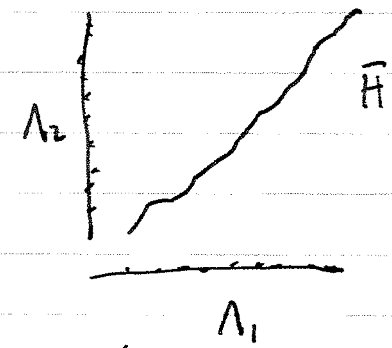
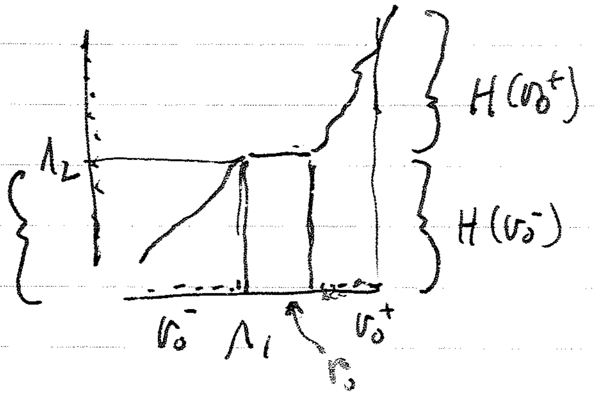
Comments on the proof of Poincaré's thm.

In how many ways is f conjugate to R_ρ?
Our choice of x_0 was arbitrary so ~~any~~ we could choose our topological conjugacy to take any point to 0. Once the image of 0 is determined the image of the orbit of zero is determined and this implies that the map is unique.



Both Λ_1 and Λ_2 are dense. $\bar H$ is invertible.



Λ_2 is dense but Λ_1 is not dense. $\bar H$ is not invertible but it is still a function.

We have proved a stronger theorem:

Define a semi-conjugacy to be a ~~map~~

$$\begin{tikzcd} X \arrow[r, "f^t"] \arrow[d, "h"'] & X \arrow[d, "h"] \\ Y \arrow[r, "g^t"] & Y \end{tikzcd}$$

a continuous function h such that

$h f^t = g^t h$. Here we don't require that h be

a homeomorphism.

Example: Consider a product of two dynamical systems $(f_1^t, f_2^t): X_1 \times X_2 \circlearrowleft$ then the projection onto the first factor is a semi-conjugacy.

Thm. If $f: \mathbb{R}/\mathbb{Z} \circlearrowleft$ has an irrational rotation number ρ then f is semi-conjugate to a rotation R_ρ.

Note that in order to define $\bar{H}$ we need the density of Λ_2 but not of Λ_1. The density of Λ_1 only comes in in showing that $\bar{H}$ is injective.

In this case the map ~~would~~ u would be monotone but not strictly increasing. Inverse images of points could be intervals.

Does this situation ever occur?

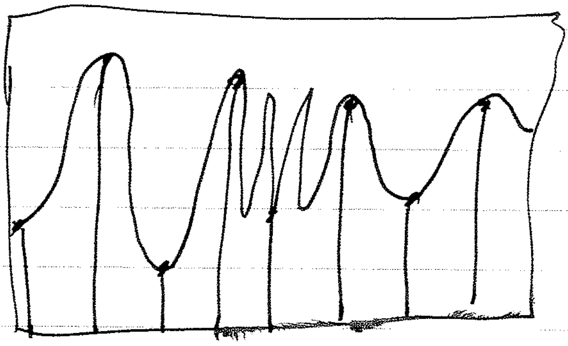
Our next objective is to establish a criterion for minimality due to Denjoy.

Definition. We define the variation of a continuous function $g : [0,1] \to \mathbb{R}$ to be

$$\mathrm{var}(g) = \sup \left\{ \sum_{i=0}^{n-1} \left| g(x_{i+1}) - g(x_i) \right| \; : \; 0 = x_0 < x_1 < \dots < x_n = \right\}$$

(n can be arbitrary).

We say that g has bounded variation if $\mathrm{var}(g) < \infty$.

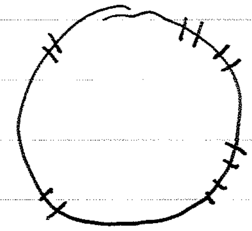


var g is the sum of the ~~jumps~~ size of the jumps.

You will show that if g is differentiable with continuous derivative then g has bounded variation.

Theorem (Denjoy) Let f be a circle homeomorphism with an irrational rotation number ρ. If $f \in C^1$ and $\log |f'|$ has bounded variation then f is topologically conjugate to R_ρ.

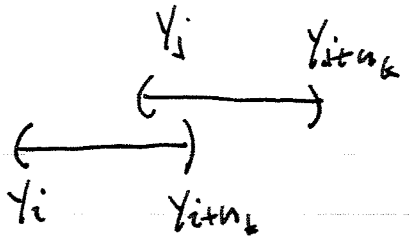
Lemma. Let f be a circle homeomorphism with an irrational rotation number. There is a sequence $n_k \to \infty$ such that for any $x \in \mathbb{R}/\mathbb{Z}$ the intervals (x_i, x_{i+n_k}) (where $x_i = f^i(x)$) for $0 \le i \le n_k$ are disjoint.



Proof. Let R_ρ be the rotation with the same rotation number as f. Let $y_k = R_\rho^k(0)$.

Let $n_0 = 1$ and define n_k recursively by

$$n_k = \min\{i \in \mathbb{N} : \mathrm{dist}(y_0, y_i) < \mathrm{dist}(y_0, y_{n_{k-1}})\}.$$



Say $i > j$

$$\begin{array}{ccc} \operatorname{dist}(y_j, y_i) & < & \operatorname{dist}(y_i, y_{i+n_k}) \\ \parallel & & \parallel \\ \operatorname{dist}(y_0, y_{i-j}) & & \operatorname{dist}(y_0, y_{n_k}) \end{array}$$

R_p is an isometry,

This would mean that y_{i-j} came back closer to y_0 than y_{n_k} but $0 < i-j < n_k$.

Say $i < j$

$$\begin{array}{ccc} \operatorname{dist}(y_j, y_{i+n_k}) & < & \operatorname{dist}(y_i, y_{i+n_k}) \\ \parallel & & \parallel \\ \operatorname{dist}(y_0, y_{i+n_k-j}) & & \operatorname{dist}(y_0, y_{n_k}) \end{array}$$

Since $0 < i+n_k-j < n_k$ y_{i+n_k-j} would be a closer return for a smaller index.

We conclude that the intervals are disjoint,
Now let f be any homeomorphism with an irrational rotation number ρ.
The monotonicity of the function H implies that for any $i, j, k \in \mathbb{Z}$, $f^i(x) \in [f^j(x), f^k(x)]$ if and only if $R_\rho^i(0) \in [R_\rho^j(0), R_\rho^k(0)]$. So the result holds for any homeomorphism and does not depend on the choice of x_0.

Lemma. Let f be a C^1 diffeomorphism of the circle with an irrational rotation number. If there is a sequence $n_k \to \infty$ and a $C > 0$ so that for all k and all $x \in \mathbb{R}/\mathbb{Z}$ we have

$$|(f^{n_k})'(x)| \cdot |(f^{-n_k})'(x)| > C$$

then f is minimal.

If f not minimal then for $\varepsilon > 0$ there is an N so that for $n \geq N$ there is an x_n s.t. $|(f^n)'(x_n)| \cdot |(f^{-n})'(x_n)| < \varepsilon$.

Proof. Suppose that there is a n_k and C as above but that f is not minimal. There is therefore an x so that $\overline{O(x)}$ is not all of $\mathbb{R}/\mathbb{Z}$. The complement of $\overline{O(x)}$ is a union of maximal intervals. The union of these intervals is invariant so if $f^j(I_n) \cap I_m \neq \emptyset$ then $I_n = I_m$. ~~A~~ A periodic interval would imply the existence of a periodic point. Thus all intervals are disjoint from their images.

Let I_0 be an maximal interval and let $I_n = f^n(I_0)$.

In particular $\sum_n |I_n| < 1$ so $|I_n| \to 0$ as $n \to ~~\pm\infty~~ +\infty$ or $n \to -\infty$.

Recall we have a sequence $n_k \to \infty$.

$$\frac{|I_{n_k}| + |I_{-n_k}|}{2} = \frac{1}{2}\int_{I_0} (f^n)'(z)\,dz + \int_{I_0} (f^{-n})'(z)\,dz \qquad (n = n_k)$$

$$= \int_{I_0} \frac{(f^n)'(z) + (f^{-n})'(z)}{2}\,dz$$

$$\geq \int_{I_0} \sqrt{|(f^n)'(z)\,(f^{-n})'(z)|}\,dz$$

$$\geq \sqrt{c}\,|I_0|.$$

But $|I_{n_k}| + |I_{-n_k}| \to 0$.