We have a criterion for minimality in terms of the uniform boundedness

$$(f^{n_k})'(x)\,(f^{-n_k})'(x) \geq C$$

for some ~~a~~ sequence $n_k \to \infty$.

We have the existence of disjoint intervals

$x_i = f^i(x_0)$

$(x_i, x_{i+n_k}) \quad i = 0 \dots n_k$ where n_k is the sequence of nearest returns of $R_\alpha^n(0)$ to 0.

Denjoy's Thm. If $\log f'$ has bounded variation and $\rho(f)$ is irrational then f is conjugate to a rotation.

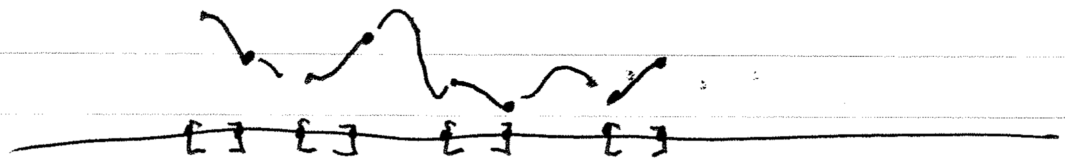
Proof of Denjoy's Thm. Let n_k be the sequence of near returns. Let $x \in \mathbb{R}/\mathbb{Z}$.

$$\left| \log \left((f^{n_k})'(x) \cdot (f^{-n_k})'(x) \right) \right| = \left| \log \frac{(f^{n_k})'(x_0)}{(f^{n_k})'(x_{-n_k})} \right|$$

$$\leq \left| \sum_{i=0}^{n_k-1} \log f'(x_i) - \sum_{i=0}^{n_k-1} \log f'(x_{i-n_k}) \right|$$

$$\leq \sum_{i=0}^{n_k-1} \left| \log f'(x_i) - \log f'(x_{i-n_k}) \right| \leq \operatorname{Var}(\log f')$$

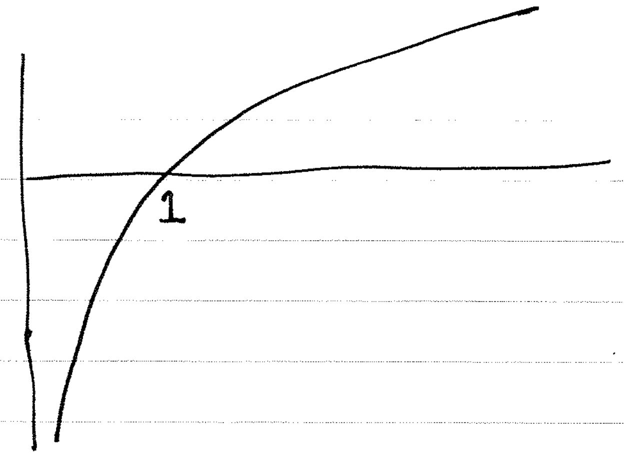
The intervals (x_{i-n_k}, x_i) are all disjoint since these are $f^{-n_k}((x_i, x_{i+n_k}))$.



We are picking up some of the terms that give the variance. Since all terms are positive we get a lower bound for the variance of $\log f'$.

According to the previous Lemma f is minimal.

log



If $\left|\log (f^{n_k})'(x)\,(f^{-n_k})'(x)\right|$ is bounded ~~below~~ above
then $\log (f^{n_k})'(x)\,(f^{-n_k})'(x)$ is bounded ~~above~~ and below
then $(f^{n_k})'(x)\,(f^{-n_k})'(x)$ is bounded ~~above~~ below.

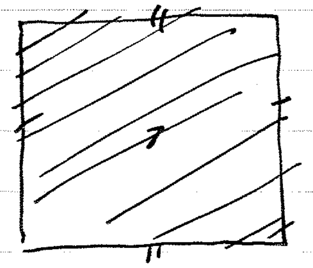
Inequality gives:

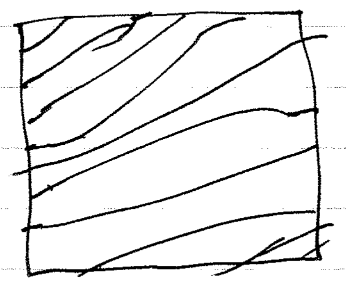
$$(f^{n_k})'(x)\,(f^{-n_k})'(x) \geq \exp(-\mathrm{Var}(\log f')).$$

Combine with minimality criterion $\Rightarrow f$ is minimal. Poincaré's thm $\Rightarrow f$ is conjugate to $R_{\rho(f)}$.

Phase locking. (model of the general situation.)

Sup we have 2 oscillators which are weakly coupled. Then the orbit of a point is either dense or asymptotic to a periodic orbit.





Assume the coef. of the equation are C^2.

Why?

Construct the first return map to the left side. This map is a circle homeomorphism of class C^2. Thus it is either minimal or has rational rotation number. If it is minimal then every orbit for the first return map is dense so every orbit of the flow in the torus is dense.

If $\rho(f)$ is rational then every point ~~converge~~ approaches to a periodic point in forward time.

Same holds for the torus.

Construction of Denjoy's example.

Thm. For any irrational α there is a C^1 diffeomorphism f of the circle with $\rho(f)=\alpha$ but f not minimal.

Assume such an f exists.
According to the second version of Poincaré's Thm. there is a semi-conjugacy $h: \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ so that

$$\begin{array}{ccc} \mathbb{R}/\mathbb{Z} & \xrightarrow{f} & \mathbb{R}/\mathbb{Z} \\ \downarrow h & & \downarrow h \\ \mathbb{R}/\mathbb{Z} & \xrightarrow{R_\alpha} & \mathbb{R}/\mathbb{Z} \end{array}$$

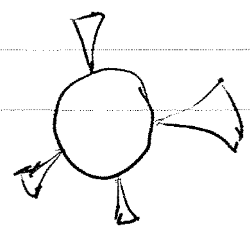
commutes. where h is monotone (but not strictly monotone.

Since f is not minimal h cannot be a homeomorphism so it must collapse some interval I_0 to a point x_0.
Let $x_n = f^n(x_0)$.
Say $h(I_0)=x_0$. ~~Now~~ Let $I_n = f^n(I_0) = h^{-1}(f^n(x_0)) = h^{-1}(x_n)$.

$h(I_n) = h f^n(I_0) = R_\alpha^n h(I_0) = R_\alpha^n(x_0)$.

In particular the I_n are all disjoint, ~~and~~

~~An~~ We can think of this as "blowing up an orbit" i.e replacing points by intervals
(Of course the orbit is infinite and dense!)



Let's assume that just one orbit is blown up.

Let's specify a rotation number, say

$\phi = \frac{\sqrt{5}+1}{2} \mod 1$
$= \frac{\sqrt{5}-1}{2}$

Say $l_n = |I_n|$.
$l_n = \frac{1}{n^2+25}$.

Let's specify lengths for I_n, say l_n. $L = \sum I_n < \infty$.

To make our map C^1 we need $\lim_{n \to \pm\infty} \frac{l_n}{l_{n+1}} = 1$.

Assume that the orbit of 0 is blown up.

This gives us enough information to determine h.

$\boxed{\frac{1}{2} \le \frac{l_n}{l_{n+1}} \le 2.}$

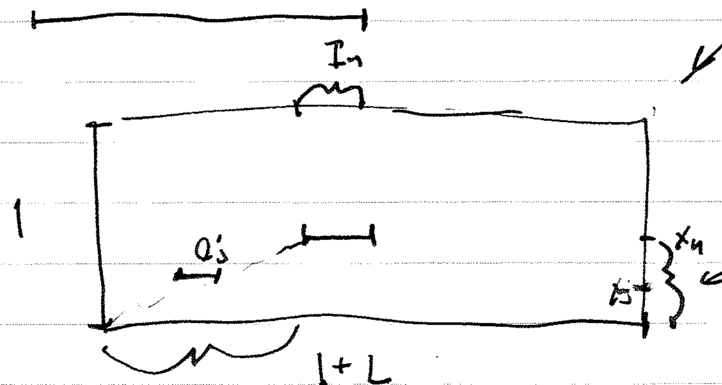
For convenience we will assume that f is defined on the circle of length $1+L$.

(This corresponds to sticking I_n into the circle of length 1.)

We can find the formula for h^{-1}.

Say $h(0)=0$.

graph of h



Say $I_n = (a_n, b_n)$.

$h([0, a_n])$

a_n

this interval maps to $[0, x_n]$.

The function we construct is monotone. Its intervals of ~~const~~ constancy correspond to ~~the~~ the orbit x_n

~~It contains intervals I_j with~~
The interval $[0, a_n]$
~~It is~~ built from the ~~interval~~ $[0, x_n]$ where we insert as many intervals I_j where ~~x_j~~ $x_j \in$ ~~$[0, x_n]$~~ $[0, x_n)$

Thus its length is ~~Σ~~ $x_n + \sum_{x_j \in [0, x_n)} l_j$. $\qquad a_n = x_n + \sum_{x_j \in [0, x_n)} l_j$

This ~~∞~~ sum makes sense because all terms ~~that~~ are positive and we have a fixed upper bound (L) for its partial sums.

$$\boxed{h^{-1}(x) = x + \sum_{x_j \in [0, x]} l_j \qquad x \notin O(x_0).}$$

The right hand endpoint b_n of the interval is

$$x_n + \sum_{x_j \in [0, x_n)} l_j + l_n = x_n + \sum_{x_j \in [0, x_n]} l_j.$$

~~Knowing~~ h ~~on~~ determines f on the complement of $\cup I_n$. If $x \notin \cup I_n$ then ~~h~~ $R_\alpha h(x) \notin O(0)$ so $h^{-1} R_\alpha h(x)$ is a unique point.

In fact we can show that $\lim_{\substack{x \to x_0 \\ x \notin \cup I_n}} \frac{f(x) - f(x_0)}{x - x_0} = 1.$