[illegible]

The next Topic is a second example of a hyperbolic diffeomorphism of a surface exhibiting chaotic behavior. This example has a different flavor from the horseshoe. Suggesting that the notion of hyperbolicity has a range of applications.

Hyperbolic automorphisms of the torus.

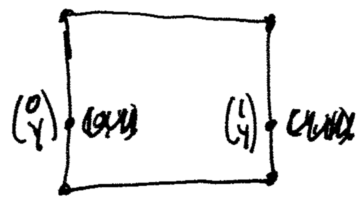
Let $\mathbb{T}^2$ be the ~~quotient~~ coset space $\mathbb{R}^2/\mathbb{Z}^2$.

An element of $\mathbb{T}^2$ is an equivalence class

$\left\{ \cancel{(x+n, y+m)} \begin{pmatrix} x+n \\ y+m \end{pmatrix} : n, m \in \mathbb{Z} \right\}$. Let $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ be the projection.
Write $\pi\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$.

Any such equivalence class has a representative in the square $[0,1]\times[0,1]$.

Certain points have more than one representative.

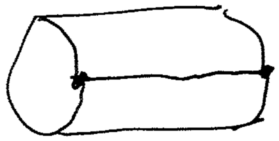


$\cancel{(0,y) \sim (1,y)}$ $\quad \begin{pmatrix} 0 \\ y \end{pmatrix} \sim \begin{pmatrix} 1 \\ y \end{pmatrix}$

$\cancel{(x,0) \sim (x,1)}$ $\quad \begin{pmatrix} x \\ 0 \end{pmatrix} \sim \begin{pmatrix} x \\ 1 \end{pmatrix}$

$\cancel{(0,0) \sim (0,1) \sim (1,0) \sim (1,1)}$

$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.



Note that $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2 = (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$. So $\pi\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x \bmod \mathbb{Z} \\ y \bmod \mathbb{Z} \end{pmatrix}$

Prop. A 2×2 matrix A with integral entries induces a linear or affine map from $\mathbb{T}^2$ to $\mathbb{T}^2$.

$f(x,y) = (ax+by, cx+dy) \bmod \mathbb{Z}$ |

$$\begin{tikzcd} \binom{x}{y} & \mathbb{R}^2 \arrow[r] \arrow[d, "\pi"] & \mathbb{R}^2 \arrow[d, "\pi"] & A\binom{x}{y} \\ \binom{\bar x}{\bar y} & \mathbb{T}^2 \arrow[r] & \mathbb{T}^2 & A\binom{x}{y} \bmod \mathbb{Z}^2 \end{tikzcd}$$

~~Proof.~~ We use the formula $f_A(\bar x, \bar y) = \pi A\binom{x}{y}$.

where $\binom{x}{y} \in \mathbb{R}^2$ and $\pi\binom{x}{y} = \binom{\bar x}{\bar y}$. makes sense

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\binom{x}{y}$

Need to check that

that if $\pi\binom{x}{y} = \pi\binom{x'}{y'}$ then $\pi A\binom{x}{y} = \pi A\binom{x'}{y'}$.

Now $\pi\binom{x}{y} = \pi\binom{x'}{y'}$ means $\binom{x}{y} - \binom{x'}{y'} \in \mathbb{Z}^2$ and

$A(\mathbb{Z}^2) \subset \mathbb{Z}^2$ so $A\left(\binom{x}{y} - \binom{x'}{y'}\right) \in \mathbb{Z}^2$ and $A\binom{x}{y} - A\binom{x'}{y'} \in \mathbb{Z}^2$.

Continuity of f follows from the continuity of the linear map A: If $p_n \to p$ in $\mathbb{T}^2$ then we can find $q_n \to q$ in $\mathbb{R}^2$ with $\pi(q_n) = p_n$ and $\pi(q) = p$. Since $A(q_n) \to Aq$ we have $f(p_n) \to f(p)$.

Cor. If A is an integral matrix with determinant ± 1 then f_A is invertible.

Proof. In this case A^{-1} has integral entries since $A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. so $f_{A^{-1}} \circ f_A = f_{A^{-1}A} = f_I = I_{\mathbb{T}^2}$.

Write elts of $\mathbb{R}^2$ as $\binom{x}{y}$.

~~$f_A(\bar x, \bar y) = A$~~

~~Write vectors~~ $\cancel{\pi\left(\binom{x}{y}\right) = (x,y) \bmod \mathbb{Z}^2}$.

$\cancel{f(\bar x, \bar y) = \pi A\binom{x}{y}}$

$$f_A\left(\binom{\bar x}{\bar y}\right) = \pi\left(A \cdot \binom{x}{y}\right).$$

Def. We say that a matrix A is hyperbolic if it has no eigenvalue on the unit circle.

If $A : V \circlearrowleft$ is hyperbolic then there are invariant subspaces V^u and V^s of V so that $V^u \cap V^s = \{0\}$, $V^u + V^s = V$ and

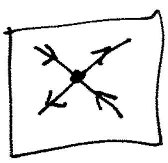
$$\lim_{n\to\infty} |A^n(v)| \to 0 \qquad \lim_{n\to-\infty} |A^n(v)| \to \infty \qquad \text{for } v \in V^s$$

$$\lim_{n\to\infty} |A^n(v)| \to \infty \qquad \lim_{n\to-\infty} |A^n(v)| \to 0 \qquad \text{for } v \in V^u.$$

In the particular case that A is 2×2 with $\det = \pm1$ and A is hyperbolic A has eigenvalues λ^u, λ^s with $|\lambda^u| > 1$, $|\lambda^s| < 1$. V^u is the eigenspace for λ^u, V^s is the eigenspace for V^s.

In general V^s and V^u are sums of generalized eigenspaces.

The image of V^u and V^s are the unstable and stable manifolds of 0.



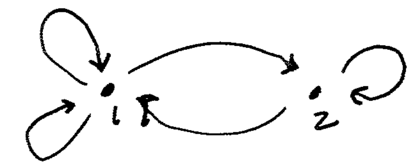
Tr $= 3$ det $= 1$
$\lambda^2 - \mathrm{Tr}\cdot\lambda + \det = 0$

2nd example of "hyperbolic behavior". If p, q suff close then $W^s_{loc}(p) \cap W^u_{loc}(q) =$ pt.

Example: $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. $\quad \lambda = \dfrac{3 \pm \sqrt{5}}{2}$.

$\lambda^u = \dfrac{3+\sqrt{5}}{2} \quad \lambda^s = \dfrac{3-\sqrt{5}}{2} \quad |\lambda^u| > 1 > |\lambda^s|$.

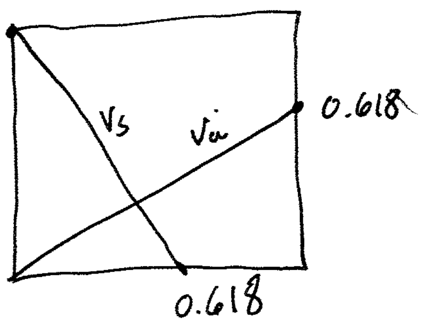
Thm. Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ then $f_A : \mathbb{T}^2 \circlearrowleft$ is ~~Topologically equivalent~~ semi-conjugate to ~~a~~ the topological Markov chain $\sigma : G_\infty \circlearrowleft$ where

G: 

Proof. 0 is a fixed point for f_A.
We will use the local stable and unstable manifolds of 0 to divide $\mathbb{T}^2$ into two rectangles.

$V_u = \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix}$. Rescale to $\begin{pmatrix} \phi \\ 0.618.. \end{pmatrix}$.

$V_s = \begin{pmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{pmatrix}$. Rescale to $\begin{pmatrix} -0.618.. \\ 1 \end{pmatrix}$

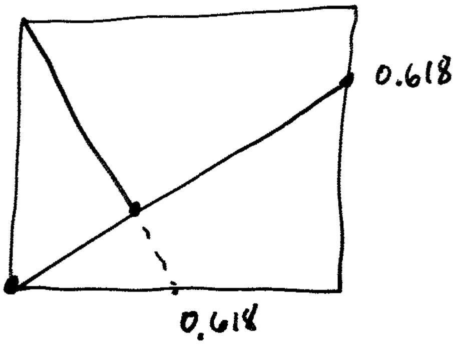


Recipe for Markov partition.

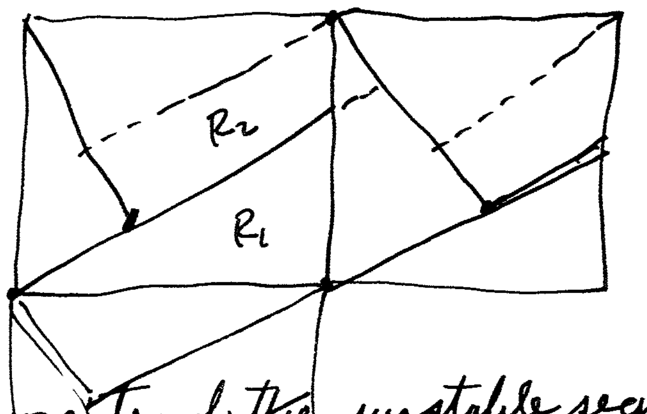
Expanding eigenvector: $\begin{pmatrix}\frac{1+\sqrt{5}}{2}\\ 1\end{pmatrix}$ Rescale $\begin{pmatrix}1\\ 0.618034\end{pmatrix}$

Contracting eigenvector: $\begin{pmatrix}\frac{1-\sqrt{5}}{2}\\ 1\end{pmatrix}$ $\begin{pmatrix}-0.6180\\ 1\end{pmatrix}$ $\begin{pmatrix}0.618\\ -1\end{pmatrix}$

Slope of the expanding eigenvector is positive but less than 1. Take the branch in the first quadrant and extend it across the square.



The contracting eigenvalue has negative slope. Extend it until it hits the ~~ef~~ expanding segment



Now extend the unstable segment in both directions until it hits the stable segment. Call the resulting rectangles R_1 and R_2. The image of union of R_1 and R_2 is ~~the~~ all of $\mathbb{T}^2$. The images of the boundaries are not disjoint.

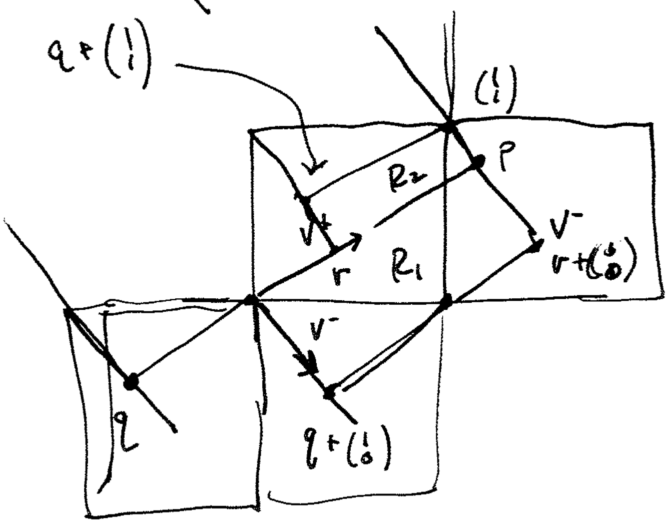
~~Let $v^+ = \begin{pmatrix} \\ \end{pmatrix}$~~

The eigenvectors of A are

~~$\frac{1}{2}(1+$~~ $\begin{pmatrix}\frac{1+\sqrt{5}}{2}\\ 1\end{pmatrix}$ and $\begin{pmatrix}\frac{1-\sqrt{5}}{2}\\ 1\end{pmatrix}$

$$\frac{1+\sqrt{5}}{2}\cdot\frac{1-\sqrt{5}}{2} = \frac{1-5}{4} = -1$$

$\begin{pmatrix}\cancel{\phi}\\ 1\end{pmatrix}$ $\begin{pmatrix}1.618..\\ 1\end{pmatrix}$ $\begin{pmatrix}-0.618..\\ 1\end{pmatrix}$



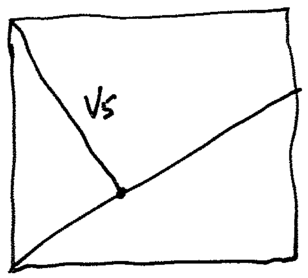
Eigenvectors give local stable and unstable manifolds of 0.

~~Note that the boundaries~~

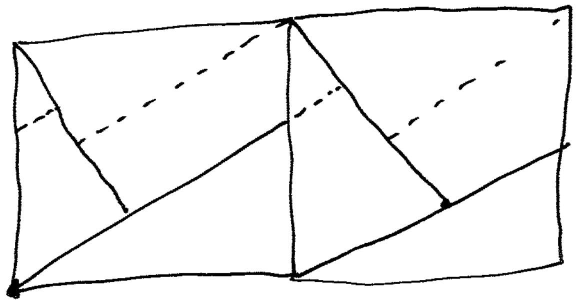
$R_1 \cup R_2$ is the whole torus.

Now ~~Note that~~

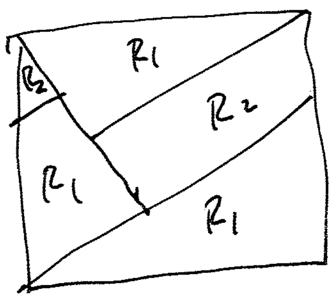
Let ∂^+R_1, ∂^+R_2 be the pieces of the boundaries which intersect the unstable manifolds. Let $\partial^-R_1, \partial^-R_2$ be the intersections of boundaries with stable manifolds. Then $f(\partial^-R_1 \cup \partial^-R_2) \subset \partial^-R_1 \cup \partial^-R_2$ and $f^{-1}(\partial^+R_1 \cup \partial^+R_2) \subset \partial^+R_1 \cup \partial^+R_2$. This is the Markov condition in the invertible case.



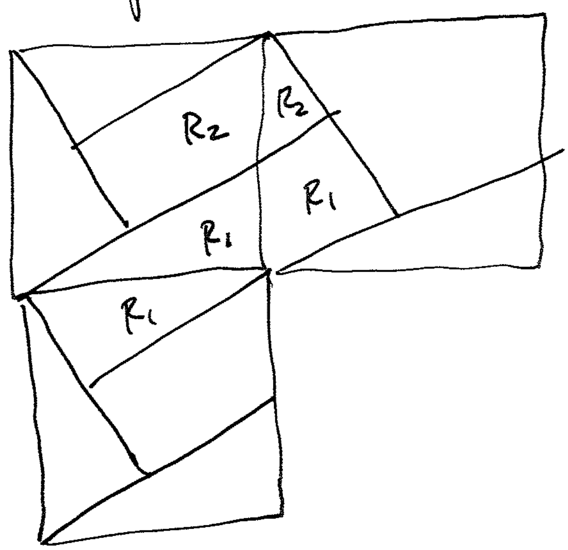
Cut off V_5 at the intersection point.
Now extend V_4 in both directions until it hits V_5 again.



This divides T^2 into two rectangles R_1 and R_2.



or



This picture shows that R_1 and R_2 fill T^2.

This picture shows that R_1 and R_2 are rectangles.

Keep track of the image as follows:

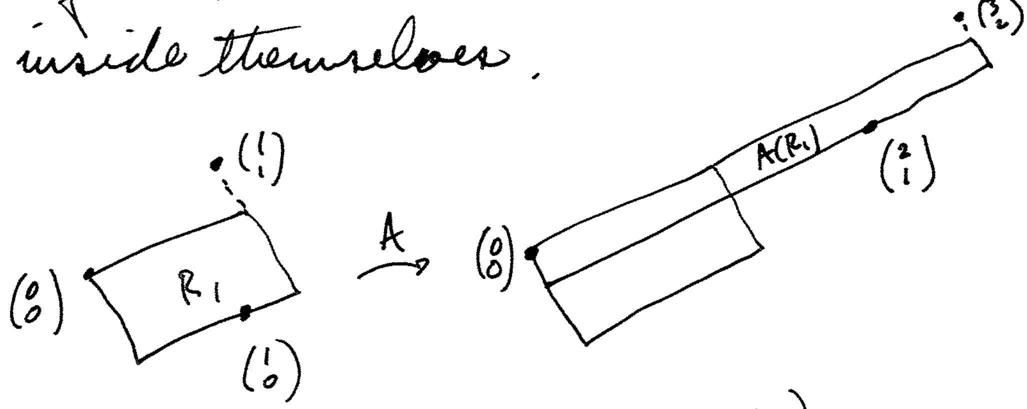
Stable boundaries

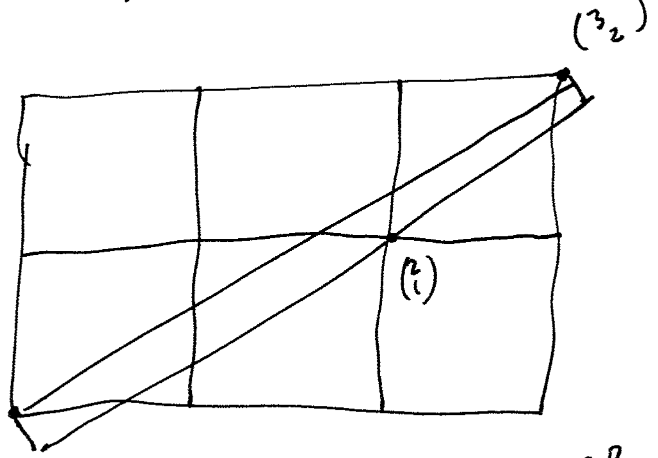
Keep track of the image of R_1. The top boundary is part of the unstable manifold of $\binom{0}{0}$. This gets mapped to a longer part of the unstable manifold of $\begin{pmatrix}2&1\\1&1\end{pmatrix}\binom{0}{0}=\binom{0}{0}$. The r.h. stable boundary is part of the stable manifold of $\binom{1}{1}$. This gets mapped to part of the stable manifold of $\begin{pmatrix}2&1\\1&1\end{pmatrix}\binom{1}{1}=\binom{3}{2}$.

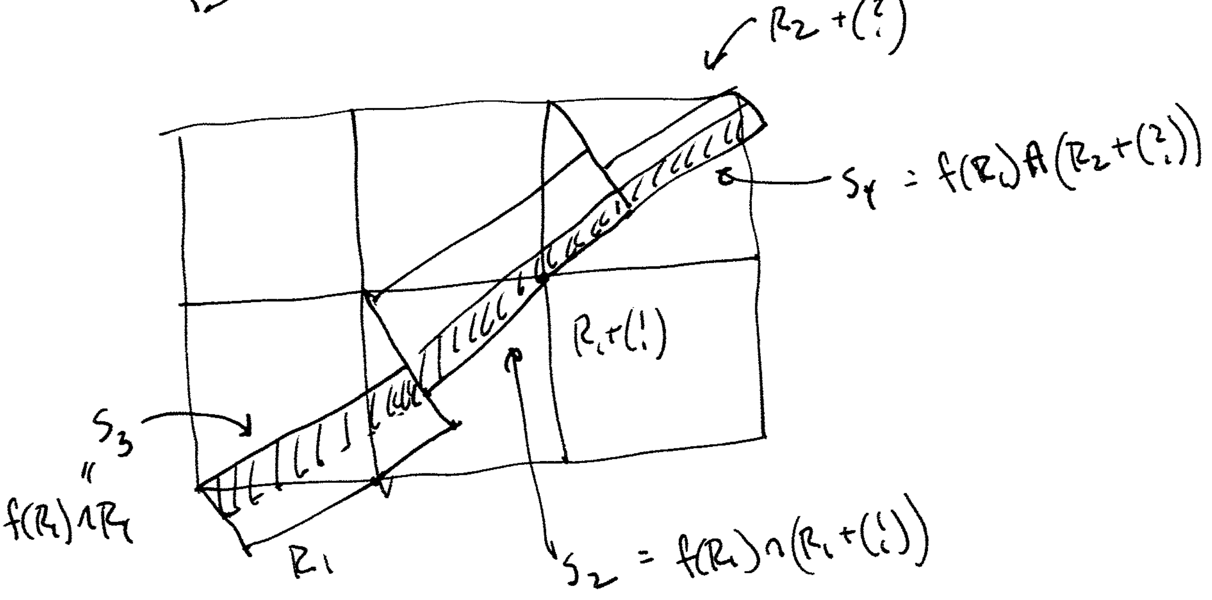
The ~~bottom~~ unstable boundary of R_1 goes through $\binom{1}{0}$. This gets mapped to an unstable segment through $\begin{pmatrix}2&1\\1&1\end{pmatrix}\binom{1}{0}=\binom{2}{1}$.

Left hand stable segment gets mapped into itself.

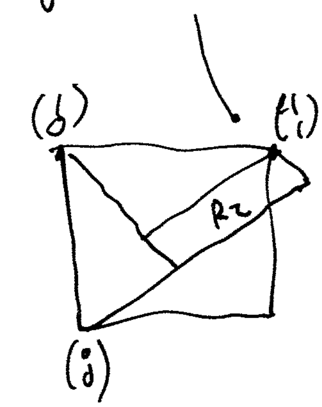
We will now calculate the images of R_1 and R_2 in $\mathbb{R}^2$. We use 3 facts. A fixes $\binom{0}{0}$ and takes integral points to integral points. A takes unstable segments outside themselves and stable segments inside themselves.

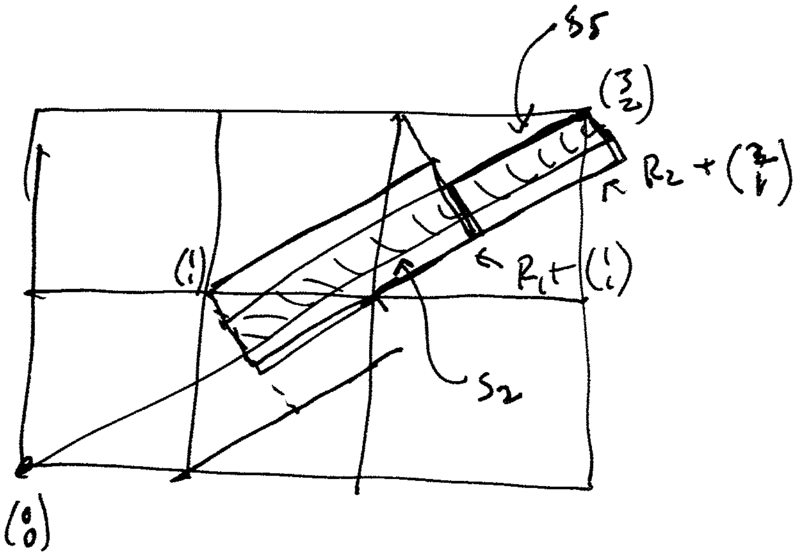


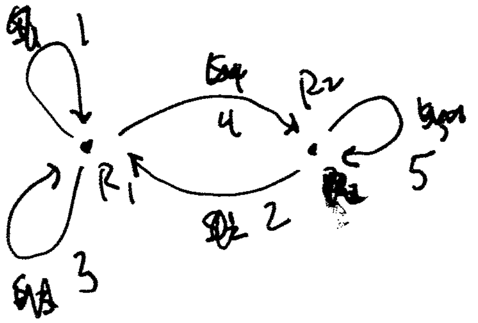




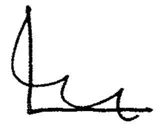
If

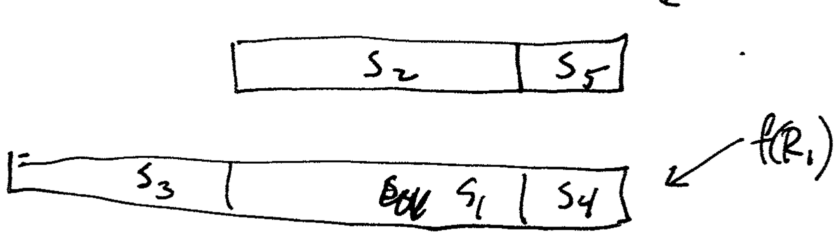




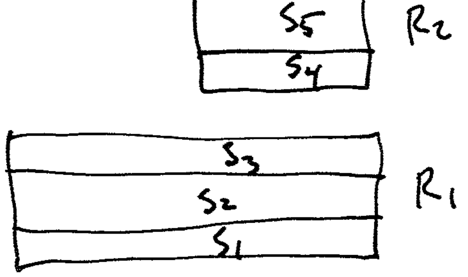


Write the images of R_1 and R_2 horizontally:





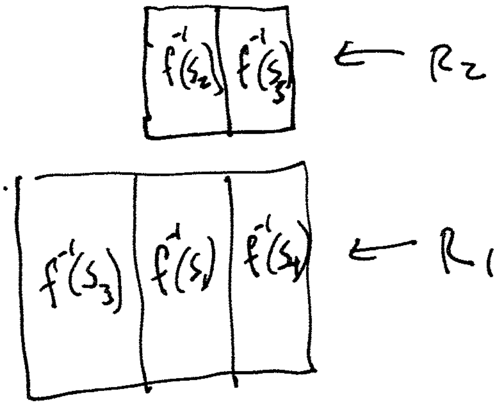
Arrange these rectangles inside R_1 and R_2



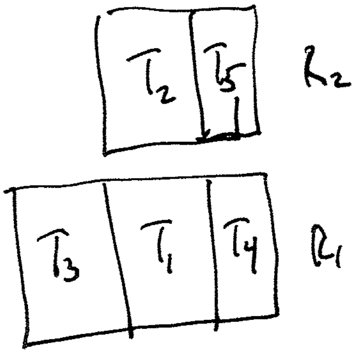
~~Part~~ The S_j give a partition of R_i's based on the location of $f^{-1}(p)$.

Corresponds to arrows landing at R_1, R_2.

Apply f^{-1} to previous picture



Let T_j denote $f^{-1}(S_j)$ so $S_j = f(T_j)$.



The T's give a partition of the R's based on the location of $f(p)$.

Partition of R_j corresponds to arrows leaving R_j.

Note that the S's have full width in the R's and the T's have full height.

Coding of orbits:

Coding of orbits by paths in G:

Say $p \in \mathbb{T}^2$. We set

$$\cdots \omega_{-3} \xrightarrow{\alpha_{-2}} \omega_{-2} \xrightarrow{\alpha_{-1}} \omega_{-1} \,\cdot \xrightarrow{\alpha_0} \omega_0 \xrightarrow{\alpha_1} \omega_1 \xrightarrow{\alpha_2} \omega_2 \cdots$$

Recall our conventions. A path in G is given by as above ~~where ω_j's~~ ω's correspond to vertices and α's correspond to edges.

Let $p \in \mathbb{T}^2$. ~~Set ω_j~~ Define $\omega_j = \omega_j(p)$ and $\alpha_j = \alpha_j(p)$ as follows.

$$\omega_j(p) = \begin{cases} 1 \\ 2 \end{cases} \text{if } f^j(p) \in \begin{cases} R_1 \\ R_2 . \end{cases}$$

$$\alpha_j(p) = \begin{cases} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{cases} \text{if } f^j(p) \in \begin{cases} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{cases} \text{or (equiv.) } f^{j-1}(p) \in \begin{cases} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 . \end{cases}$$

$\alpha_j(p)$ determines $\omega_{j-1}(p)$, $\omega_j(p)$

Applying f to p shifts the sequence to the left.

Now say that we have a finite word

$$w = \omega_j \overset{\alpha_{j+1}}{\curvearrowright} \cdots \omega_{-1} . \omega_0 \cdots \overset{\alpha_k}{\curvearrowright} \omega_{k} \qquad j \le 0 \quad k \ge 0,$$

w determines a cylinder set $C(w)$ in G_∞ of infinite words which agree with w where w is defined.

w determines a set $B(w)$ in $\mathbb{T}^2$ of points whose codings agree with the entries of w where they are defined.

Example:

$w = .1$	$B(w) = R_1$
$w = .2$	$B(w) = R_2$
$w = 1 \overset{1}{\curvearrowright} .1$	$B(w) = S_1$
$w = 2 \overset{2}{\curvearrowright} .1$	$B(w) = S_2$
$w = .1 \overset{1}{\curvearrowright} 1$	$B(w) = T_1$
$w = .2 \overset{2}{\curvearrowright} 1$	$B(w) = T_2$.

The strategy for the construction of the semi-conjugacy h is the same as that for the horseshoe. We will show that ~~as~~ ~~$i \to \infty, j \to -\infty$~~ ~~the set $B(w)$~~ $B(\bar{w})$ is a rectangle and as $i \to \infty$, $j \to -\infty$ the height and width of $B(\bar{w})$ goes to 0. We then define $h(\bar{w})$ as an intersection of a nested sequence of rectangles.

Let $\vec{\omega} = \omega_j \ldots \omega_{-1} . \omega_0 \ldots \omega_k \qquad j \le 0,\ k \ge 0$

be a word. Note that the ω_0 location is always included, $|j|$ is the number of specified positions to the left of 0, k is the number of specified positions to the right of 0.

Let h_j be the width of R_j and w_j be the height of B_j.

Claim. $B(\vec{\omega})$ is a rectangle with height

$$\frac{h_{\omega_j}}{\lambda^{|j|}} \text{ and width } \frac{w_{\omega_k}}{\lambda^k}.$$

Note that the height and width depend only on the ω's not the α's.

Note that the claim implies that if $j=0$ then $B(\vec{\omega})$ has full height and if $k=0$ then $B(\vec{\omega})$ has full width.

$S_2 = B(\overset{2}{2.1}) \qquad k=0 \ j=-1 \qquad S_2$ has full width

$T_2 = B(.\overset{2}{21}) \qquad j=0 \ k=1 \qquad T_2$ has full height.

To find the width of T_2 we can apply f. f multiplies widths by λ^u. $f(T_2) = S_2$ has width w_1. So T_2 has width $\frac{w_1}{\lambda}$ as was predicted by the claim.

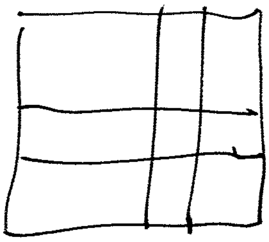
Proof of the claim by induction on the length of the word. If the word has length 1 then the claim is true.

If the word ~~is~~ has length 2 say $.\overset{2}{\vec{2}}1$ then it has full ~~leng~~ height h_2. If we shift left ~~then~~ we get $\overset{2}{\vec{2}}.1$ corresponds to shrinking vertically by λ so the height is $\frac{h_2}{\lambda}$ as predicted.
$j=-1$

$$\frac{h_2}{\lambda^{|j|}} = \frac{h_2}{\lambda}$$

If the assertion is true for words of length $n-1$ and we have a word $\omega_j \ldots \omega_{-1}.\omega_0 \ldots \omega_k$ with ~~$i \geq 1$~~ $i \leq -1$ $|j|$ $k \geq 1$
$i \geq 1$ then we consider the words

$\omega_{-j} \ldots \omega_{-1}.\omega_0$ and $.\omega_0 \ldots \omega_k$. These words are shorter. One corresponds to a rectangle of full width, the other to a rectangle of full height.



The intersection is again a rectangle. ~~Its height and width~~ Its height corresponds to the word $\omega_{-j} \ldots \omega_0$. Its width is the width of $.\omega_0 \ldots \omega_k$.

If we have a word with say $k=0$

$w_{-j} \dots w_{-1} \xrightarrow{\alpha_{-1}} w_0$ then it is contained in a rectangle $S_{\alpha=1}$. If we apply f^{-1} we shift the word right. ~~and we~~ $f^{-1}|S$ multiplies widths by λ and contracts heights by λ. Furthermore we can analyze this set by means of the previous argument.