MA424 Example Sheet 5

23 November 2015

1. Let f be a symmetric unimodal map. Let $\mathcal{R}$ be the operator which takes a symmetric unimodal map f to a symmetric unimodal map $g = \mathcal{R}(f)$ for which the periods of periodic points are doubled. Show the sequence of maps $\mathcal{R}^n(f)$ converges to a limit uniformly as $n \to \infty$. Show that this limit is independent of f.

2. Let f_∞ be the period doubling limit described in the first problem. Show that f_∞ has periodic points with period 2^n for all n. Show that f_∞ has an invariant Cantor set. Show that the limit set of every point is either a periodic point or the invariant Cantor set.

3. Let $f_A : \mathbb{T}^2 \to \mathbb{T}^2$ be a toral automorphism where A is an integral matrix. Show that a point $p \in \mathbb{T}$ is periodic if and only if the coordinates of p are rational.

4. Let $\mathcal{G}$ be a finite directed graph and let $\Sigma_\mathcal{G}$ be the corresponding topological Markov chain. Define a metric on $\Sigma_\mathcal{G}$ by

$$d_\lambda(\omega, \omega') = \frac{\max \varepsilon(\omega_j, \omega'_j)}{\lambda^n}$$

where $\varepsilon(j,k) = 0$ if $j = k$ and 1 otherwise and $\lambda > 1$. For $\omega \in \Sigma_\mathcal{G}$ describe the ball of radius r around ω in this metric. Show that the topology determined by the metric d_λ is independent of λ.

5. We say $\mathcal{G}$ is irreducible if any two edges e and f there is a path of directed edges of some length starting at e and ending at f. Show that if $\mathcal{G}$ is irreducible then periodic points of the shift map are dense in $\Sigma_\mathcal{G}$.

6. Show that if $\mathcal{G}$ is irreducible then the shift map has a dense orbit.