Definition. Let (X, d) be a metric space and let $f: X \circlearrowleft$ be a map (continuous function) or homeomorphism. Let (p_i) be a sequence of points in X where $i \in \mathbb{N}$ in the map case and $i \in \mathbb{Z}$ in the homeomorphism case.

We say that (p_i) is an ε-pseudo-orbit if

$$d(p_{i+1}, f(p_i)) \leq \varepsilon.$$

Note that when $\varepsilon = 0$ an ε-pseudo-orbit is an orbit.

Def. f has the shadowing property if for any $\varepsilon > 0$ there is a $\delta > 0$ so that for any δ-pseudo-orbit there is an orbit q_i with

$$d(p_i, q_i) \leq \varepsilon \quad \text{for all } i$$

We further assume that if ε is sufficiently small this orbit is unique.

Theorem (warm up). An expanding map of the circle has the shadowing property

Definition. Let (X,d) be a metric space and $f: X \circlearrowleft$ a ~~fixx~~ or homeo. map. We say that (p_i) is an ε-pseudo-orbit if

$$d(p_i, f(p_{i-1})) \le \varepsilon \quad \text{for all } i \in \mathbb{Z} \quad (f \text{ invertible})$$
$$i \in \mathbb{N} \quad (f \text{ not invertible})$$

~~a o-pse~~ $\varepsilon = 0$ corresponds to an orbit $f(p_i) = p_{i+1}$.

Def. We say that f has the shadowing property if for any $\varepsilon > 0$ there is a $\delta > 0$ such that for any δ-pseudo orbit $(p_i)_{i\in\mathbb{Z}}$ there is a ~~unique~~ orbit (q_i) $(f(q_{i-1}) = q_i)$ with

$$d(p_i, q_i) < \varepsilon \quad \text{for all } i \in \mathbb{Z}.$$

"Any δ-pseudo is ε-shadowed."

If ε is suff. small this orbit is unique.

~~Theorem.~~ Exercise: A topological Markov chain has the shadowing property.

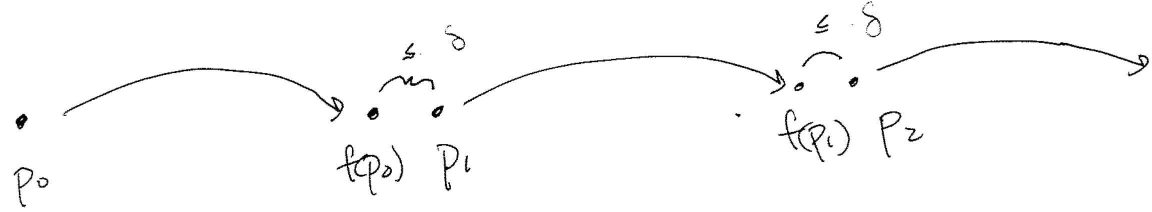
Warm-up Theorem. An expanding map of the circle with $|f'| \ge \lambda > 1$ has the shadowing property. ~~when~~

Remark. Say $f(x) = 2x + 1/10 \sin(x/2\pi)$. It is very hard to know the forward orbit of a point nevertheless shadowing (like the Markov property) says that we ~~can find~~ if we describe the behavior first then ~~w~~ there will be some point with that behavior

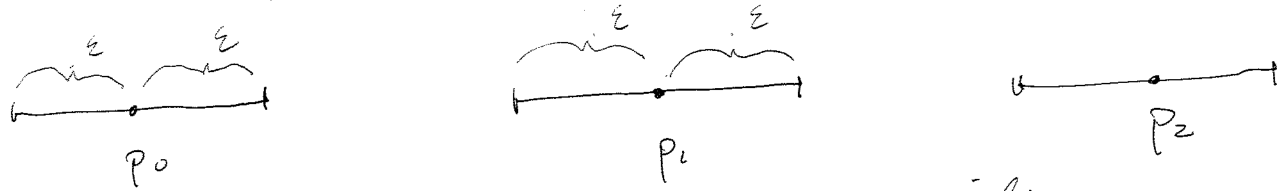
Proof. Assume $|f'(p)| \geq \lambda > 1$ for all $p \in \mathbb{R}/\mathbb{Z}$.

~~Ask~~ Given ε choose $\delta < (\lambda - 1) \cdot \varepsilon$

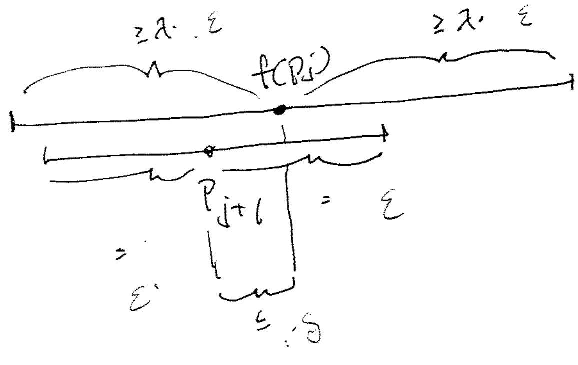
Consider an ε pseudo orbit



Now draw ~~an~~ balls $B_\varepsilon(p_j)$. These are intervals of length 2ε centered at p_j.



We want to find an actual orbit (q_j) with $q_j \in B_\varepsilon(p_j)$. Claim that $f(B_\varepsilon(p_j))$ contains $B_\varepsilon(p_{j+1})$.



need ~~ε~~ $\lambda \cdot \varepsilon \geq \varepsilon + \delta$ or $(\lambda - 1)\varepsilon \geq \delta$.

Now, as in the proof of the Sarkovskii Thm., there is a sequence of intervals $I_0 = B_\varepsilon(p_j) \supset I_1 \supset I_2 \ldots$ where $f^\ell(I_j) \subset B_\varepsilon(p_\ell)$ ~~and~~ for $\ell < j$ and $f^\ell(I_\ell) = B_\varepsilon(p_\ell)$.

$|f'| \geq \lambda$

In addition we have $|I_j| \leq \frac{\varepsilon}{\lambda^j}$ since $f^j(I_j) = B(p_j, \delta)$.

Let $q_0 = \bigcap_{j=0}^{\infty} I_j$. Then by construction $f^j(q_0) \in I_j$ for all j and q_0 is the unique point with this property.

Remarks. Quantitative uniqueness:

If $d(f^\ell(q_i), f^\ell(q_i')) \leq \delta$ for $0 \leq \ell \leq n$ then $d(q_i, q_i') \leq \frac{\delta\varepsilon}{\lambda^n}$.

In this case q_i, q_i' are both in I_n.

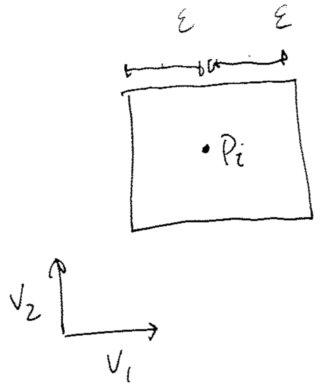
Thm. Let f_A be a linear hyperbolic diffeomorphism of the torus then f_A has ~~the~~ shadowing property.

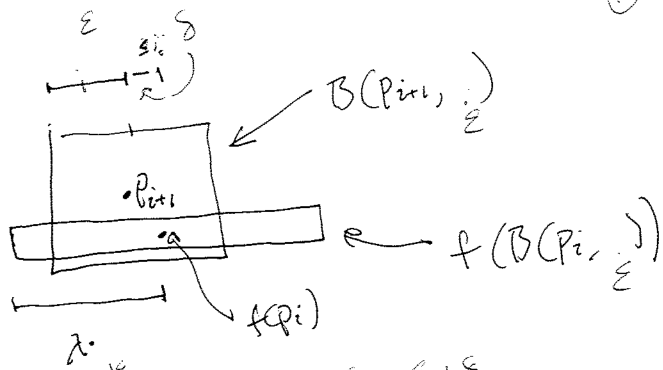
Proof. If f_A has the shadowing property with respect to one metric metric then it has the shadowing property (with different ε, δ). with respect to any equivalent metric. Let us tho show that f has the shadowing property with respect to the distance that comes from the max norm

$|av_1 + bv_2|_{max} = \max\{|a|, |b|\}$ where v_1 and v_2 are eigenvectors of unit length, λ_1, λ_2 are the eigenvalues $|\lambda_1| > 1 > |\lambda_2|$ and $\lambda = |\lambda_1|$.

Given $\varepsilon > 0$ choose $\delta < \underbrace{(\lambda-1)}_{\text{positive}}\varepsilon$.

Let p_i be an δ-pseudo-orbit. Let $B(p_i, \varepsilon)$ be the unit ball around p_i with respect to the max norm. We claim that $f(B(p_i, \varepsilon))$ crossed $B(p_{i+1}, \varepsilon)$ in a "Markov fashion".

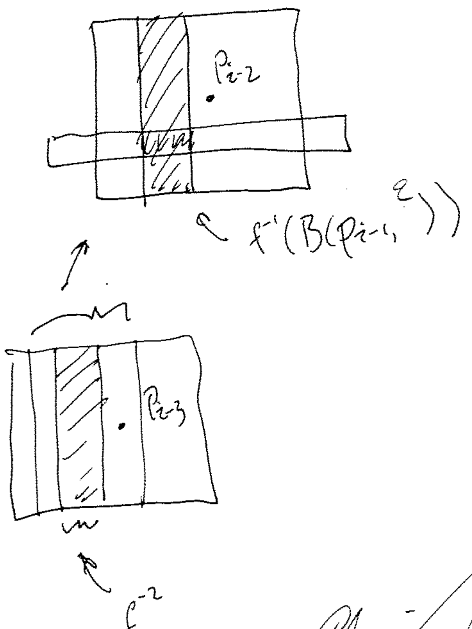




need to check $\lambda\varepsilon > \varepsilon + \delta$

or $(\lambda - 1)\varepsilon > \delta$.

Now the set of points in $B(p_i, \varepsilon)$ that map to $B(p_i, \varepsilon)$ is a vertical (full height) rectangle inside $B(p_i, \varepsilon)$



Let I_j for $j \geq 0$ be the set of points in $v \in B(p_0, \delta)$ so $f^\ell(v) \in B(p_\ell, \delta)$ for $0 \leq \ell \leq j$.

Let $I_{j,k}$ be the set of points v for which $f^\ell(v) \in B(p_\ell, \delta)$ for $j \leq \ell \leq k$.

Claim that $I_{j,k}$ is a full width rectangle in $B(p_k, \delta)$ and a full height rectangle in $B(p_j, \delta)$. Proof by induction following the Markov partition proof.

Furthermore the width of

Let I_j for $j \geq 0$ be the set of points $r \in B(p_0, \varepsilon)$ so that $f^\ell(r) \in B(p_\ell, \varepsilon)$ for $0 \leq \ell \leq j$.

Claim that I_j is a full height rectangle in $B(p_0, \varepsilon)$ and $f^j(I_j)$ is a full width rectangle in $B(p_j, \varepsilon)$. Furthermore width of I_j is $\frac{\varepsilon}{\lambda^j}$.

Proof by induction on j.

For $j \leq 0$ let I_j'' be the set of points $r \in B(p_0, \varepsilon)$ so that $f^\ell(r) \in B(p_\ell, \varepsilon)$ for $j \leq \ell \leq 0$. The height of I_j'' is $\frac{\varepsilon}{\lambda^{|j|}}$.

Conclude that there is a unique point q in $\bigcap_{j=-\infty}^{\infty} I_j$ and this is the unique point that ε-shadows the pseudo-orbit p_i.

Quantitative uniqueness:

If we have two orbits p_i, p_i' and

$d(p_i, p_i') \leq \varepsilon$ for $-N \leq i \leq N$ then $d(p_0, p_0') \leq \dots \frac{\varepsilon}{\lambda^N}$.