On the representations of GL3(Fp) (UG colloquium talk).

- Supporting slides -

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Background - rough definitions

- $GL_3(\mathbb{F}_p) = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} : a_{ij} \in \mathbb{F}_p \right\}$
- A **finite field** of order p is $\mathbb{F}_p = \{\overline{0}, \overline{1}, \overline{2}, ..., \overline{p-1}\} \cong \mathbb{Z}/p\mathbb{Z}$.
- A **group** G is a set equipped with a binary operation (denoted by '.') such that:
  - G has an identity, usually denoted 'e' where $e.g = g.e = g$ for all $g \in G$
  - every element in G has an inverse (for all $g \in G$ there is $g^{-1} \in G$ with $g.g^{-1} = g^{-1}.g = e$
  - The operation is associative ie $f.(g.h) = (f.g).h$ for all $f, g, h \in G$
  - (EXTRA) if $g.h = h.g$ for all $g, h \in G$ we say the group is abelian, this isn't required to be a group though.
- A **group homomorphism** (say from group G to group H) $\phi : G \to H$, is a map between groups which "respects group structure", that is: $\phi(g.g') = \phi(g).\phi(g')$ for all $g, g' \in G$
- A **vector space V** over a field F (examples of fields are $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{F}_p$ from above) is an abelian group under addition with "nice" scalar multiplication on elements of V by elements in F
- A **subspace** U of V is a subset of V which is closed under addition and scalar multiplication inherited from V.

Restriction to the torus subgroup

There is a subgroup of $GL_3(\mathbb{F}_p)$ called the torus which is defined as:

$$T := \left\{ \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} : t_1, t_2, t_3 \in \mathbb{F}_p^\times \right\}$$

If we have a representation of $GL_3(\mathbb{F}_p)$, say V, then we can restrict this to a representation of T. We can then decompose V into $V = \bigoplus V(\lambda)$ where each λ is a weight. A weight space is basically a generalisation of an eigenspace, and a weight a generalisation of an eigenvalue.

Example.

Let $V = \mathbb{F}_p^3$. Define $\rho : GL_3(\mathbb{F}_p) \to GL_3(\mathbb{F}_p)$ by $\rho(A) = A$ for $A \in GL_3(\mathbb{F}_p)$.

Restricting to T only we have:

$$\begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t_1 x \\ t_2 y \\ t_3 z \end{pmatrix} = \begin{pmatrix} t_1^1 t_2^0 t_3^0 x \\ t_1^0 t_2^1 t_3^0 y \\ t_1^0 t_2^0 t_3^1 z \end{pmatrix}$$

So our weight spaces will be $< e_1 >$, $< e_2 > < e_3 >$. As T reps we have:

$$Res_T^{GL_3(\mathbb{F}_p)}(\rho) = V(1,0,0) + V(0,1,0) + V(0,0,1)$$

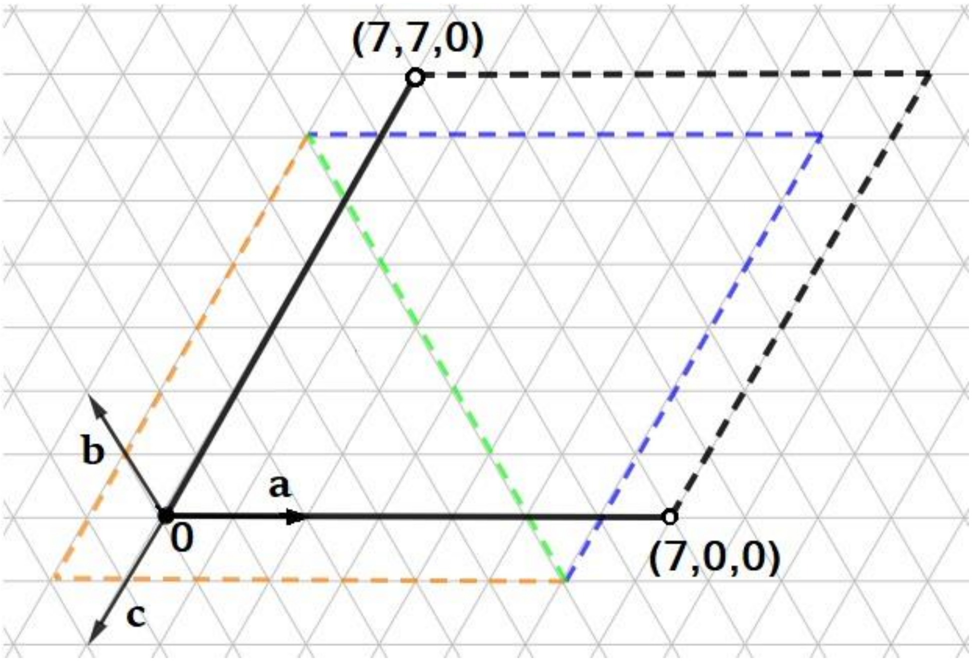
Labelling representations of GL3.

For each weight $\lambda = (a, b, c)$ such that $a \geq b \geq c$ (called a dominant weight) we have a "natural" representation with highest weight λ (when considered as a T rep and decomposed) which we call $W(\lambda)$.

Inside each $W(\lambda)$ is a sub-rep called $\mathrm{F}(\lambda)$. If λ lives in the p-restricted region then $\mathrm{F}(\lambda)$ is irreducible (see below).

There are a number of theorems and methods to decompose these W representations which include:

- Strong linkage principal
- Schur polynomials
- Littlewood-Richardson Rule
- Translation principal



Littlewood-Richardson rule

This rule gives useful way to express a tensor product $W(\lambda) \otimes W(\mu)$ as a linear combination of other W representations as follows:

$$W(\lambda) \otimes W(\mu) = \sum_{\nu} N^{\nu}_{\lambda\mu} W(\nu)$$

where $N^{\nu}_{\lambda\mu}$ is the Littlewood-Richardson coefficient.
$N^{\nu}_{\lambda\mu}$ is the number of ways to μ-extend a Young diagram of shape λ to a diagram of shape ν with the following rules:

- rows must be weakly increasing
- columns must be strictly increasing
- row length must be weakly decreasing (so weights are dominant)
- reading boxes top right to bottom left we must have a lattice word, that is there is at least as many 1s as 2s as 3s for every prefix.

Example
$W(2,0,0) \otimes W(2,1,1) = W(4,1,1) + W(3,2,1)$

p=2 and p=3 table. (1/2)

$$\sigma_1 = F(p-1,0,0) + F(p-1,p-1,0) - 2F(0,0,0)$$

$$\sigma_2 = F(0,0,0) - (F(p-1,0,0) + F(p-1,p-1,0)) + F(2(p-1),(p-1),0))$$

F(weight) $\otimes$ F(weight)	$(0,0,0)$	$(1,0,0)$	$(1,1,0)$	$(2,1,0)$
$(0,0,0) \otimes (0,0,0)$	1	0	0	0
$(0,0,0) \otimes (1,0,0)$	0	1	0	0
$(0,0,0) \otimes (1,1,0)$	0	0	1	0
$(0,0,0) \otimes (2,1,0)$	0	0	0	1
$(1,0,0) \otimes (1,0,0)$	0	1	2	0
$(1,0,0) \otimes (1,1,0)$	1	0	0	1
$(1,0,0) \otimes (2,1,0)$	1	3	2	1
$(1,1,0) \otimes (1,1,0)$	0	2	1	0
$(1,1,0) \otimes (2,1,0)$	1	2	3	1
$(2,1,0) \otimes (2,1,0)$	4	6	6	3
$\sigma_1 \otimes (0,0,0)$	-2	1	1	0
$\sigma_2 \otimes (0,0,0)$	1	-1	-1	1
$\sigma_1 \otimes (1,0,0)$	1	-1	2	1
$\sigma_2 \otimes (1,0,0)$	0	3	0	0
$\sigma_1 \otimes (1,1,0)$	1	2	-1	1
$\sigma_2 \otimes (1,1,0)$	0	0	3	0
$\sigma_1 \otimes (2,1,0)$	2	5	5	0
$\sigma_2 \otimes (2,1,0)$	2	1	1	2

Table 1: Table displaying multiplicity of irreps in tensor calculations for $p = 2$.

For example:
$F(1,0,0) \otimes F(2,1,0) = 1F(0,0,0) + 3F(1,0,0) + 2F(1,1,0) + 1F(2,1,0)$.

P=2 and p=3 table. (2/2)

$F(weight)\otimes F(weight)$	$(0,0,0)/(1,1,1)$	$(1,0,0)/(2,1,1)$	$(2,0,0)/(3,1,1)$	$(1,1,0)/(2,2,1)$	$(2,1,0)/(3,2,1)$	$(3,1,0)/(4,2,1)$	$(2,2,0)/(3,3,1)$	$(3,2,0)/(4,3,1)$	$(4,2,0)/(5,3,1)$
$(1,0,0)\otimes(2,0,0)$	0/1	1/0	0	0	2/0	0	0	0	0
$(2,0,0)\otimes(2,0,0)$	0	0	1/0	1/0	0	1/0	2/0	0	0
$(1,1,0)\otimes(2,0,0)$	0	0/1	0	0	0	1/0	0	0	0
$(2,1,0)\otimes(2,0,0)$	0/2	0	0	0/1	1/0	0	0	2/0	0
$(3,1,0)\otimes(2,0,0)$	2/0	0/2	0	2/0	0/4	1/0	1/0	0	1/0
$(2,2,0)\otimes(2,0,0)$	2/0	0	0	0	0/1	0	0	0	1/0
$(3,2,0)\otimes(2,0,0)$	0/4	1/0	0	0/1	2/0	0/3	0/1	1/0	0
$(4,2,0)\otimes(2,0,0)$	5/0	0/3	3/0	1/0	0/4	1/0	2/0	0/3	1/0
$(1,0,0)\otimes(2,2,0)$	0	0	0	0/1	0	0	0	1/0	0
$(1,1,0)\otimes(2,2,0)$	1/0	0	0	1/0	0/2	0	0	0	0
$(2,1,0)\otimes(2,2,0)$	0/2	1/0	0	0	1/0	0/2	0	0	0
$(3,1,0)\otimes(2,2,0)$	4/0	0/1	1/0	1/0	0/2	1/0	0	0/3	0
$(2,2,0)\otimes(2,2,0)$	0	0/1	2/0	0	0	0	1/0	0/1	0
$(3,2,0)\otimes(2,2,0)$	0/2	2/0	0/1	0/2	4/0	0	0	1/0	0/1
$(4,2,0)\otimes(2,2,0)$	5/0	0/1	2/0	3/0	0/4	3/0	3/0	0/1	1/0
$(1,0,0)\otimes(4,2,0)$	0/4	0	0	0/1	2/0	0/3	0	1/0	0
$(1,1,0)\otimes(4,2,0)$	4/0	0/1	0	0	0/2	1/0	0	0/3	0
$(2,1,0)\otimes(4,2,0)$	0/5	4/0	0/2	0/4	7/0	0/2	0/2	2/0	0/1
$(3,1,0)\otimes(4,2,0)$	15/0	0/6	4/0	6/0	0/12	7/0	4/0	0/6	1/0
$(3,2,0)\otimes(4,2,0)$	0/15	6/0	0/4	0/6	12/0	0/6	0/4	7/0	0/1
$(4,2,0)\otimes(4,2,0)$	25/0	0/10	8/0	10/0	0/20	10/0	8/0	0/10	4/0
$\sigma_1\otimes(0,0,0)$	-2/0	0	1/0	0	0	0	1/0	0	0
$\sigma_2\otimes(0,0,0)$	1/0	0	−1/0	0	0	0	−1/0	0	1/0
$\sigma_1\otimes(1,0,0)$	0/1	−1/0	0	0/1	2/0	0	0	1/0	0
$\sigma_2\otimes(1,0,0)$	0/3	0	0	0	0	0/3	0	0	0
$\sigma_1\otimes(2,0,0)$	2/0	0	−1/0	1/0	0/1	1/0	2/0	0	1/0
$\sigma_2\otimes(2,0,0)$	3/0	0/3	3/0	0	0/3	0	0	0/3	0
$\sigma_1\otimes(1,1,0)$	1/0	0/1	0	−1/0	0/2	1/0	0	0	0
$\sigma_2\otimes(1,1,0)$	3/0	0	0	0	0	0	0	0/3	0
$\sigma_1\otimes(2,1,0)$	0/4	1/0	0	0/1	0	0/2	0	2/0	0
$\sigma_2\otimes(2,1,0)$	0/1	3/0	0/2	0/3	6/0	0	0/2	0	0/1
$\sigma_1\otimes(3,1,0)$	6/0	0/3	1/0	3/0	0/6	0	1/0	0/3	1/0
$\sigma_2\otimes(3,1,0)$	9/0	0/3	3/0	3/0	0/6	6/0	3/0	0/3	0
$\sigma_1\otimes(2,2,0)$	2/0	0/1	2/0	0	0/1	0	−1/0	0/1	1/0
$\sigma_2\otimes(2,2,0)$	3/0	0	0	3/0	0/3	3/0	3/0	0	0
$\sigma_1\otimes(3,2,0)$	0/6	3/0	0/1	0/3	6/0	0/3	0/1	0	0/1
$\sigma_2\otimes(3,2,0)$	0/9	3/0	0/3	0/3	6/0	0/3	0/3	6/0	0
$\sigma_1\otimes(4,2,0)$	10/0	0/4	5/0	4/0	0/8	4/0	5/0	0/4	0
$\sigma_2\otimes(4,2,0)$	15/0	0/6	3/0	6/0	0/12	6/0	3/0	0/6	3/0

Table 2: Multiplicity of irreps in decomposition of $F(\lambda)\otimes F(\mu)$, for the various λ and μ, and $\sigma_i\otimes F(\lambda)$, $\lambda\in X_3(T)$ for p=3.

General F tensor F formula (for 0 < a < p)

$$F(a,a-1,0)\otimes F(p-1,0,0) = \begin{cases} F(a,a-1,0)+F(a-1,a-1,1)+F(p-2,a,a) \\ +2F(a+p-2,a,0)+F(a+p-2,a-1,1)+F(p-2+\frac{a}{2},a,\frac{a}{2}) \\ +\sum_{i=2}^{\frac{a}{2}}\Big(F(a+p-1-i,a-1,i)+2F(i+p-2,a-1,a+1-i)\Big) \\ +\sum_{i=1}^{\frac{a}{2}-1}\Big(F(a+p-2-i,a,i)+2F(i+p-2,a,a-i)\Big) & \textbf{if a even} \\ \\ F(a,a-1,0)+F(a-1,a-1,1)+2F(a+p-2,a,0) \\ +F(a+p-2,a-1,1)+F(p-2+\frac{a+1}{2},a-1,\frac{a+1}{2}) \\ +\sum_{i=2}^{\frac{a-1}{2}}\Big(F(a+p-1-i,a-1,i)+2F(i+p-2,a-1,a+1-i)\Big) \\ +\sum_{i=1}^{\frac{a-1}{2}}\Big(F(a+p-2-i,a,i)+2F(i+p-2,a,a-i)\Big) & \textbf{if a odd} \end{cases}$$

$$F(a,0,0)\otimes F(p-1,0,0) = \begin{cases} \sum_{i=0}^{\frac{a}{2}}(2-\delta_{i,0}-\delta_{i,\frac{a}{2}})F(a-i,i,0)+2\sum_{i=1}^{\frac{a}{2}-1}F(i+p-1,a-i,0) \\ +\sum_{i=1}^{\frac{a}{2}-1}F(a+1-i,i,1)+\sum_{i=\frac{a}{2}+1}^{\min\{a,p-2\}}F(p-2,i,a+1-i) \\ +F(p-1+\frac{a}{2},\frac{a}{2},0 & \textbf{if a even} \\ \\ \sum_{i=0}^{\frac{a-1}{2}}\Big((2-\delta_{i,0})F(a-i,i,0)+2F(i+p-1,a-i,0)\Big) \\ +\sum_{i=1}^{\frac{a-1}{2}}F(a-1-i,i,1)+\sum_{i=\frac{a+1}{2}}^{a}F(p-2,i,a+1-i) & \textbf{if a odd} \end{cases}$$