Hasselblatt & Katok: A first course in dynamics.
MA 424 notes by Vassili Gelfreich.
" " " Mark Pollicott → Mark's website
Devaney, "Chaotic Dynamical Systems".

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A *dynamical system* is something that evolves in time.

MA 4240
10:05 – 10:55.

We mean by this that there is some notion of the state of the system (which we think of as being ~~given~~ determined by finitely many parameters) and the state of the system changes with time.

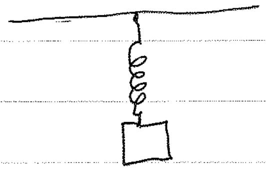
Let X be the set of all possible states.
We call X the *state space*.
If $p_0 \in X$ let $\phi_t(p_0)$ be the state of ~~p~~ the system t time ~~units later~~ given that it is in state p_0 at $t=0$.
Def.^{n} $\phi_t(p_0)$ is a *trajectory* of our system

We make two additional assumptions on our ~~dynamical systems~~ evolving system:

We assume it is *deterministic*: the current state of the system completely determines the state at all future times.

We assume our system is *autonomous*: the rule determining the evolution of the system does not change with time.

Let me give a simple example.
Consider a weight hanging on a spring:



The motion of the weight is governed by the ODE

$$\ddot{x}(t) = -k\,x(t)$$

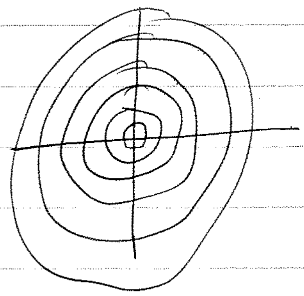
where $x(t)$ is the deviation from the rest position and k is determined by the spring constant and mass. System is called a *harmonic oscillator*.

If we know both position and velocity at the current time then we know position and velocity at all future times so in order to have the determinism property X should record both position x and velocity v.

In terms of x and velocity v our equation becomes the system of first order equations:

$$\dot{x} = v$$
$$\dot{v} = -kx.$$

The solution ϕ_t with initial condition $\begin{pmatrix} a \\ b \end{pmatrix}$ is $a\begin{pmatrix} \cos\sqrt{k}t \\ -\sqrt{k}\sin\sqrt{k}t \end{pmatrix} + b\begin{pmatrix} \sin\sqrt{k}t \\ \sqrt{k}\cos\sqrt{k}t \end{pmatrix}$.



Solutions give a family of ellipses that fill up $\mathbb{R}^2$.

The equations are autonomous in that time does not appear explicitly on the right hand side of the equation.
This is what we mean by saying that the evolution rule does not change with time.
This implies that if $t \mapsto \phi_t$ is a solution then $t \mapsto \phi_{t+c}$ is also a solution.

Let $f^t(p) = \phi_t(p)$ for $p \in X$. We can think of $f^t : X \to X$ as the "time advance map".

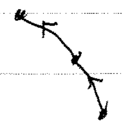
We have:

Thm. Consider an autonomous first order ODE so that solutions are defined for all time then:

(1) $f^{s+t} = f^s \circ f^t$

(2) $f^0 = \mathrm{Id}$.

Follows from 2 facts:
f^s takes orbits to orbits (autonomous)
We can join orbits together to create orbits (1st order).



where f^t is the time advance map.

Example:
$$f^t = \begin{pmatrix} \cos\sqrt{k}t & \sin\sqrt{k}t \\ -\sqrt{k}\sin\sqrt{k}t & \sqrt{k}\cos\sqrt{k}t \end{pmatrix}$$
linear map given by matrix.

Connection between dynamical systems and topology!

If the coef. of the ~~eqs~~ ODE are ~~C~~ C^k functions then

Cor. f^t is a C^k diffeomorphism.

Proof. $f^{-t} \circ f^t = f^t \circ f^{-t} = f^0 = \mathrm{Id}$. f^t and f^{-t} are ~~diff~~ C^k if the coef. of the equations are C^k.

We take these 2 properties as the defining properties of a dynamical system.

Definition. A dynamical system is a family of diffeomorphisms $\{f^t : t \in \mathbb{R}\}$ ~~such that~~ of $X \subset \mathbb{R}^n$ such that $f^{s+t} = f^s \circ f^t$ and $f^0 = \mathrm{id}$.

Actually we allow ourselves some freedom with the set T where our time parameter takes its values. Let $\mathbb{R}_+ = \{v : v \in \mathbb{R}, v \geq 0\}$ $\mathbb{Z}_+ = \{n \in \mathbb{Z}, n \geq 0\}$

dynamical systems for which:

then we allow $\{f^t : t \in \mathbb{R}\}$ flow

$\{\quad t \in \mathbb{R}_+\}$ semi-flow

$\{\quad t \in \mathbb{Z}\}$ ~~diffeomo~~ iteration of a diffeo

$\{\quad t \in \mathbb{Z}_+\}$ iteration of a map.

For $t \in \mathbb{Z}$ or $\mathbb{Z}_+$, $f^n = \underbrace{f^1 \circ f^1 \circ \dots f^1}_{n}$ ~~and $f^{-n} = f^{-1} \circ \dots f^{-1}$~~

If $t \in \mathbb{Z}$ then $f^n \circ f^{-1} = f^0 = \mathrm{Id}$ so f is invertible.

Definition. An orbit of p of a dynamical system is $\{f^s(p) : p \in X,\ s \in T\}$.

Example: Ellipses of the harmonic oscilator

Definition. An orbit of p is periodic if $f^s(p) = p$ for some $s \neq 0$. The period of an orbit is the smallest positive s such that $f^s(p) = p$.

For flows periodic orbits are circles.

Example: For the harmonic oscilator the orbits (of the flow) ~~[illegible]~~ are periodic with period $\sqrt{k}$ (they all have the same period.)

Example of a discrete time dynamical system

Let $X = \{z \in \mathbb{C} : |z| = 1\}$.

Any $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.

Let $M_\lambda : X \to X$ be defined by $M_\lambda(z) = \lambda z$.

Write M_λ^n for f^n. Defined for $n \in \mathbb{Z}$.

Note that this system is closely related to the harmonic oscillator. In fact $w_\lambda = \phi_{t_0}$ where ~~t=~~ $\lambda = \cos t_0 + i \sin t_0$.

Can think of a strobe light going off every t_0 seconds. Rotation of the circle

It is sometimes useful to write this system additively.

Let $\mathbb{R}/\mathbb{Z}$ ~~=~~ be the quotient space of $\mathbb{R}$ modulo the equivalence relation $x \sim y$ if $x - y \in \mathbb{Z}$.

We can identify $\mathbb{R}/\mathbb{Z}$ with the interval $[0,1]$ with $0 \sim 1$ since every equivalence class has one representative in $(0,1)$.

~~Define $\exp: \mathbb{R}/\mathbb{Z} \to$~~ Define $\exp: \mathbb{R} \to \mathbb{C}$

$\exp(v) = 2^{\pi i v}$. exp induces a map from $\mathbb{R}/\mathbb{Z}$ to $\mathbb{C}$ since $\exp(x) = \exp(y)$ iff $x - y \in \mathbb{Z}$.

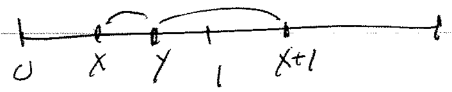
Let ~~$R_\alpha(x)=x$~~ $R_\alpha : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be defined

by $R_\alpha(x) = x + \alpha \mod 1$.

$\mathbb{R}/\mathbb{Z}$ becomes a metric space by defining

a distance; say $0 \le x \le y \le 1$ then $d(x,y) =$

$\min(y-x,\ x+1-y)$



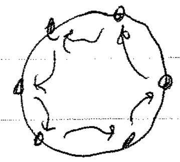
exp is a continuous map

There is a close connection between rotations (discrete time) of the circle and pairs of harmonic oscilators with different periods. (flow). But before we discuss this connection we will establish some properties of R_α.

Note that an orbit for a discrete time dynamical system is closed if and only if it is finite.

$f^n(p) = p$.

Periodicity.



$f^6(p) = p$.

Proposition. (1) α is rational iff (2) R_α has one periodic point iff (3) all points on $\mathbb{R}/\mathbb{Z}$ are periodic.

$$\begin{tikzcd} & (1) \arrow[dl, shift left] \arrow[dr] & \\ (2) \arrow[ur, bend left] & & (3) \arrow[ll] \end{tikzcd}$$

$(3) \Rightarrow (2)$

Proof. $(1) \Rightarrow (2) \& (3)$ If $\alpha = m/n$ then $R_\alpha^n = R_{m/n}^n = R_m = Id$.

If x is periodic for R_α then $R_\alpha^n(x) = x$ so

$n\alpha x = x \mod \mathbb{Z}$ so $n\alpha = 0 \mod \mathbb{Z}$ so $n\alpha = m$

and $\alpha = m/n$. $(2) \Rightarrow (1)$.