At the end of the last class we started to construct a Denjoy counterexample $f: \mathbb{R}/\mathbb{Z} \circlearrowleft$ where $\rho(f)$ is irrational, f is C^1 and f is not minimal. Let $x_n = R_\alpha^n(0)$, α irrational. Let $l_n > 0$ satisfy $\sum_{n=-\infty}^{\infty} l_n = L < \infty$.

Our key tool is the following function

$$\gamma(x) = x + \sum_{k:\, x_k \in [0,x)} l_k .$$

γ is increasing.

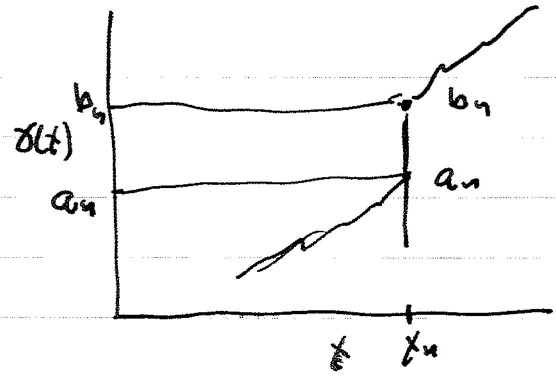
$\gamma(0) = 0$, $\gamma(1) = 1 + L$

γ has jump type discontinuities at the points x_n.

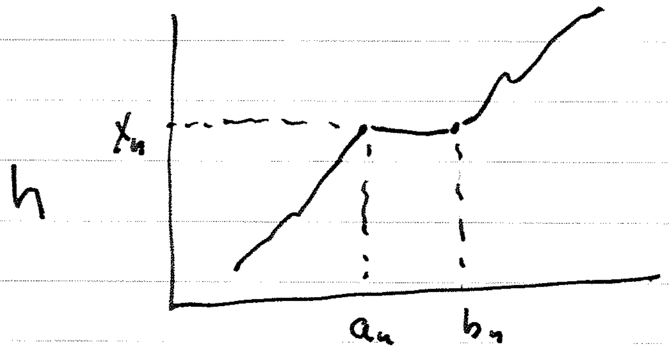
$$\lim_{x \uparrow x_n} \gamma(x) = x_n + \sum_{k:\, x_k \in [0, x_n)} l_k \overset{\text{def}}{=} a_n$$

$$\lim_{x \downarrow x_n} \gamma(x) = x_n + \sum_{k:\, x_k \in [0, x_n]} l_k \overset{\text{def}}{=} b_n .$$

So $b_n = a_n + l_n$.



If we flip the graph we get a graph of a continuous function h which has plateaus instead of jumps.



Let $I_n = [a_n, b_n]$ then for $x \in I_n$, $h(x) = x_n$.

Otherwise $h(x) = \gamma^{-1}(x)$.

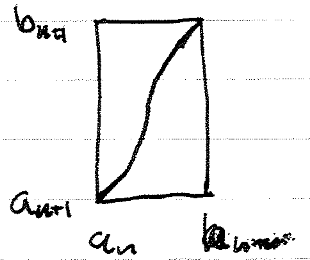
$$\begin{tikzcd}
\mathbb{R}/(1+L)\mathbb{Z} \arrow[r, "``f"'] \arrow[d, "h"] & \mathbb{R}/(1+L)\mathbb{Z} \arrow[d, "h"] \\
\mathbb{R}/\mathbb{Z} \arrow[r, "R_\alpha"] & \mathbb{R}/\mathbb{Z}
\end{tikzcd}$$

h plays the role of the semi-conjugacy.

"f" is yet to be defined.

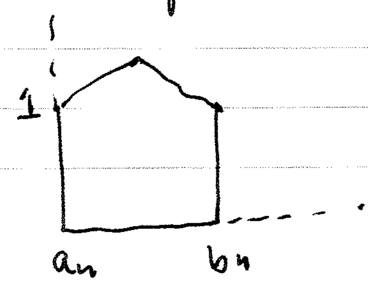
Now if $h(x) \notin \{x_j : j \in \mathbb{Z}\}$ then we can define $f(x)$ to be $h^{-1}(R_\alpha(h(x)))$. ~~Thus~~ In this case $h^{-1}(R_\alpha(h(x)))$ is a single point.

If $h(x) = x_j$ then $x \in I_j$. In order for the diagram to commute $f(x) \in h^{-1}(R_\alpha(h(x))) = h^{-1} R_\alpha x_j = h^{-1} x_{j+1} = I_{j+1}$.

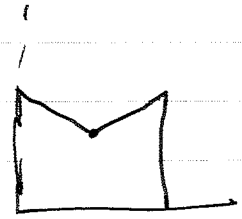


We define f on I_n to be piecewise quadratic with $f'(a_n) = f'(b_n) = 1$.

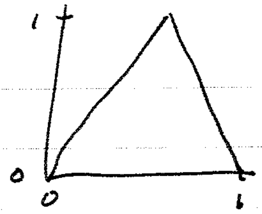
So f' is piecewise linear



or



Let $\chi(x) = 1 - |1-2x|$



$c_n = 2\left(\frac{\ell_{n+1}}{\ell_n} - 1\right)$

For $x \in [a_n, b_n]$ define

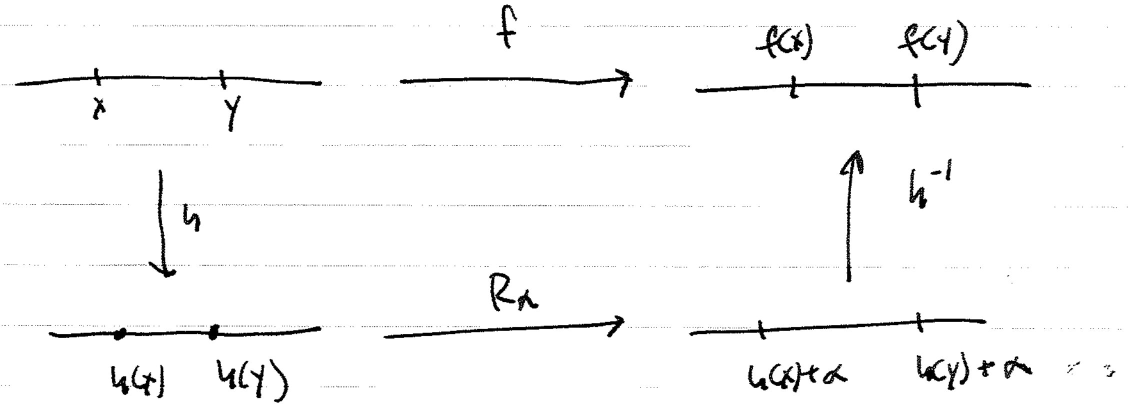
$$f(x) = \int_{a_n}^{x} c_n \chi\left(\frac{x-a_n}{b_n-a_n}\right) dx$$

Then $f(b_n) = b_{n+1} \Rightarrow c_n = 2\left(\frac{\ell_{n+1}}{\ell_n} - 1\right)$.

Assume that $\frac{1}{2} < \frac{\ell_{n+1}}{\ell_n} < 2$ and $\frac{\ell_{n+1}}{\ell_n} \to 1$ as $n \to \pm\infty$.

Now f is defined. We can check that f is continuous and invertible by showing that it is strictly monotone. It is semi-conjugate to R_α. This implies that $\rho(f) = \alpha$ (in particular it has an orbit with the same order type as x_n).

Why is f differentiable with a continuous derivative?



Assume $x, y \notin \bigcup I_n$.

Want to show that $f'(x)=1$ or

$$\lim_{y \to x} \frac{f(y)-f(x)}{y-x} = 1$$

Assume $x<y$

Now $x = h(x) + \sum_{i : x_i \in [0, h(x)]} l_i$ $\qquad$ $y = h(y) + \sum_{i : x_i \in [0, h(y)]} l_i$

$$y - x = h(y) - h(x) + \sum_{i : x_i \in (h(x), h(y)]} l_i$$

$$f(y) - f(x) = \big(h(y) + \alpha\big) - \big(h(x) + \alpha\big) + \sum_{i:\, x_i \in [h(x)+\alpha,\, h(y)+\alpha]} l_i$$

$$= h(y) - h(x) + \sum_{i:\, x_i \in [h(x),\, h(y)]} l_{i+1}$$

so
$$\frac{f(y) - f(x) - (y - x)}{y - x} = \frac{\sum l_{i+1} - \sum l_i}{h(y) - h(x) + \sum l_i}$$

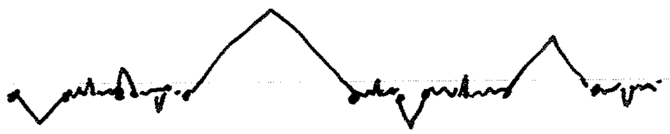
$$\leq \frac{M \cdot \sum l_i - \sum l_i}{\sum l_i}$$

where $M = \max_{i:\, x_i \in [h(x),\, h(y)]} \frac{l_{i+1}}{l_i}$ $\qquad \leq M - 1.$

Now as $y \to x$, $h(y) \to h(x)$ and $M \to 1$.

To get the lower bound argue with $\min \frac{l_{i+1}}{l_i}$.

Picture of f'

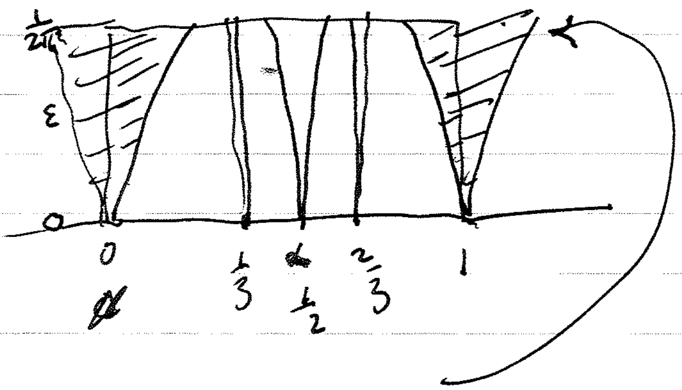


For any $\varepsilon > 0$ only finitely many peaks are above $1+\varepsilon$ or below $1-\varepsilon$.
Thus f' is continuous.
At any $x \notin \bigcup I_n$ we have heights of wedges converging to 0 but the sum of the heights of all wedges is infinite.

Definition. A monotone ~~increase~~ cts from $\mathbb{R}\to\mathbb{R}$ function which is constant on an open dense subset is called a "Devil's Staircase".

We have seen one example already in the semi-conjugacy h.

Consider ~~the~~ the family of maps $f:\mathbb{R}/\mathbb{Z}\to\mathbb{R}/\mathbb{Z}$ defined by $f_{\alpha,\varepsilon}(x) = x+\alpha+\varepsilon\sin(2\pi x)$. ~~Let $\rho(\varepsilon,\alpha)$ be the~~ For $|\varepsilon| < \frac{1}{2\pi}$ these maps are diffeomorphisms of $\mathbb{R}/\mathbb{Z}$. Let $\rho(\varepsilon,\alpha) = \rho(f_{\alpha,\varepsilon})$.



For any fixed $\varepsilon = \varepsilon_0$, $\alpha \mapsto \rho(\varepsilon_0,\alpha)$ is an example of a Devil's staircase.

Arnold tongues.

Arnold tongues are dense.

$\rho(\varepsilon,\alpha)$ being rational corresponds to phase locking.