Recall that if, for a lift F of an orientation preserving homeomorphism,

$$F^n(0) \geq k \text{ then } \rho(F) \geq \frac{k}{n},$$
$$F^n(0) \leq k \text{ then } \Rightarrow \rho(F) \leq \frac{k}{n}.$$

(We also implicitly assumed that $n \geq 0$.)

We will need an extension of this result:

Lemma. Let F be a lift of an orientation preserving homeo. of $\mathbb{R}/\mathbb{Z}$.

If $F^n(x) \geq k$ then $\begin{cases} \rho(F) \geq \frac{k}{n} & \text{if } n > 0 \\ \rho(F) \leq \frac{k}{n} & \text{if } n < 0. \end{cases}$

If $F^n(x) \leq k$ then $\begin{cases} \rho(F) \leq \frac{k}{n} & \text{if } n > 0 \\ \rho(F) \geq \frac{k}{n} & \text{if } n < 0. \end{cases}$

Proof. Say $F^n(x) \geq x+k$ then

$$F^n(x) \geq T^k(x)$$

$$F^n T^{-k}(x) \geq x$$

$$(F^n T^{-k})^p(x) \geq x$$

$$F^{np} T^{-kp}(x) \geq x$$

$$F^{np}(x) \geq T^{kp}(x)$$

$$\frac{F^{np}(x)}{np} \geq \frac{x+kp}{np}$$

$$\frac{F^{np}(x)-x}{np} \geq \frac{k}{n}$$

If $n>0$ then $\dfrac{F^{np}(x)-x}{np} \to \rho(F)$ as $p \to \infty$.

If $n>0$ but $F^n(x) \leq x+k$ then same argument gives $\rho(F) \leq \frac{k}{n}$.

If $n<0$ then let $z = F^n(x)$.

$$F^n(x) \geq x+k$$

$$z \geq F^{-n}(z)+k \qquad \text{where } -n>0.$$

$$F^{-n}(z) \leq z-k$$

$$\rho(F) \leq \frac{-k}{-n} = \frac{k}{n}.$$

Lemma (with altered notation).

Let f be a homeomorphism with $\rho(f) = \rho$ irrational. Let F be a lift of f with $\rho(F) = \rho_0$.

Let $G(x) = x + \rho_0$ so G is a lift of R_ρ. Pick $x_0 \in \mathbb{R}$.

Let $\Lambda_1 = \{ F^n(x_0) + m \, ; \, m, n \in \mathbb{Z} \}$

$$\Lambda_2 = \{ \underbrace{G^n(0) + m}_{= n\rho_0 + m} \, ; \, m, n \in \mathbb{Z} \}$$

Define $H : \Lambda_1 \to \Lambda_2$ by $H(F^n(x_0) + m) = n\rho_0 + m$.

Then H is bijective, strictly increasing and $H(x+1) = H(x) + 1$

$HF(x) = GH(x)$. $(= H(x) + \rho_0)$.

Proof of Lemma. Consider the maps

$L_1(m,n) = F^n(x_0)+m \qquad L_1 : \mathbb{Z}^2 \to \Lambda_1$

$L_2(m,n) = G^n(0)+m \qquad L_2 : \mathbb{Z}^2 \to \Lambda_2$.

If $L_1(m,n) = L_1(m',n')$ then $F^n(x_0)+m = F^{n'}(x_0)+m'$ (*)

so, reducing mod 1, $f^n(x_0) = f^{n'}(x_0)$ mod 1.

This gives $f^{n-n'}(x_0)=x_0$ and $n-n'=0$ since f has no periodic points. Follows from (*) that $m=m'$!

Same argument shows L_2 is injective so $H = L_2 \circ L_1^{-1}$ is injective.

Now take $x_1, x_2 \in \Lambda_1$ with $x_1 < x_2$.

$x_1 = F^{n_1}(x_0)+m_1 < F^{n_2}(x_0)+m_2 = x_2$ for some n_1, n_2, m_1, m_2.

Let $y = F^{n_2}(x_0)$. Then $F^{-n_2}(y) = x_0$ so

$$F^{n_1-n_2}(y) < y + m_2 - m_1 \qquad \text{(recall } F, T \text{ commute)}$$

If y were 0 then we could something about p_0 and hence G.

$$
\begin{tikzcd}[column sep=small]
 & F^{n_1-n_2}(y) < y + m_2 - m_1 \arrow[dl, "n_1-n_2>0"'] \arrow[dr, "n_1-n_2<0"] & \\
\rho(F) < \dfrac{m_2-m_1}{n_1-n_2} \arrow[dr] & & \rho(F) > \dfrac{m_2-m_1}{n_1-n_2} \arrow[dl] \\
 & (n_1-n_2)\rho_0 < m_2-m_1 &
\end{tikzcd}
$$

$$n_1\rho_0 + m_1 < n_2\rho_0 + m_2 .$$

5A

$$H\big(F(F^n(x_0)+m)\big) = H\big(F^{n+1}(x_0)+m\big) = G^{n+1}(0)+m$$
$$= G\big(G^n(0)+m\big) = G\big(H(F^n(x_0)+m)\big).$$

$$H\big((F^n(x_0)+m)+1\big) = H\big(F^n(x_0)+m+1\big) = G^n(x_0)+m+1$$
$$= H\big(F^n(x_0)+m\big)+1 .$$

Thm. If f is minimal it is conjugate to a rotation.

Poincarés

Proof of theorem. Since f is minimal f has no periodic points. Consequently ~~ρ~~ $\rho(f)$ is irrational. Let F be a lift of f, $x_0 \in \mathbb{R}$ and

$\Lambda_1 = \Lambda_{x_0} = \{ F^n(x_0) + m \}$

Let $\rho_0 = \rho(f)$. $\rho_0 \in \mathbb{R}$.

Let $\Lambda_2 = \{ n\rho_0 + m \} = \{ G^n(0) + m \}$ where $G(x) = x + \rho_0$.

By the Lemma we have an strictly ~~monotone~~ increasing ~~orientation~~ preserving map from $\Lambda_1 \subset \mathbb{R} \xrightarrow{H} \Lambda_2 \subset \mathbb{R}$.

We can extend H to a $\bar{H} : \mathbb{R} \to \mathbb{R}$ as follows.

If $v_0 \in \mathbb{R} - \Lambda_1$ let $v_0^- = \{ v < v_0,\ v \in \Lambda_1 \}$

$v_0^+ = \{ v > v_0,\ v \in \Lambda_1 \}$

Then every element of $H(v_0^-)$ is less than every element of $H(v_0^+)$ so $\sup(H(v_0^-)) \le \inf(H(v_0^+))$.

~~On the other hand~~ We can ~~clearly~~ extend H by sending v_0 to any element of $[\sup(H(v_0^-)), \inf(H(v_0^+))]$

On the other hand the union of $H(v_0^-)$ and $H(v_0^+)$ is $H(\Lambda_1) = \Lambda_2$ so it is dense. This means $\sup(H(v_0^-)) = \inf(H(v_0^+))$.

This extension is strictly increasing and hence injective. This follows from the denseness of Λ_1. If $\bar{H}(x) = \bar{H}(y)$ then choose $\lambda_1, \lambda_2 \in \Lambda_1$ with $x < \lambda_1 < \lambda_2 < y$. We have $\bar{H}(x) \le \lambda_1 < \lambda_2 \le \bar{H}(y)$.

An increasing function is continuous. (Check that the inverse image of an interval contains a non-trivial interval.)

Since $H(x+1) = H(x)+1$ and $H \circ F = G \circ H$ we have

(1) $\bar{H}(x+1) = \bar{H}(x)+1$ and $\bar{H} \circ F = G \circ \bar{H}$. An increasing function satisfying (1) is a lift of a homeomorphism. Let h be that homeomorphism. Reducing mod 1 we get $h \circ f = R_\rho \circ h$ so h is the conjugacy we are seeking.