Our next topic is topological entropy.

This will be a real valued invariant of a dynamical system which measures the "amount of chaos".

For dynamical systems with Markov partitions the topological entropy should be the exponential growth rate of the number of words of length n.

$f: X \circlearrowleft$ $\qquad h(f) \overset{?}{=} \lim_{n\to\infty} \frac{\log \#\text{ of words of length } n}{n}$

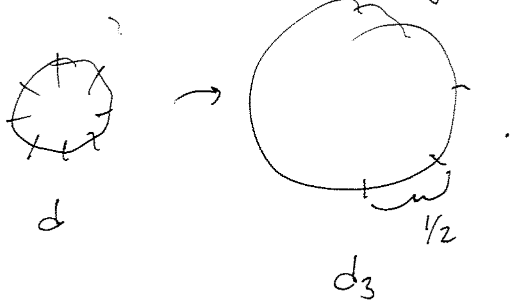
so $\#\text{ of words of length } n \approx \exp(n \cdot h)$
$\approx \exp(h)^n$.

Example:? If ~~we~~ our system is the shift map on 2 symbols $\sigma: \Sigma_2 \circlearrowleft$ then $\#\text{ of words of length } n = 2^n$ so we want $h(\sigma) = \lim_{n\to\infty} \frac{\log 2^n}{n} = \frac{n \log 2}{n} = \log 2$.

Similarly the entropy of the doubling map on the circle should be $\log 2$.

On the other hand we would like the entropy to be defined whether or not our dynamical system has a Markov partition.

We will proceed by defining a family of metrics d_n on X which reflect the dynamical behavior of f. In the case of the doubling map of the circle the metric d_n has the effect of stretching the circle so that the pieces of the Markov partition into words of length n have size $1/2$.



The number of words of length n will then correspond to the size of the metric space measured appropriately.

~~Now consider~~

To get an idea of how to define these metrics d_n consider the case of the shift map on Σ_2.

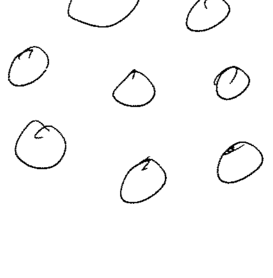
Consider words of length n

$.w_0 \dots w_{n-1}$

Say ω and ω' are sequences ~~where~~ which correspond to different words of length n. Then $d(\omega, \omega') = \max \frac{\delta(\omega_i, \omega_i')}{2^i}$ depends on the position of the symbol which is different. We would like d_n to treat each of these coordinates equally.

Define $d_n(\omega, \omega') = \max_{0 \le \ell \le n-1} d(f^\ell(\omega), f^\ell(\omega'))$.

With respect to this metric $d_n(\omega, \omega') = 1$ if ω and ω' correspond to different words and $d_n(\omega, \omega') \le \frac{1}{2}$ if the correspond to the same word. ~~Σ_2 is covered by 2~~ Σ_2 is covered by 2^n balls of radius $1/2$.



Let (X, d) be a compact metric space and $f: X \circlearrowleft$ a continuous map (possibly invertible).

We use f to define a family of new metrics on X.

Let

$$d_n(p, q) = \max_{0 \le l < n} d(f^l(p), f^l(q)).$$

There are several ways of measuring a metric space. Here is one.

Let $B(p, n, \varepsilon) = \{ q \in X : d_n(p, q) < \varepsilon \}$ be the ball of radius ε. A set E is an (n, ε) <u>spanning set</u> if $X \subset \bigcup_{p \in E} B(p, n, \varepsilon)$.

Let $S(n, \varepsilon)$ be the least cardinality of an (n, ε) spanning set.

Finite (n, ε) spanning sets exist since X is compact so $S(n, \varepsilon) < \infty$.

Define $h(f, \varepsilon) = \limsup_{n \to \infty} \frac{1}{n} \log S(n, \varepsilon)$.

The function $h(f,\varepsilon)$ is monotone in the sense that if $0<\varepsilon'<\varepsilon$ then $h(f,\varepsilon') \geq h(f,\varepsilon)$.

This follows from $S(n,\varepsilon') \geq S(n,\varepsilon)$ as every (n,ε') spanning set is also an (n,ε) spanning set.

Define $h(f) = \lim\limits_{\varepsilon \to 0} h(f,\varepsilon)$.

Monotonicity implies that the limit exists but it can be infinite. Since $S(n,\varepsilon) \geq 1$, $h(f,\varepsilon) \geq 0$ and $h(f) \geq 0$.

Examples:

Let $R_\alpha : \mathbb{R}/\mathbb{Z} \circlearrowleft$ be the rotation by α then $h(R_\alpha) = 0$.

Note that R_α preserves distances between points so $d_n(x,y) = \max_{0 \le i \le n-1} d(f^i(x), f^i(y)) = d(x,y)$.

Consequently $B(x,n,\varepsilon) = B(x,\varepsilon)$ for all n

Thus $S(n,\varepsilon)$ is independent of n. It follows that

$$h(f,\varepsilon) = \limsup_{n \to \infty} \frac{\log S(n,\varepsilon)}{n} = 0$$

and $h(f) = \lim_{\varepsilon \to 0^+} h(f,\varepsilon) = 0$.

Let $f : \mathbb{R}/\mathbb{Z} \circlearrowleft$ be defined by $f(x) = mx \mod 1$ where $m \ge 2$. $h(f) = \log m$.

Let $0 < \varepsilon < \frac{1}{2m}$ if $d(p,q) < \frac{1}{2m}$ then $d(f(p), f(q)) = m \cdot d(p,q)$.

So either $d_n(p,q) \ge \frac{1}{2m}$ or $d_n(p,q) = m^{n-1} \cdot d(p,q)$

The smallest (n,ε) spanning set consists of $\left[\frac{m^{n}}{\varepsilon}\right] + 1$ elements.

for $\varepsilon \le \frac{1}{2m}$

Then

$$h(f,\varepsilon) = \limsup_{n\to\infty} \frac{1}{n} \log S(n,\varepsilon) = \lim_{n\to\infty} \frac{1}{n} \log\left(\left[\frac{m^n}{\varepsilon}\right]+1\right)$$

$$= \lim_{n\to\infty} \frac{1}{n} \log\left(m^n \cdot \frac{1}{m^n}\left(\left[\frac{m^n}{\varepsilon}\right]+1\right)\right)$$

$$= \lim_{n\to\infty} \frac{1}{n} \log m^n + \lim_{n\to\infty} \frac{1}{n} \log\left(\frac{1}{m^n}\underbrace{\left(\left[\frac{m^n}{\varepsilon}\right]+1\right)}_{\leq m^n + 1}\right)$$

$$= \log m.$$

Thus $h(f) = \lim_{\varepsilon\to 0} h(f,\varepsilon) = \log m.$