne \omega'_k$. Thus for $n \ge |k|+1$ $A(\omega_{-n} \ldots \omega_n)$ and $A(\omega'_{-n} \ldots \omega'_n)$ are disjoint. But $h(\omega)$ is in the first set and $h(\omega')$ is in the second.

h is surjective. Any point p in $\bigcap_{n=-\infty}^{\infty} f^n(\Delta)$ is in $\bigcap_{n=-\infty}^{\infty} f^n(L \cup R)$. So it has ~~a coding~~ an itinerary ω. We have $h(\omega) = p$.

h is a homeomorphism: since Σ and Λ are compact and h is a continuous bijection, h^{-1} is ~~a homeomorphism~~ continuous.

h is a topological conjugacy. ~~Lots~~ ~~If The image of p~~ The inverse image of p is determined by its itinerary. But the itinerary of $f(p)$ is the shift applied to the itinerary of p.