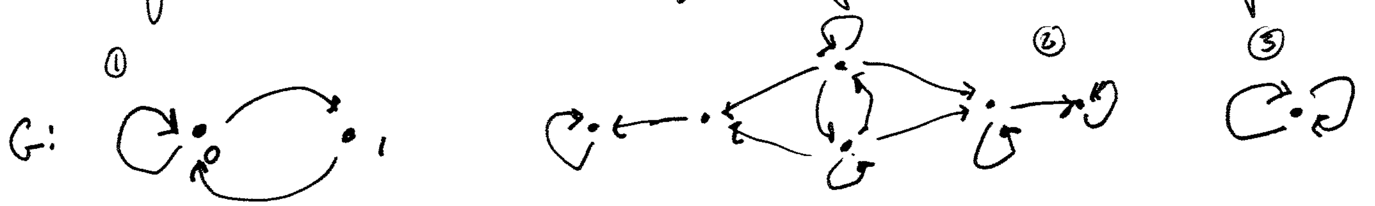
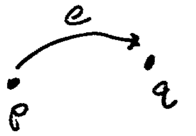
Definition. A directed graph G ~~is~~ consists of a finite set V of vertices and E ~~of edges~~ directed edges:



This data can be encoded by a pair of maps

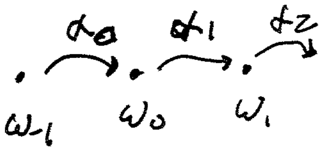
$h, t : E \to V$.



$h(e) = q$

$t(e) = p$.

We have seen examples before in connection with coding dynamical systems.



α_j goes from ω_{j-1} to ω_j

$h(\alpha_j) = \omega_j$

$t(\alpha_j) = \omega_{j-1}$

A path in G is a sequence of edges $\alpha = e_1 e_2 e_3 \ldots$ where $h(e_i) = e_{i+1}$.

Paths can be finite or ∞.

$\boxed{\alpha \text{ determines } \omega.}$

Let G_∞ be the set of bi-infinite paths in G. $= \{(\alpha_i)_{i=-\infty}^{\infty} : h(\alpha_j) = t(\alpha_{j+1})\}$.

Define the shift $\sigma : G_\infty \circlearrowleft$ by $\sigma(\omega) = \omega'$ where $\omega'_{i-1} = \omega_i$. $\sigma(\alpha) = \alpha'$ $\alpha'_{i+1} = \alpha_i$

Example ① above correspond to sequence of 0's and 1's ~~where~~ without consecutive 1's.

③ is another way of writing the 2-shift.

When edges are uniquely determined by their vertices it suffices to write the ∞ sequence of vertices ω. Otherwise we need to ~~to~~ write the sequence of edges α.

¶ Just as for n-shifts the space G_∞ has a metric:

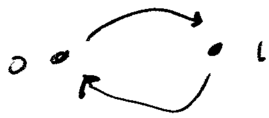
Let $\lambda > 1$. Let $\omega, \omega' \in G_\infty$ then

$$d_\lambda(\omega, \omega') = \cancel{\varepsilon(\omega_j, \omega'_j)} \max \frac{\varepsilon(\omega_j, \omega'_j)}{\lambda^{|j|}}.$$

Prop. G_∞ is compact and σ is continuous.

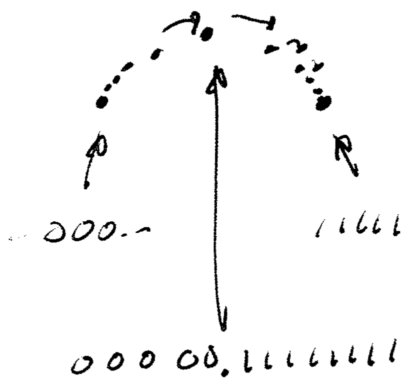
Examples: G_∞ might be empty:

G_∞ might be finite:



or

G_∞ might be countable and not have dense periodic points.



Cylinder sets:

Let ω be a word of length ℓ in G.

~~Let~~ Say $\omega = \omega_0 \ldots \omega_{\ell-1}$. Let $n < m$ satisfy $m = n + \ell$.

We define $C_\omega^{n,m} \subset G_\infty$ to be

$$C_\omega^{n,m} = \{\alpha' \in G_\infty : \alpha'_{n+j} = \omega_j \text{ for } j = 0 \ldots m-n-1\}.$$

The particular cylinder sets $C_\alpha^{-n,n}$ are the unit balls with respect to the metric d_λ.

Specifically if $\alpha = \alpha_{-n} \ldots \alpha_n$ is a finite word of length $2n+1$ and α' is an ∞ word with $\alpha'_j = \alpha_j$ for $-n \le j \le n$ then

$$B(\alpha', \tfrac{1}{\lambda^{(n+1)}}) = C_\omega^{-n,n}.$$

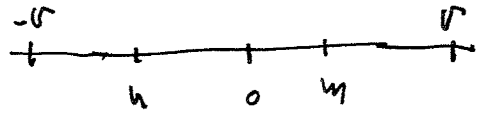
In particular any point of the cylinder set serves as a center of the ball.

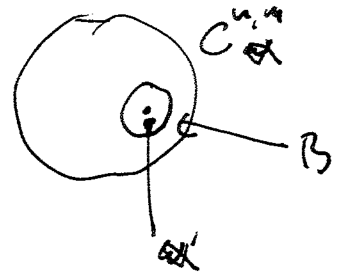
Cor. The topology on G_∞ is independent of λ.

Prop. $C_\alpha^{n,m}$ is open, ~~and closed~~.

Proof. (open) Let $r > |n|, |m|$. Then $\alpha' \in C_\alpha^{n,m}$ satisfies

$\alpha' \in B(\omega', \frac{1}{\lambda^{|r|+1}}) \subset C_\alpha^{n,m}$





Prop. $C_\alpha^{n,m}$ is closed.

Proof. Let $\alpha = \alpha_0, \alpha_1, \dots, \alpha_k$ be the words of length $l = m-n$.

Then $G_\infty = \cancel{\bigcup_{\alpha}} \bigcup_{\alpha_j} C_{\alpha_j}^{n,m}$ and these sets are disjoint. But each $C_{\alpha_j}^{n,m}$ is open so $(C_\omega^{n,m})^c$ is a union of finitely many open sets.

Cor. G_∞ is totally disconnected.

Cor. If G_∞ ~~is~~ contains no isolated points then it is homeomorphic to a Cantor set.

Proof. Any compact perfect ~~metric space~~ totally disconnected metric space is homeomorphic to the Cantor set.

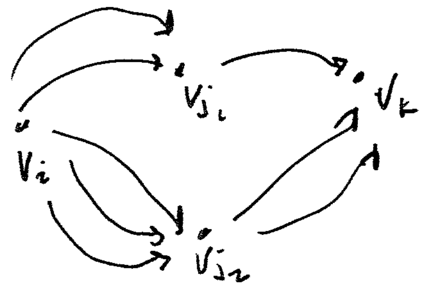
To each directed graph G we can associate a transition matrix A.

If n is the # of vertices then then A is $n \times n$.

a_{ij} ~~fay~~ we number the ~~ass~~ vertices $v_1 \dots v_n$ then let a_{ij} = # of edges from v_i to v_j.

Prop. A^n counts the number of paths of length n.

Proof. Consider paths of length 2.



The number of paths from v_i to v_k is the sum over all j of the product of the number of paths from v_i to v_j times the number of paths from v_j to v_k. That is $\sum_j a_{ij} a_{jk} = [A^2]_{ik}$.

In general the # of paths of length n from v_i to v_k is $\sum_{j_1 \dots j_{n-1}} a_{ij_1} \cdots a_{j_{n-1}k} = [A^n]_{ik}$.

Cor. The number of ~~periodic~~ fixed points of σ^n is Trace A^n. $\sum [A^n]_{ii}$ where ~~is~~ $[A^n]_{ij}$ counts paths of length n from i to j.

Remarks: If A has a unique largest eigenvalue λ_1 then

Trace $A^n = \lambda_1^n + \lambda_2^n + \dots \lambda_k^n \sim \lambda_1^n$.

Example: $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ eigenvalues $\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$

$\lambda_1, \lambda_2 \quad |\lambda_1| > 1 \quad |\lambda_2| < 1.$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$Tr(A^n) = \lambda_1^n + \lambda_2^n$$
$$\lambda_2^n \to 0$$

$$Tr(A^n) \approx \lambda_1^n.$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$$

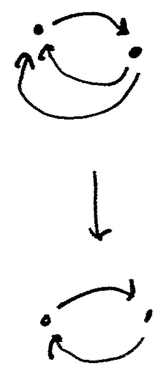
Entries correspond to Fibonacci numbs

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ Eigenvalues $1, -1$

$$Tr(A^n) = 1^n + (-1)^n \begin{cases} = 0 & \text{if } n \text{ even} \\ = 2 & \text{if } n \text{ odd.} \end{cases}$$

~~Perron-Frobenius~~

Example:



Every loop has even length.

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \quad A^2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad A^3 = \begin{bmatrix} 0 & 2 \\ 4 & 0 \end{bmatrix}$$

$a_{n,m}$ counts the number of paths from n to m. The number of loops of length n is

$\sum a_{n,n} = Tr(A)$.

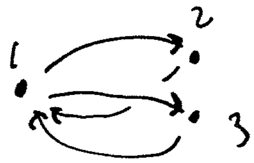
Definition. A transition matrix is ~~aperiodic~~ <u>irreducible</u> if for any i, j there is an n so that $(A^n)_{ij} > 0$. A transition matrix is aperiodic if there is an n such that $(A^n)_{ij} > 0$ for all i, j.

Example:  aperiodic.

 not irreducible

 irreducible but not aperiodic.

Exercise: If A_G is ~~aw~~ irreducible then periodic points are dense and there is a dense orbit. G_∞ is a Cantor set.



$$\begin{array}{c c} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array} = A$$

$$A^2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

For each i, j there is some n with $[A^n]_{ij} \neq 0$. but there is no n so that all entries of A^n are non-zero.

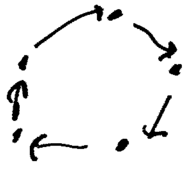
Proposition. If G is a directed graph then we can choose an ordering on the vertices so that A_G has block upper triangular form where each block corresponds to a transitive subgraph or the 0 block.

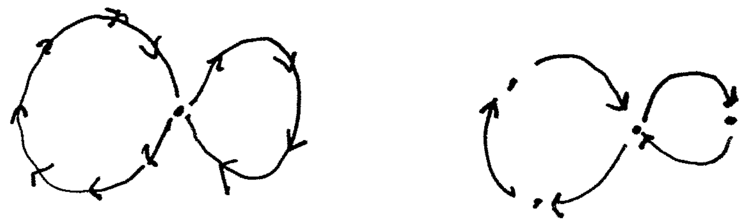
Definition. A graph map from a directed graph G to a directed graph G' is a function f from the vertices of and edges of G to the vertices an edges of G' so that if e is an edge of G from v_1 to v_2 then $f(e)$ is an edge from $f(v_1)$ to $f(v_2)$.

Proposition. If f is a graph map from G to G' then f induces a semi-conjugacy from $\sigma : G_\infty \circlearrowleft$ to $\sigma : G'_\infty \circlearrowleft$.

Proposition. If G is transitive but not aperiodic then there is a

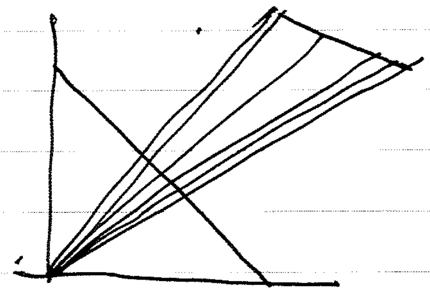
from G to the cyclic graph G' with $m \geq 1$ vertices.

$G' =$ 

Example: 

join a ~~loop of~~ of length n and a loop of length m.

Peron-Frobenius. If A is a non-negative aperiodic irreducible matrix then the largest eigenvalue of A is real and positive and of multiplicity 1. The corresponding eigenvector has positive entries.



Picture of proof in dim 2:

Define a map from the interval to the interval by mapping and projecting.

~~a fixed~~ A map from the interval into itself has a fixed point. A fixed point of this map corresponds to an eigenvector.