Entropy of a Topological Markov chain.

Let G be a directed graph. We want to show that the entropy of the shift $\sigma : G_\infty \circlearrowleft$ is given by the ~~growth~~ log of the growth rate of the number of words of length n.

For this discussion I will make the technical hypothesis that each ~~node of~~ vertex of G has an entering edge an an exiting edge. We will call such a G nondegenerate.

~~The non-degenerate~~

degenerate G

Let $W_n(G)$ be the set of paths of length n in G also call words. If G is non-degenerate then every $\alpha \in W_n(G)$ is contained in ~~an~~ a bi-infinite word and in particular $G_\infty \neq \phi$.

If l is the # of vertices in G then $A = A_G = (a_{ij})$ is the $l \times l$ matrix where a_{ij} is the number of edges from v_i to v_j.

If we write $A^n = (a^n_{ij})$ then a^n_{ij} is the number of paths of length n from v_i to v_j.

Thus
$$\# W_n(G) = \sum_{i,j} a^n_{ij}.$$

Define a norm on the vector space of $l \times l$ matrices
$$\|M\| = \sum_{i,j} |m_{ij}|.$$

So
$$\# W_n(G) = \|A^n\|.$$

The spectral radius thm. says that
$$\lim_{n \to \infty} \|A^n\|^{1/n} = \rho(A)$$

where $\rho(A) = \max_i |\lambda_i|$ and $\lambda_i \in \mathbb{C}$ ranges over the complex eigenvalues of A ie the solutions of $Av = \lambda v$ for $\lambda \in \mathbb{C}$ and $v \in \mathbb{C}^l$.

~~One consequence~~ This result is independent of our choice of norm.

Now taking logs of both sides we have

$$\log\left(\lim_{k\to\infty} \|A^n\|^{1/n}\right) = \log(\rho(A))$$

$$\lim_{k\to\infty} \frac{\log\|A^n\|}{n} = \log(\rho(A)).$$

Thus $$\lim_{k\to\infty} \frac{\log \# W_n(G)}{n} = \log(\rho(A_G)).$$

Proposition. Let G be a non-degenerate graph then and $\sigma : G_\infty \circlearrowleft$ the shift Then:

$$h(\sigma) = \log(\rho(A_G)).$$

Remarks. We can consider 1-sided or 2-sided shifts and entropy is the same in either case. We will give the proof in the 2-sided case.

Proof. Let $\lambda > 1$. Consider the metric d_λ on G_∞.

$$d_\lambda(\alpha, \alpha') = \max \frac{\varepsilon(\alpha_i, \alpha_i')}{|\lambda|^i}$$

Pick $k \geq 1$. We wish to calculate $h(\sigma, \varepsilon)$ where $\varepsilon = 1/\lambda^k$.

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Let $\alpha_{-k} \dots \alpha_k \in W_{2k+1}(G)$.

Let $C_{-\alpha_k \dots \alpha_k} = \{ \alpha' \in G_\infty : \alpha'_j = \alpha_j \quad -k \le j \le k \}$.

~~Then $B(\alpha, k, \lambda^{-k}) = C_{\alpha_{-k} \dots \alpha_k}$~~ $\quad \lambda^{-k-1} < \varepsilon < \lambda^{-k}$

$\cancel{B(\alpha, \lambda^{-k})} \quad B(\alpha, \varepsilon) = C_{-\alpha_k \dots \alpha_k}$.

~~Case~~

Let $d_n(\alpha, \alpha') = \max_{0 \le \ell \le n} d_\lambda(\sigma^\ell(\alpha), \sigma^\ell(\alpha'))$

$$\begin{array}{cccc} -k & 0 & -k & k+n \\ [& &][&] \end{array}$$

$B(\alpha, n, \varepsilon)$ is the cylinder set $C_{-\alpha_k \dots \alpha_{k+n}}$.

$\lambda^{-k-1} < \varepsilon < \lambda^{-k}$.

As $-\alpha_k \dots \alpha_{k+n}$ ranges over the words in W_{2k+n+1} we get ~~an ε covering~~ a covering of G_∞ by disjoint ε balls so $S(n, \varepsilon) \le \# W_{2k+n+1}$.

On the other hand we can find an ε separated set by choosing one point in each ball so $N(n, \varepsilon) \ge \# W_{2k+n+1}$

Thus

$$h(\sigma,\varepsilon) = \limsup_{n\to\infty} \frac{\log(S(n,\varepsilon))}{n} \geq \limsup_{n\to\infty} \frac{\log(\# W_{2k+n+1})}{n}$$

$$= \limsup_{n\to\infty} \frac{\log(\# W_{2k+n+1})}{2k+n+1} \cdot \lim_{n\to\infty} \frac{2k+n}{n}$$

$$= \log(\rho(A).$$

$$h(\sigma,\varepsilon) = \limsup_{n\to\infty} \frac{\log(N(n,\varepsilon))}{n} \leq \limsup_{n\to\infty} \frac{\log(\# W_{2k+n+1})}{n}$$

$$= \log(\rho(A)).$$

So then $h(\sigma) = \lim_{\varepsilon\to 0} h(\sigma,\varepsilon) = \log(\rho(A_G))$.

Remark. We have seen that the ~~growth~~ ~~rate~~ of the number of fixed points of σ^n is $\mathrm{Tr}(A^n)$.

$$\mathrm{Tr}(A^n) = \lambda_1^n + \dots \lambda_k^n.$$

If A_G is aperiodic then the Perron-Frobenius theorem says that ~~then~~ A has a unique eigenvalue λ_{pf} of maximum modulus and that this eigenvalue is real and positive.

In this case
$$\lim_{n\to\infty} \frac{1}{n}\log(\#\text{fixed pts of } \sigma^n) = \lim_{n\to\infty}\frac{1}{n}\log \lambda_{pf}^n + \lambda_2^n + \dots$$
$$= \lim_{n\to\infty}\frac{1}{n}\log\lambda_{pf}^n$$
$$= \log\lambda_{pf}$$
$$= \rho(A)$$
$$= h(\sigma).$$

Consider the ~~linear hyperbo~~ hyperbolic linear toral automorphism $f:\mathbb{T}^2 \circlearrowleft$ given by $\begin{pmatrix}2&1\\1&1\end{pmatrix}$. We have a semi-conjugacy from ~~the~~ a top. Markov chain ~~the~~ $\sigma:\Sigma_{\infty}\circlearrowleft$ to ~~the~~ $f:\mathbb{T}^2\circlearrowleft$. A priori this means that $\underbrace{h(\sigma)}_{=\log(\phi^2)} \geq h(f)$

by our result on semi-conjugacies.

In fact we can use our analysis of the sizes of rectangles to show the equality of entropy.

~~Let A be a 2×2 integral matrix~~

Let A be a hyperbolic integral matrix with $\det A = \pm 1$. Let $f_A : \mathbb{T}^2 \circlearrowleft$ be the corresponding linear hyperbolic ~~automorph~~ diffeomorphism.

Let λ be the larger eigenvalue of A. $\lambda = \lambda_1$ where $|\lambda_1| > 1 > |\lambda_2|$ are the eigenvalues of A.

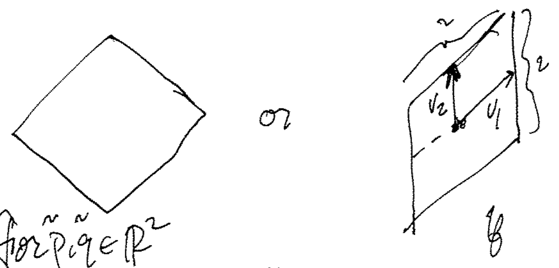
~~Thm. $h(f_A$~~

Thm. $h(f_A) = \log(\lambda)$.

Proof. I want to construct a convenient norm on $\mathbb{R}^2$. Let v_1 and v_2 be unit length eigenvectors for A associated to λ_1 and λ_2.

Define $|v|_{max} = |av_1 + bv_2|_{max} = \max\{|a|, |b|\}$.

The unit ball for this norm is a tipped parallelogram or diamond (rhombus?).



Let $\pi : \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2 = \mathbb{T}^2$.

For $\tilde{p}, \tilde{q} \in \mathbb{R}^2$

Define $\tilde{d}_{max}(\tilde{p}, \tilde{q}) = |p - q|_{max}$.

Define a metric d_{max} on $\mathbb{T}^2$ by

$$d_{max}(p, q) = \min_{\substack{\tilde{p}, \tilde{q} \\ \pi(\tilde{p}) = p \\ \pi(\tilde{q}) = q}} \tilde{d}_{max}(\tilde{p}, \tilde{q}) \text{ where}$$

The map π is a local ~~homeomorphism~~ isometry. ~~For~~

There is a $c > 0$ so that $\tilde{d}_{max}(\tilde{p},\tilde{q}) < c \Rightarrow d_{max}(p,q) = d_{max}(\tilde{p},\tilde{q})$.

$c \approx 1/4$.

Now define $d_n(p,q) = \max_{0 \le \ell < n} d_{max}(f^\ell(p), f^\ell(q))$, $\tilde{d}_n(\tilde{p},\tilde{q}) = \max_{0 \le \ell \le n} \tilde{d}_{max}(\tilde{f}^\ell(\tilde{p}), \tilde{f}^\ell(\tilde{q}))$

Claim that π is also a local isometry w.r.t. $\tilde{d}_n$ and d_n.

~~If~~ $\tilde{d}_n(\tilde{p},\tilde{q}) < c \Rightarrow$ ~~...~~ $\max(\tilde{d}_{max}(f^\ell(\tilde{p}), f^\ell(\tilde{q}))) < c$

$$\Rightarrow \tilde{d}_{max}(A^\ell(\tilde{p}), A^\ell(\tilde{q})) < c$$

$$\Rightarrow d_{max}(A^\ell(\tilde{p}), A^\ell(\tilde{q})) < c$$

$$\Rightarrow \tilde{d}_{max}(A^\ell(\tilde{p}), A^\ell(\tilde{q})) = d_{max}(f_A^\ell(p), f_A^\ell(q))$$

$$\Rightarrow d_n(p,q) = \tilde{d}_n(\tilde{p},\tilde{q}).$$

Now ~~we can~~ the advantage of dealing with $\tilde{d}_n$ is that we can write it down more explicitly:

Say $v = av_1 + bv_2$ then $A^\ell(v) = a\lambda_1^\ell v_1 + b\lambda_2^\ell v_2$

~~So~~

$$|A^\ell(v)| = \max(|a\lambda_1^\ell|, |b\lambda_1^{-\ell}|)$$

$$|A^\ell(v)|_n = \max\{|a|, |b|, |\lambda_1 a|, |\lambda_1^{-1} b| \ldots |\lambda_1^{n-1} a|, |\lambda_1^{1-n} b|\}.$$

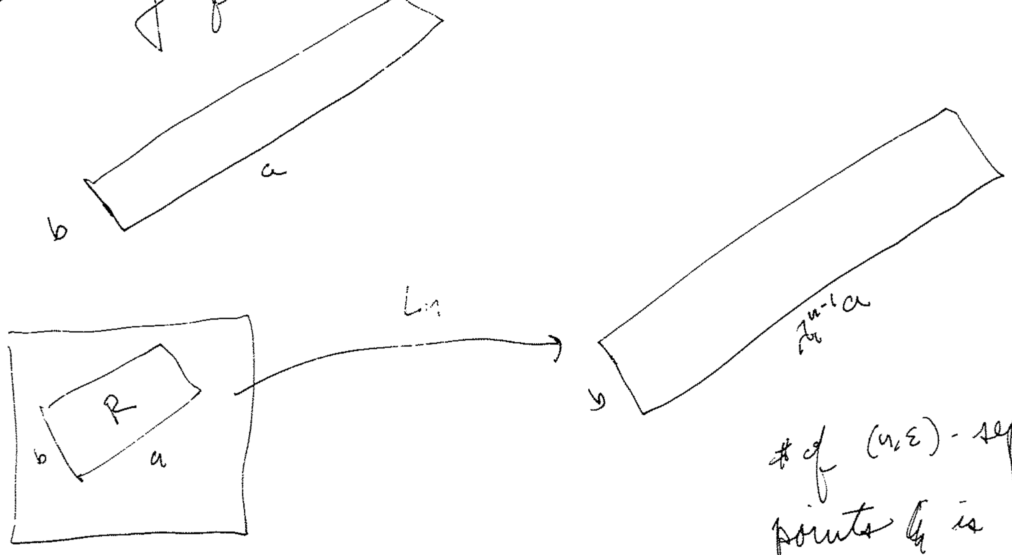
$$|A^{\ell}(v)|_n = \max\{|\lambda_1^{n-1}a|, |b|\}.$$

Define $L(av_1+bv_2) = \cancel{(\lambda_1^{n-1}av_1, bv_2)}\ \cancel{(\lambda_1^{n-1}a, b)}$
$$= (\lambda_1^{n-1}av_1,\ bv_2).$$

Then $\cancel{|av_1+bv_2|_{max}}$
$$|L_n(av_1+bv_2)|_{max} = |av_1+bv_2|_n.$$

The advantage of describing $|\cdot|_{max}$ in terms of the map L_n (L_n?) is the following. If we want to see how many d_n^{ε}-balls it takes to cover a given region R this will be the same as the number of d_{max} balls it takes to cover the region $L_n(R)$. (Similarly for (n,ε)-separated sets.)

~~Similarly if we want to~~

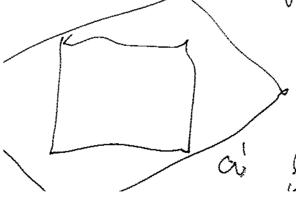


fund. domain

\# of (n,ε)-separated points Q is
$$\left[\frac{b}{\varepsilon}\right]\cdot\left[\frac{\lambda_1^{n-1}a}{\varepsilon}\right]$$

$$\limsup_{n\to\infty}\frac{1}{n}\log N(n,\varepsilon) \geq \lim_{n\to\infty}\frac{1}{n}\log\left[\frac{b}{\varepsilon}\right]\cdot\left[\frac{\lambda_1^{n-1}a}{\varepsilon}\right]$$
$$\geq \log(\lambda_1).$$

\# of (n,ε)-spanning points



$$\leq \left(\left[\frac{b'}{\varepsilon}\right]+1\right)\left(\left[\frac{\lambda_1^{n-1}a'}{\varepsilon}\right]+1\right)$$

a' $\quad\limsup_{n\to\infty}\frac{1}{n}\log S(n,\varepsilon) \leq \log(\lambda_1).$