Comments on the entropy proof.

On a small scale the metric d_n on the torus expands distances by λ^n in the unstable direction and preserves distances in the stable direction. To get a global idea of its behavior we compare it to the $\tilde{d}_n$ metric on $\mathbb{R}^2$.

~~On~~ The idea of the proof was to compare the d_n metric on the torus with the $\tilde{d}_n$ metric on $\mathbb{R}^2$. The $\tilde{d}_n$ metric expands distances by λ^n in the unstable direction and preserves distances in the stable direction and, on a small scale, it agrees with the d_n metric on the torus.

say we have a differential equation on a torus $\mathbb{T}^2$ that gives rise to a flow $f^t : \mathbb{T}^2 \circlearrowleft$.

We want to capture the notion that every point $p \in \mathbb{T}^2$ has the same behavior under the flow.

Boltzmann made a definition that f^t is ergodic if f^t has only one orbit.

This is <u>not</u> the modern definition of the notion of ergodicity. Why not?

If there is some $p \in \mathbb{T}^2$ so that every point in $\mathbb{T}^2$ lies on the orbit of p then

$\mathbb{T}^2 = \bigcup_{j=-\infty}^{\infty} \{ f^t(p) : j \le t \le j+1 \}$. The theory of O.D.E. tells ~~us~~ that every interval is closed and nowhere dense. Then Baire's thm. tells us that $\mathbb{T}^2$ is not the union of ~~closed now~~ a countable collection of closed nowhere dense sets.

Baire's thm. holds in a compact metric space. Let us use it.

It has the following equivalent formulation:
Any countable collection of open dense sets has a non-empty intersection.

A criterion for the existence of dense orbits ~~in ran~~ (in a compact metric space).

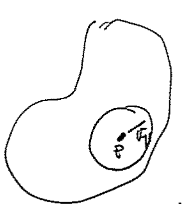
Thm. Let $f^t: X \circlearrowleft$. If for any open sets U, V there is a t so that $f^t(U) \cap V \neq \phi$ then dense orbits exist.

Remark. I referred to the conclusion of this theorem as "topological transitivity". ~~tht~~ I should have ~~referre~~ defined "top. trans." in terms of the hypothesis.

Remark. Converse is also true.

Proof. Construct a countable collection of open balls as follows.

For each n we have a cover of X by open balls of radius $\frac{1}{n}$. Choose a finite subcover. Let $B_1, B_2 \ldots$ be ~~the~~ a list of sets in these ~~union of~~ these finite subcovers. This collection of sets has the property that for each non-empty open set $U \subset X$ there is a $B_n \subset U$ for some n.



Consider $\frac{1}{j}$ cover where $\frac{1}{j} < \frac{r_n}{2}$. p is in some ball B_k in this cover. $B_k \subset$ ball of radius r V.

assume the hypothesis. Point p has a dense orbit if for each j there is a k_j so that $f^{k_j}(p) \in B_j$.

Now let $A_n = \bigcup_{n=1}^{\infty} f^{-n}(B_n)$. A_n is open. ~~Given~~ any U there is some m with $f^m(U) \cap B_n \neq \phi$ so $U \cap f^{-m}(B_n) \neq \phi$.

Pick $q \in$ set $q = f^m(p)$ so p

Thus A_n is ~~dense~~ open and dense. By the Baire property $\bigcap_n A_n \neq \phi$. Any point in this intersection has a dense orbit.

Now assume that for any u, v there is an n with $f^n(u) \cap V \neq \phi$. (4.5)

Let $A_n = \bigcup_{n=1}^{\infty} f^{-n}(B_n)$. This is the set of points that visit B_n at some point in the future. $p \in f^{-n}(B_n)$ means $f^n(p) \in B_n$.

A_n is open. A_n is also dense. Given V we have $f^m(V) \cap B_n \neq \phi$ for some m. If $p \in V$ then $f^m(p) \in B_n$ so $p \in f^{-m}(B_n)$.

Since each A_n is open and dense there is a $q \in \bigcap_n A_n$. q visits every set B_n so q visits every open set. That is to say the orbit of q is dense.

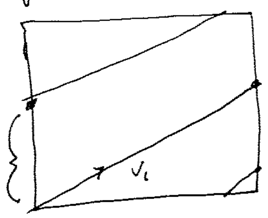
Theorem. A linear hyperbolic diffeomorphism of the torus has a dense orbit.

Proof.

Lemma. The stable and unstable manifolds of 0 are dense.

Lemma. Let $f_A : \mathbb{T}^2 \circlearrowleft$ be a hyperbolic toral automorphism. The stable and unstable manifolds of 0 are both dense in $\mathbb{T}^2$.

Proof.



~~Grace If we follow~~ The unstable manifold of 0 is the image of an eigenspace in $\mathbb{R}^2$. We can consider successive intersections with the vertical circle $\{\binom{0}{y} : 0 \le y \le 1\}$.

If $y_0 = 0$ and y_j is the jth intersection then $y_j = R_\theta^j(0)$ where R_θ is rotation by θ = slope of v.

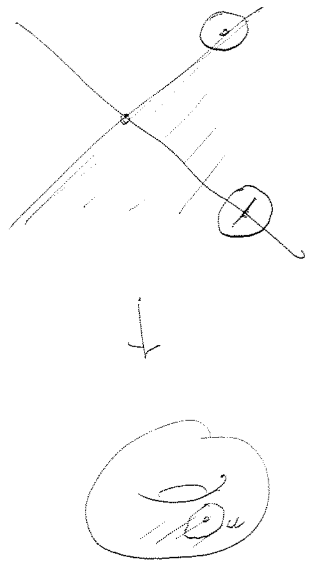
Denseness of the unstable manifold is equivalent to the irrationality of θ. If θ were rational then there would be an element of $\mathbb{Z}^2$ ~~which is~~ whose length is contracted by f_A^{-1}. This is impossible since $f_A^{-1}(\mathbb{Z}^2) = \mathbb{Z}^2$ and $\mathbb{Z}^2$ does not contain arbitrarily short non-zero vectors.

Proof of theorem.

Let U and V be open sets in the torus.

We want to show $f^n(U) \cap V$ for some n.

Now U intersects the stable manifold of 0 so there is some $\tilde{p}$ in $\mathbb{R}^2$ contained in ~~$\mathbb{R}$~~ $\mathbb{R}V_1$ so that an ε ball around $\tilde{p}$ maps into U.



Similarly there is a $\tilde{q}$ contained in $\mathbb{R}V_2$ so that the image of an ε ball around $\tilde{q}$ is contained in V.

There is an n so that $f^n(B_\varepsilon(\tilde{p})) \cap B_\varepsilon(\tilde{q})$.