Contramodules for Algebraic Groups.

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There are two main aims of this talk:

1) Briefly introduce contramodules

2) Construct projective covers of simple $G$-modules (in the category $k[G]$-Conba.)

Note: This is a contra-analog of the “Inverse limit theorem.”

Let $A$ be an algebra over a field $k$. Let $M$ be an $A$-module. Two natural ways to view this are:

\[ \rho: A \otimes M \to M \quad \text{and} \quad \tau: M \to \text{Hom}_k(A, M) \]

- $\rho: a \otimes m \mapsto a \cdot m$
- $\tau: m \mapsto (a \mapsto a \cdot m)$

Reversing arrows in this defn (and replacing $A$ algebra with $C$ coalgebra) defines $C$-comodules.

Reversing arrows here (with algebra $A \to$ coalgebra $C$) defines $C$-contramodules.
Contramodules of the coalgebra $k[G]$.

$G$ algebraic group over $k = \overline{k}$ with co-ord ring $k[G]$.

$m : G \times G \rightarrow G \xrightarrow{\ast} m^* : k[G] \rightarrow k[G] \otimes k[G].$

or "Δ"


• Comodules: $V \rightarrow V \otimes k[G]$

Gives $G$-repn. via $G \times V \rightarrow G \times V \otimes k[G] \xrightarrow{\text{eval.}} V$

$\{\text{rational } G\text{-modules}\} \leftrightarrow \{k[G]\text{-comodules}\}$

• Contramodules: $\text{Hom}_k(k[G], V) \rightarrow V$

$G \times V \subset k[G]^* \otimes V$

This composition is a $G$-action on $V.$
An important example of contramodules

If $M$ is a $k[G]$-comodule, $\rho: M \rightarrow M \otimes k[G]$ then $\text{Hom}_k(M, V)$ is a contramodule via:

\[ \text{Hom}_k(k[G], \text{Hom}_k(M, V)) \otimes \text{Hom}_k(M \otimes k[G], V) \xrightarrow{\rho^*} \text{Hom}_k(M, V) \]

→ In particular for $M = k[G]$ (as a comodule over itself)

**Lemma**: For any contramodule $B$ we have:

\[ \text{Hom}_k(k[G], \text{Hom}_k(k[G], V), B) \cong \text{Hom}_k(V, B). \]

**Consequences**: 1. $\text{Hom}_k(k[G], V)$ is a free contramodule.

2. From contra-adjunction $\text{Hom}_k(k[G], B) \rightarrow B$: $B$ projective $\iff$ $B$ & a free contramodule.
Part 2: Constructing projective covers.

- $G$ simply connected, semisimple alg group over $k = \bar{k}$ with $\text{char} k = p$.

  The $r$th Frobenius kernel.

  Frobenius morphism.

  Have $F : G \rightarrow G$, with $G_r := \ker(F^r)$, $r > 0$

- $\lambda$ dominant weight $\sim L(\lambda)$ simple $G$-module.

- $\lambda$ $p$-restricted dom. weight $\sim L(1, \lambda)$ simple $G_1$-module

- $G$-structures: If $V \in G_1$-Mod and $W \in G$Mod s.t. $W|_{G_1} \cong V$ then $W$ is a $G$-structure on $V$

Assume $P(\lambda)$ is a $G$-structure on $P(1, \lambda)$ - the $G_1$-projective cover of $L(1, \lambda)$.

- For $p > 2^{h-4}$ they exist (Jantzen, C. Bendel, D. Nakano, '22)

- Conjectured to always exist (Humphreys & Vermani)
Fix $\lambda$ - dominant. Write $\lambda = \sum_{i=0}^{\xi} p^i \lambda_i$ (p-restricted).

For $m > s$ define:

$$P_{\lambda,m} := P(\lambda_0) \otimes P(\lambda_1)^{Fr} \otimes \ldots \otimes P(\lambda_s)^{Fr^s} \otimes P(0)^{Fr^{s+1}} \otimes \ldots \otimes P(0)^{Fr^{m-1}}.$$ 

Fix projection $q : P(0) \rightarrow L(0)$ (this induces projections $q_{m+1} : P_{\lambda,m+1} \rightarrow P_{\lambda,m}$).

Define $P_{\lambda} := \lim P_{\lambda,m} = \text{limit of diagram } (P_{\lambda,m} \leftarrow P_{\lambda,m+1} \leftarrow P_{\lambda,m+2} \leftarrow \ldots)$

**Theorem**

$P_{\lambda}$ is the projective cover of $L(\lambda)$. 
Remainder of talk: Idea behind the proof.

- In 1980, Donkin gave a "short simple minded proof of the direct limit theorem."

- Try to modify this proof in our "dual case."

⚠️ Problem: In Donkin's proof he makes use of the fact that every comodule is a direct limit (a.k.a union) of its finite dimensional subcomodules.

❌ The analogous statement for contramodules is not true. (c.f. Positselski: Contramodules, §1.5).

This suggests that using $\text{Hom}^k(k[G], P^\lambda)$ to check projectiveness is not the way to go... So?
Need construction known as $\text{Cohom}_{k[G]}$:

- Takes a comodule $M$, and a contramodule $B$.
- $\text{Cohom}_{k[G]}(M, B) = \text{Hom}_R(M, B) / \text{stuff}$

**Lemma**

Let $V, W \in k[G]$-Comod f.d. then:

$$\text{Cohom}_{k[G]}(V, W) \cong \text{Hom}_{k[G]}(W, V)^*$$

(viewed as a contramodule via $\text{Hom}_k(k[G], W) = k[G]^* \otimes W \to k[G]^* \otimes k[G] \otimes W \to W$.)

In particular, one can show for a fixed $\lambda$:

$$\dim (\text{Cohom}_{k[G]}(V, P_\lambda, m)) = f(V), \ m \gg 0$$

where $f(V) =$ composition multiplicity of $L(\lambda)$ in $V$. 

Think $\text{M}_{R \otimes N}$ for $R$-modules.
Q: How is Cohom useful to us?

Answer:

Lemma (Positselski, ’15)

A $k[G]$-contramodule $P$ is projective

$\iff \text{Cohom}_{k[G]}(-, P): k[G]-\text{Comod}_{f.d.} \to k-\text{Vect}$

is exact.

By using $\text{Cohom}_{k[G]}$ we only need to check exactness for finite-dimensional objects!
proof sketch of main Theorem

- Define $\text{Rad}^k[G](P_\lambda) = \lim_m \text{Rad}_{k[G_m]}(P_\lambda, m)$.  

- This satisfies conditions we want $\frac{P_\lambda}{\text{Rad}(P_\lambda)} \cong \lambda(\mu)$.  

- To see that $P_\lambda$ projective:

  For $V$ f.d. comodule, one can show:\n  $$\text{Cohom}_{k[G]}(V, P_\lambda) = \lim \text{Cohom}_{k[G]}(V, P_\lambda, m).$$

  But by lemma: $\dim \left( \text{Cohom}_{k[G]}(V, P_\lambda, m) \right) = f(V), \ m \gg 0$

  $\Rightarrow \dim \left( \text{Cohom}_{k[G]}(V, P_\lambda) \right) = f(V)$
Finally, $f$ is additive on SESs of finite dimensional comodules

\[(0 \to U \to V \to W \to 0 \text{ exact } \Rightarrow \ f(V) = f(U) + f(W))\]

we conclude that $\text{Cohom}_{k[G]}(-, P_{\lambda})$ is exact as functor from $k[G]-\text{Comod}^{\text{op}}_{fd} \to k-\text{Vect}$

Positselski
\[\Rightarrow \ P_{\lambda} \text{ projective.} \]

Thank you!