Methods of Mathematical Modelling II

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These notes accompany the module *MA144: Methods of Mathematical Modelling II* for first-year undergraduates reading mathematics at Warwick.

Some useful textbooks for this course:

Multivariable calculus, Stewart, J., Watson, S. and Clegg, D., 9^{th} ed., Brook/Cole (2020).

Thomas' calculus, Hass, J., Heil, C. Weir, M. and Thomas, G., 14^{th} ed., Pearson (2020).

Both of these book have lots of examples, exercises and excellent graphics. Stewart et. al. is available as an e-book from the library (simultaneous access is limited).

Mathematical methods for the physical sciences, Boas, M. L., 3^{rd} ed., Wiley (2006).

A classic textbook with excellent explanations of vector-calculus concepts. Available as hardcopies from the library. The 2^{nd} edition is just as good.

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CHAPTER 1

CURVES

We are all familiar with the concept of functions from school. In fact, you would have already mastered calculus techniques (differentiation and integration) for *real-valued* functions. You will remember that *practice* was the key to success. However, you may not have been taught *why* and *when* you can differentiate and integrate functions.

So what's calculus like at university?

- In *MA139 Analysis II*, you will relearn differentiation of real-valued functions. This time, with ______________ (*i.e.* not based on pictures, but logic). You will learn precisely what it means for a function to be differentiable.
- In *MA144 Methods of Mathematical Modelling II*, you will learn calculus techniques for *vector-valued* functions. We won't necessarily go deeply into why and when you can differentiation or integrate *multivariable* functions. ______________ is important.
- Towards the end of *MA139* you will relearn integration of functions - this time more rigorously. In Year-2 *Analysis III* You will also study calculus for complex-valued functions.
- In Year-2 *Multivariable Analysis*. you will revisit the content of this course - this time more rigorously, and prove *MA144* theorems using a more abstract language.

I guess what I'm saying is, this module is easy and fun, so let's go!

Definition. A function $\mathbf{r}$ is said to be ____________________ if it maps a number $t \in \mathbb{R}$ to a vector $\mathbf{r}(t) \in \mathbb{R}^n$, where $n = 2, 3, 4 \ldots$

We usually use the letter t as the independent variable (often thought of as "time"). A vector–valued function can also be viewed in terms of its component functions:

$$\mathbf{r}(t) =$$

where each component function $r_i(t)$ is a real-valued function. Another way to say "real-valued" is "____________________" (*i.e.* the opposite of vector-valued).

In this module, we will primarily be interested in the case $n = 2$ or 3. We sometimes write $\mathbf{r}(t) = (x(t), y(t), z(t))$ rather than $(x_1(t), x_2(t), x_3(t))$.

You should already be familiar with the standard unit vectors $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, and $\mathbf{k} = (0, 0, 1)$. Thus, we can also write

$$\mathbf{r}(t) =$$

Sometimes, where there is no confusion from mixing row and column vectors, we will write vectors vertically to save space and avoid commas.

1.1 Parametric curves

Definition. Consider the function $\mathbf{r}\colon I \to \mathbb{R}^n$, where $I \subseteq \mathbb{R}$ is an interval. Then the set of image points

$$\mathcal{C} =$$

is called a **curve** in $\mathbb{R}^n$. The function $\mathbf{r}$ is called a ____________________ of the curve $\mathcal{C}$.

Think of $\mathbf{r}(t)$ as the position of a particle at time t. Its trajectory traces out the curve $\mathcal{C}$.

A curve can have many parametrisations, so we have to be careful what we say.

Question. Which of the following sentences makes more sense?

☐ "A curve is parametrised by $\mathbf{r}(t) = \ldots$"

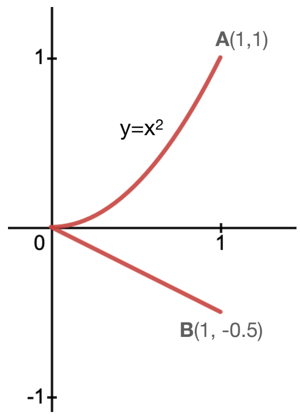
☐ "The curve $\mathbf{r}(t) = \ldots$"

Example 1. A line in $\mathbb{R}^3$ can be parametrised by $\mathbf{r}(t) =$ ______________ where $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b}$ are constant vectors in $\mathbb{R}^3$ and $t \in \mathbb{R}$. The vector _____ determines the direction of the line.

Example 2. (*Line joining two points*) Let $\mathbf{a}$ and $\mathbf{b}$ be the position vectors of points A and B. Write down a parametrisation of the line, beginning at A and ending at B, using the following parameters: a) $t \in [0, 1]$ b) $u \in [0, 2\pi]$ c) $v \in [-1, 1]$.
Are your answers unique?

Example 3. (*Parabolas*) a) Write down a parametrisation of the curve $y = ax^2 + bx + c$.
b) Consider the curve parametrised by $\mathbf{r}(t) = (2t^2, t)$ where $t \in [-2, 2]$. Find its Cartesian equation and sketch the curve. Include an arrow to indicate increasing t.
c) Sketch the curve parametrised by $\mathbf{r}(t) = \big(2\sin^2 t, \sin t\big)$ where $t \in [0, 2\pi]$.

Example 4. (*Curve defined piecewise*) The curve shown below starts at A and ends at B. Write down a parametrisation of the curve using parameter $t \in [-1, 1]$.



Example 5. (*Circle*) Write down two parametrisations of the semi-circle $x^2 + y^2 = 2$ and $y \geq 0$, traversed in the anti-clockwise direction.

Example 6. (________________) a) Find the Cartesian equation of the curve parametrised by $\mathbf{r}(t) = (3\cos t, 2\sin t)$, where $t \in [0, 2\pi]$. Sketch the curve.

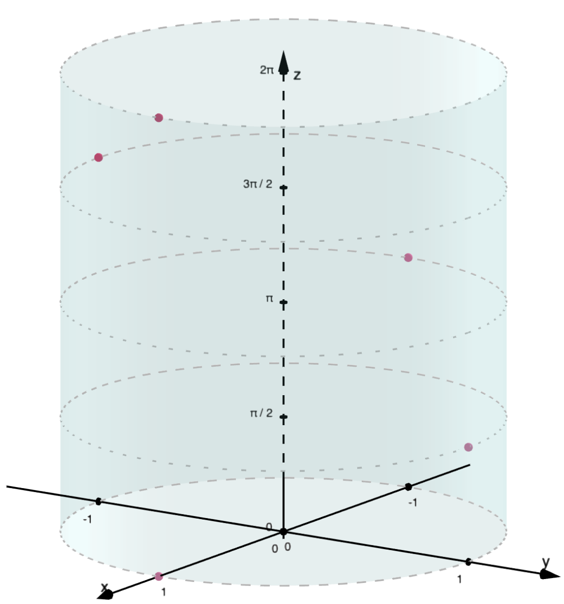
b) Sketch curve parametrised by $\mathbf{r}(t) = (3\sin t - 3, 2\cos t + 2)$ where $t \in [0, \pi]$.

Example 7. (_______________) a) Find the Cartesian equation of the curve parametrised by $\mathbf{r}(t) = (2\sinh t, \cosh t)$, where $t \in \mathbb{R}$. Sketch the curve.

b) Find a parametrisation for the curve $x^2 - 4y^2 = 1$, where $x > 0$. Sketch the curve.

Example 8. (_______________) Sketch the curves in $\mathbb{R}^3$ parametrised by

$$\mathbf{r}(t) = (\cos t, \sin t, t)\,, \text{ where } t \in [0, 2\pi].$$



Example 9. (*Polar curves*) Sketch these polar curves (where $\theta \in [0, 2\pi]$). Assume $r \geq 0$.
a) $r(\theta) = 1 - \cos\theta$,
b) $r(\theta) = 3\sin 2\theta$.

Example 10. Sketch the curve parametrised by $\mathbf{r}(t) = (e^t + 1,\, t^2), \quad t \in \mathbb{R}$.
Tip: Think about the behaviour at $t = 0$ and at $t \to \pm\infty$.

Example 11. On the same set of axes, sketch the curves parametrised by

$$\mathbf{r}(t) = (\cos^n t,\, \sin^n t)\,,$$

where $n = 1$ and 2, and $t \in [0, \pi/2]$. (Also think about $n = 3$. See Quiz 1.)

1.2 Plotting curves on a computer

There are lots of online tools to help you visualise parametric curves in $\mathbb{R}^2$ and $\mathbb{R}^3$. These include *Desmos*, *GeoGebra* and `math3d.org`. These tools are very intuitive and don't require you to be an expert in special syntaxes.

Python can also produce beautiful, interactive graphics in $\mathbb{R}^2$ and $\mathbb{R}^3$. This will require a bit of coding, and you need to learn Python commands. But the skills you gain from being able to produce Python visualisation are much more beneficial in the long run compared to quick online tools. Python fluency stands out on your CV, and any future work you do on data analysis will require these visualisation skills. I strongly encourage you to use Python throughout this course to help you visualise curves and surfaces, and to verify answers that you've worked out by hand. But remember: you won't have access to the computer during the exams, so don't become overly dependent on it.

Here are Python code snippets for plotting curves in $\mathbb{R}^2$ and $\mathbb{R}^3$. I have given two of several possible coding variations to plot a parametric curve in $\mathbb{R}^3$. Both require the keyword argument `projection='3d'`.

Plotting parametric curves in $\mathbb{R}^2$ and $\mathbb{R}^3$

	`import numpy as np`
	`import matplotlib.pyplot as plt`
Create array $t \in [0, 2\pi]$ (50 values)	`t = np.linspace(0, 2*np.pi)`
Plot $\mathbf{r}(t) = (\cos t, \sin t)$ (a circle)	`plt.plot(np.cos(t), np.sin(t))`

Plot $\mathbf{r}(t) = (\cos t, \sin t, t)$ (a helix)	`ax = plt.axes(projection='3d')`
	`ax.plot(np.cos(t), np.sin(t), t)`
	`plt.show()`
	`# OR`
	`fig = plt.figure()`
	`ax= fig.add_subplot(projection='3d')`
	`ax.plot(np.cos(t), np.sin(t), t)`
	`plt.show()`

Polar coordinates

Recall that polar coordinates (r, θ) are related to Cartesian coordinates (x, y) by $x = $ __________, $y = $ __________, where $r \geq 0$.

To plot a polar curve $r = f(\theta)$ in Python, use either of the following methods

Plotting a curve in polar coordinates

	`import numpy as np` `import matplotlib.pyplot as plt`
Create array $\theta \in [0, 2\pi]$ (100 values) $r = 3\sin 2\theta$	`theta = np.linspace(0, 2*np.pi, 100)` `r = 3*np.sin(2*theta)`

Method I: Convert to Cartesian and use θ as the parameter	`# Method I`
	`plt.plot(r*np.cos(theta),`
	`         r*np.sin(theta))`
	`plt.show()`

Method II: Plot on polar axes	`# Method II`
	`ax = plt.axes(projection='polar')`
	`ax.plot(theta, r)`
	`plt.show()`

Both methods allow the possibility that $r < 0$, but Method II displays the curve in an unconventional way when $r < 0$ (try it), so Method I is usually preferred. To exclude the part $r < 0$ when using Method I, insert `r = r.clip(min=0)` before plotting.

Example 12. Plot the following famous curves using Python.

(a) *Lemniscate of Bernoulli*

$$\mathbf{r}(t) = \left(\frac{\cos t}{1 + \sin^2 t}, \frac{\cos t \sin t}{1 + \sin^2 t}\right), \quad t \in [0, 2\pi].$$

(b) *Viviani's curve*

$$\mathbf{r}(t) = \left(1 + \cos t, \sin t, 2\sin\frac{t}{2}\right), \quad t \in [-2\pi, 2\pi].$$

(c) *Conchoid*

$$r = 3 + 1\sec\theta.$$

(Watch out where $\cos\theta$ is small. The answer has two pieces: a curve plus a loop)

1.3 Discussions

- In this course we will assume that $\mathbf{r}(t)$ is a continuous vector-valued function. This means that each component is a continuous function. Furthermore, we will assume that each component function can be *differentiated infinitely many times.* When this happens, we say that the curve parametrised by $\mathbf{r}(t)$ is a _______________ curve. If the curve is defined piecewise, we assume that each piece is smooth.

  But what does it *really* mean for a function to be continuous or differentiable? These questions are answered in *Analysis I* and *II*.

- In this course, a curve $\mathcal{C}$ usually comes with a *direction* in which the curve is traced out (*i.e.* "start from A and end at B", or "traversed in an anti-clockwise direction"). We say that the curve $\mathcal{C}$ has an _________________. We can also say that $\mathcal{C}$ is an _______________ curve.

- A curve that does not intersect itself is said to be a _______________ curve, or an _______________ curve.

  Observation: Suppose that the function $\mathbf{r}$ satisfies the property

  $$\mathbf{r}(t_1) = \mathbf{r}(t_2) \implies t_1 = t_2$$

  (*i.e.* $\mathbf{r}$ is _______________) on the interval (a, b). Then $\mathbf{r}(t)$ is clearly a simple curve. We allow the possibility that $\mathbf{r}(a) = \mathbf{r}(b)$.

- A curve parametrised by $\mathbf{r}(t)$ where $t \in [a, b]$ is said to be _______________ if $\mathbf{r}(a) = \mathbf{r}(b)$.

Example 13. In each case, draw a rough sketch of a curve $\mathcal{C}_i$ in $\mathbb{R}^2$ with the stated properties.

a) $\mathcal{C}_1$ is a simple closed curve

b) $\mathcal{C}_2$ is closed but not simple

c) $\mathcal{C}_3$ is simple but not closed

d) $\mathcal{C}_4$ is neither closed nor simple.

1.4 Can-do checklist

- ☐ Given the Cartesian equation of a curve, find a parametrisation for it.
- ☐ Given a parametrisation of a curve, sketch it and find its Cartesian equation.
- ☐ Be very familiar with parametric and Cartesian equations of straight lines and ____________________, *i.e.* parabolae, hyperbolae, circles and ellipses.
- ☐ Plot polar curves and parametric curves in $\mathbb{R}^2$ and $\mathbb{R}^3$ using Python.
- ☐ Understand what it means for a curve to be *smooth*, *oriented*, *closed* and *simple*.

CHAPTER 2

DIFFERENTIATION AND ARC LENGTH

2.1 Vector differentiation and the tangent vector

Let f be a scalar-valued function. Recall that the derivative of f at $x = c$ can be defined as a limit:

$$f'(c) = \lim_{\Delta x \to 0}$$

The value $f'(c)$ is the slope of the ________________ to the curve at $x = c$.
Let $\mathbf{r}$ be a vector-valued function. The derivative of $\mathbf{r}$ at $t = c$ can be similarly defined:

$$\mathbf{r}'(c) = \lim_{\Delta t \to 0}$$

This simply means that to differentiate a vector function, we just differentiate each component in the usual way. The geometric meaning of the derivative of a vector-valued function is also analogous to the usual derivative:

> The vector $\mathbf{r}'(c)$ is a __________________________ to the curve at $t = c$.

Example 1. Sketch the curve parametrised by $\mathbf{r}(t) = (\cos t, 2\sin t)$ where $t \in [0, 2\pi]$. Find the *unit tangent* to the curve at $t = 0$ and at $t = \pi/4$. Include them on your sketch. [*Note:* the standard symbol for the unit tangent is $\mathbf{T}(t)$.]

Example 2. Find the unit tangent to the helix $\mathbf{r}(t) = (\cos t, \sin t, t)$ at $t = 0$.

2.2 Regular curves

Definition. Consider a curve $\mathcal{C}$ parametrised by $\mathbf{r}(t)$ where $t \in I$. $\mathbf{r}$ is said to be a *regular parametrisation* if _______________ (or, equivalently, _______________) at all points on the curve

Definition. A curve is said to be __________ if it has a regular parametrisation.

Curves defined in segments can also be described as *piecewise regular*.

We have already seen examples of regular curves.

Example 3. Show that the curve with equation $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ (where a and b are nonzero) is a regular curve.

Example 4. Find a regular and a non-regular parametrisation for the line $y = x$.

Just because a curve "looks nice" doesn't guarantee that its parametrisation is regular. See Quiz 2.

It's useful to study regular curves because they are guaranteed to have a special parametrisation which will greatly simplify equations and proofs in vector calculus. More about this soon.

2.3 Differentiation rules for vectors

Let $\mathbf{u}(t)$ and $\mathbf{v}(t)$ be differentiable vector functions. The following identities follow from those of their real-valued components. Try proving them by writing everything out in component form.

I. For all $\lambda \in \mathbb{R}$, $\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathbf{u}(t) + \lambda\mathbf{v}(t)\right] =$

II. Let $f(t)$ be a differentiable real-valued function, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[f(t)\mathbf{u}(t)\right] =$$

III. $\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathbf{u}(t) \cdot \mathbf{v}(t)\right] =$

IV. $\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathbf{u}(t) \times \mathbf{v}(t)\right] =$

V. $\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathbf{u}\left(f(t)\right)\right] =$

When working with vector functions and the arguments (t) start to get in the way of your working, you may tastefully omit them with the understanding that all vectors are treated as functions of t.

Example 5. Prove property III.

Example 6. Find the derivative of $f(t) = |\mathbf{r}(t)|$ (assuming that $f(t) \neq 0$).

The following result is simple to prove, yet it is one of the most useful results in vector calculus, so I've labelled it as a Lemma.

Lemma 2.1. Let $\mathbf{r}(t)$ be a vector-valued function such that $|\mathbf{r}(t)| =$ constant. Then $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are orthogonal.

Proof:

2.4 Arc length

Given a curve parametrised by $\mathbf{r}(t)$, how long is the curve measured between two given t values?

Here is a sketch of the argument. Let Δs be the arc-length element of the curve between two parameter values t and $t + \Delta t$.

Lemma 2.2. The *arc length* of the curve parametrised by $\mathbf{r}(t)$ between $t = t_0$ and $t = t_1$ is given by

$$s = \qquad (2.1)$$

We always have $t_1 > t_0$ to ensure that the arc length s is always positive.

Definition. The *arc-length function* is defined as

$$s(t) = \qquad \text{for some constant } t_0 \qquad (2.2)$$

Important: (2.1) is a number, but (2.2) is a function.

If we think about $\mathbf{r}(t)$ physically as the trajectory (displacement) of a particle. Then the vector $\mathbf{r}'(t)$ is its _______________, and the magnitude $|\mathbf{r}'(t)|$ is its _______________. The vector $\mathbf{r}''(t)$ is its ____________________.

Example 7.

(a) Calculate the length of the helix $\mathbf{r}(t) = (\cos t, \sin t, t)$ where $t \in [0, 2\pi]$.

(b) Obtain the arc-length function, $s(t)$, with $t = 0$ at the starting point.

(c) Write down the parametrisation of the helix using s instead of t. Find the speed of the curve in this parametrisation.

Definition. The *arc-length parametrisation* of $\mathbf{r}(t)$ is denoted $\mathbf{r}(s)$, where s is the ________________________ (with some initial parameter value t_0).

Lemma 2.3. Let $\mathbf{r}(s)$ be the arc-length parametrisation of a curve parametrised by $\mathbf{r}(t)$. Suppose that $|\mathbf{r}'(t)| \neq 0$, then $\mathbf{r}(s)$ has unit speed.
Proof:

This means that *every regular curve has a unit-speed parametrisation.*

Example 8. Find the arc-length parametrisation of the semicircle radius $R > 0$:

$$\{(x, y) : x^2 + y^2 = R^2,\ y > 0\}.$$

It's always good to check that the range of s is from 0 to the entire length of the curve.

Example 9. Find the arc-length parametrisation of the ellipse:

$$\mathbf{r}(t) = (\cos t, 2\sin t),\ t \in [0, 2\pi].$$

The moral of the story: it's often not possible to find the arc-length parametrisation explicitly. Most of the time we won't need to know the explicit form of the arc-length parametrisation, $s(t)$, but just the fact that it *exists* is enough to simplify many proofs.

Finally, Lemma 2.3 says that an arc-length parametrisation has unit speed. But does a unit-speed parametrisation have to be an arc-length parametrisation? See if you can deduce the answer from the following result.

Example 10. Let $\mathbf{r}(s)$ and $\mathbf{r}(u)$ be two unit-speed parametrisations of a curve. Find a relation between u and s.

2.5 Can-do checklist

- ☐ Find the (unit) tangent vector for a parametric curve.
- ☐ Show that a curve is regular. Show that a given parametrisation is regular.
- ☐ Differentiate vector equations containing dot and cross products.
- ☐ Calculate the speed of a parametrised curve.
- ☐ Find the arc length of a curve.
- ☐ Find the arc-length parametrisation of a curve explicitly where possible.

Example 11. Let $t \in (0, \pi/2)$. Consider the curves with the following parametrisations.

$$\mathbf{r}_1(t) = \left(\sin^2 t, 2\cos^2 t\right), \qquad \mathbf{r}_2(t) = \left(\sin t, 2\sin^2 t\right).$$

i) Sketch the curves. ii) Are these curves regular? iii) Calculate their arc lengths.

Hint: Here is a useful integral that you should prove once in your life. Let $\alpha > 0$.

$$\int \sqrt{1 + \alpha^2 x^2}\, \mathrm{d}x = \frac{x}{2}\sqrt{1 + \alpha^2 x^2} + \frac{1}{2\alpha} \sinh^{-1} \alpha x + C.$$

CHAPTER 3

PARTIAL DIFFERENTIATION AND APPLICATIONS

Suppose we want to define a function, T, that tells us about the temperature at each point in this lecture theatre, what type of function would this be?

Well, this function would need to map each point (x, y, z) to a number $T(x, y, z)$. If we let $U \subseteq \mathbb{R}^3$ be the set of all points in this lecture theatre, then $T : U \to$ ____ is a ____________________ function, (sometimes called a *scalar field*). In this chapter, we will learn to differentiate this kind of *multivariable* scalar-valued function.

Example 1. Here's an example of a familiar multivariable function. Consider $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \sqrt{x^2 + y^2}.$$

Can you think of two ways to represent this function visually?
On your visual representations, indicate the point where $(x, y) = (3, 4)$

.

Definition. Let $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}$. The __________ of f is the set of points in $\mathbb{R}^3$ such that

$$=$$

A graph is a surface in $\mathbb{R}^3$. Think of z as the height associated with each point (x, y).

Example 2. Describe the surface of the unit hemisphere (centred at the origin) as a graph $z = f(x, y)$ where $z \geq 0$. Write down the domain of f.

Example 3. a) Sketch the surface $z = f(x, y) = 6 - 3x - 2y \quad$ where $x, y \geq 0$.
b) Locate all points on the domain that are mapped by f to 0, 3 and 6.
These are called the _______________ of f.

Example 4. a) Sketch $z = f(x, y) = 1 - x^2 - y^2$ where $(x, y) \in \mathbb{R}^2$. How would you name this surface? b) On $\mathbb{R}^2$, locate all points that are mapped by f to 1, 0 and -1.

Example 5. Let $T : \mathbb{R}^3 \to \mathbb{R}$ be a function defined by $T(x, y, z) = x^2 + y^2 + z^2$. Describe all points that are mapped by T on to 0, 1 and 2. These are called the ____________________ of T.

Definition. The *level sets* of a function $f : U \subset \mathbb{R}^n \to \mathbb{R}$ are the sets of points:

for each constant k in the range of f.

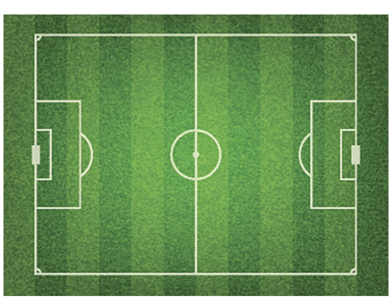
When $n = 2$, the level sets are called *contours*. When $n = 3$, the level sets are called *isosurfaces*. See Quiz 3 for visualisations of interesting level sets.

3.1 Partial differentiation

Suppose that the temperature at the point (x, y) on a football pitch is given by $f(x, y)$. If you stand at a point P with coordinates (a, b), how would you measure the *rate of change* of the temperature at P?

Of course you will want to measure the temperature at *nearby* points, but where?

Since the football pitch is a two-dimensional world, we probably need *two* derivatives to completely describe the rate of change at P: one along the x and one along the y direction. So perhaps we can measure the following numbers (where $h > 0$ is a tiny step).



Definition. The *partial derivatives* of $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ at the point (a, b) are defined by

$$\frac{\partial f}{\partial x}(a, b) = \lim_{h \to 0}$$

Another way to write $\frac{\partial f}{\partial x}(a, b)$ is _____

Sometimes we leave the partial derivatives as functions of (x, y), just like when leave the usual derivative of f as $f'(x)$. In other words, partial derivatives can be expressed as functions of (x, y). We then substitute $(x, y) = (a, b)$ to obtain numerical values of the partial derivatives if required.

We will seldom need to use the limit definition to calculate partial derivatives. All we need is our usual differentiation techniques, but pretend that **all variables, except the one we're differentiating with respect to, are constant**.

Example 6. Suppose the temperature at point (x, y) on the football pitch is given by $f(x, y) = 10x - x^2y$, and point P has coordinates $(1, 1)$.
a) Find the temperature at P, b) Evaluate the partial derivatives of f at P.

Example 7. Let $f(x, y, z) = 2xy - \ln(1 + y) + 3\sin(y + 3z)$.
Find all first-order partial derivatives of f. Evaluate them at the origin.

Example 8. Let $g(u,v) = e^{uv} + \sqrt{u+3v}$. Find all first and second-order partial derivatives of g.

Note: In this course, you can assume that mixed second derivatives are symmetric, *i.e.* the partial differentiation can be done in any order. Thus we can use the notation $\frac{\partial^2 g}{\partial u \partial v}$ without ambiguity about which variable is differentiated first. (Proof next year.)

Strong warning: If a function depends on more than one variable, say $f(x,y)$, then you must **not** write $\frac{\mathrm{d}f}{\mathrm{d}x}$ or $\frac{\mathrm{d}f}{\mathrm{d}y}$ because these quantities don't make sense.
Equally, if a function depends on one variable, say $f(x)$, then you should write $\frac{\mathrm{d}f}{\mathrm{d}x}$ and not $\frac{\partial f}{\partial x}$. Using the wrong d or ∂ will result in loss of marks!

3.2 The chain rule for partial differentiation

Almost all differentiation rules you know for one-variable functions can be applied (carefully) to multivariable functions. For instance, the product rule certainly follows trivially. Let f and g be functions of (x,y,z), then

$$\frac{\partial}{\partial x}(fg) =$$

This holds because we are just treating y and z as constants.

What about the Chain Rule for partial differentiation? Recall the one-variable version

$$\frac{\mathrm{d}}{\mathrm{d}t} f\left(x(t)\right) =$$

Consider the function $f(x, y)$ where x and y are themselves functions of t. Then it's possible to consider f as a function of *one* variable t. This means that we can calculate the derivative of f with respect to t, denoted _____.

The result is the following. (Tip: draw a diagram to keep track of variables.)

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \tag{3.1}$$

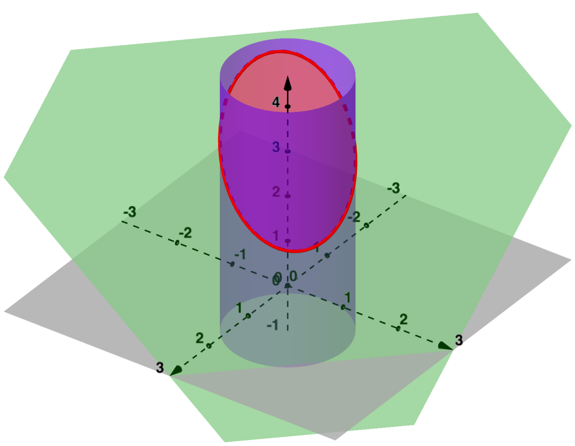
Similarly, consider $g(x, y, z)$ where x, y, z are themselves functions of t, then

$$\frac{\mathrm{d}g}{\mathrm{d}t} = \tag{3.2}$$

The proof of these results will need a bit more knowledge of analysis and will be discussed next year. For now it is more important to get the hang of applying the Chain Rule to perform simple calculations.

Top tip: Draw a diagram. Go via all possible paths towards the independent variable.

Example 9. Let $f(x, y) = 3 - x - y$, where $x = \cos t$, $y = \sin t$. Calculate $\frac{\mathrm{d}f}{\mathrm{d}t}$ in 2 ways. Interpret what $\frac{\mathrm{d}f}{\mathrm{d}t}$ means given the following figure.



Example 10. Let $w = xe^{y/z}$, where $x = t^2$, $y = 1 - t$, $z = 1 + 2t$
Find an expression for dw/dt. Hence, evaluate it at $t = 1$.

Example 11. Let $w = xy + yz + zx$, where $x = r\cos\theta$, $y = r\sin\theta$, $z = r\theta$
Find $\dfrac{\partial w}{\partial r}$ using the Chain Rule.

[*Hint: The partial derivative we want implicitly assumes that we are viewing w as a function of (r, θ). So your final expression should contain only r and θ, not x, y, z.*]

3.3 The grad operator and directional derivatives

Definition. Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$. The ______________ of f, denoted ∇f or __________ is defined as the vector

$$\nabla f =$$

In other words ∇f (read "grad f") is a vector containing all the information about the rates of change of f along all the coordinate directions. Here are some examples.

Example 12. a) Let $f(x, y) = x \sinh y + 1$. Find ∇f.
b) Let $g(x, y, z) = \ln(5xyz)$. Find ∇g.

Example 13. Consider $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ defined by $g(t) = f(\mathbf{r}(t))$, where $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$ is a parametrised curve in $\mathbb{R}^3$. Show that $g'(t) = \nabla f \cdot \mathbf{r}'(t)$.

We can also regard ∇f as the result of an *operator* ∇ acting on a scalar-valued function f. For example, in $\mathbb{R}^3$, we identify the ________________________ as

$$\nabla := \tag{3.3}$$

This is analogous to saying that $\frac{\mathrm{d}f}{\mathrm{d}x}$ is the result of the differential operator $\frac{\mathrm{d}}{\mathrm{d}x}$ acting on a function f. The operator needs an input (a function) to make sense.

The grad operator acts on a scalar/vector, and returns a scalar/vector. It is extremely important that you are confident which objects are scalar and which are vectors.

> The grad operator ∇ takes a ______________ and returns a ______________.

We will meet the operator ∇ again in various guises the weeks to come.

Let's study one application of the grad operator. How fast does the temperature at a point on the football pitch change if we were to walk in a given direction $\mathbf{u}$ (not necessary a coordinate direction)?

Definition. Define $f : U \subset \mathbb{R}^3 \to \mathbb{R}$ and let $\mathbf{u}$ be a unit vector. The ____________________ of f in the direction of $\mathbf{u}$, denoted $D_{\mathbf{u}}f$, is defined as

$$= \lim_{h \to 0}$$

Note that the directional derivative is a scalar (a number, telling us how fast the temperature changes). **The direction u must be a unit vector**. If not, you must first normalise it. We use the hat symbol $\hat{\mathbf{u}}$ to emphasise the fact that it is a unit vector.

In practice, we don't usually need the limit definition to evaluate direction derivatives. Use this result instead.

Theorem 3.1. $D_{\mathbf{u}}f(\mathbf{x}) = \nabla f \cdot \mathbf{u}$
Proof:

The above result shows that the direction derivative of f is simply ∇f ____________ in the direction of $\mathbf{u}$.

Also note that if $\mathbf{u} = \mathbf{i}$, then $D_{\mathbf{u}}f$ reduce to _______ as expected.

Example 14. a) In Example 6 (temperature on the football pitch), what is the rate of change of the temperature at P if you walk in the direction of $\mathbf{u} = \mathbf{i} + \mathbf{j}$?
b) Find the directional derivative of $f(x, y, z) = x\sin(yz)$ at the point $(1, 3, 0)$ in the direction of $\mathbf{v} = (1, 2, -1)$.

3.4 Discussions

- **Practice to gain confidence.** We have just taken a first glimpse into the world of multivariable calculus. This is a topic that requires a lot of practice to gain confidence, much more than these notes and the Quizzes can give you. Do find additional practice in the recommended textbooks.

- **Subscript notation.** The notation $\dfrac{\partial}{\partial x}$ can get quite clunky and burdensome. Just as we abbreviate $\dfrac{\mathrm{d}f}{\mathrm{d}x}$ by $f'(x)$, we can use the *subscript* notation to save time and make our working cleaner:

$$f_x \quad \text{means} \quad \frac{\partial f}{\partial x}.$$

Similarly, we can write __________ to mean $\dfrac{\partial f}{\partial y}$, __________ to mean $\dfrac{\partial^2 f}{\partial x^2}$, and __________ to mean $\dfrac{\partial^2 f}{\partial x \partial y}$.

You are not obliged to use this notation. In fact, I would recommend you use it only when you are very confident with partial derivatives (*e.g.* done > 100 partial derivatives).

- **Are mixed derivatives always equal?** See *Counterexamples in Analysis* by Gelbaum and Olmsted.

3.5 Can-do checklist

- ☐ Identify the level sets of a given scalar-valued multivariable function (including sketching of contours in $\mathbb{R}^2$, and identifying isosurfaces in $\mathbb{R}^3$).
- ☐ Sketching simple surfaces of the form $z = f(x, y)$ (a 'graph').
- ☐ Define and calculate partial derivatives of a given multivariable function. Intuitive understanding of what partial derivatives mean.
- ☐ Use the Chain Rule to differentiate a given multivariable function. Use the correct d or ∂. Use a diagram to help formulate the correct Chain Rule.
- ☐ Calculate the gradient ∇f for a given f. Define the grad operator ∇.
- ☐ Define and calculate the directional derivative of f in a given direction $\mathbf{u}$. Intuitive understanding of what the directional derivative means.

CHAPTER 4

GEOMETRY AND APPLICATIONS

Let's study 3 applications of partial differentiation to the geometry of surfaces in $\mathbb{R}^3$, a) finding linear approximations, b) finding normal vectors, c) classifying critical points.

4.1 Linear approximations and tangent planes

Recall the Taylor expansion of the function $f(x)$ around $x = a$. To linear order:

$$f(x) \approx c_0 + c_1(x - a). \tag{4.1}$$

To find the constant c_0, we substitute $x = a$ in (4.1) to obtain $c_0 =$ __________.

To find c_1, we differentiate wrt x then substitute $x = a$.

Thus we obtain the linear approximation of $f(x)$ for x near a:

$$f(x) \approx \tag{4.2}$$

The meaning of this equation is that as we zoom in towards the curve $y = f(x)$ around $x = a$, the function is well approximated by a __________ with equation (4.2).

Now let's use the same technique to find a linear approximation for the two-variable function $f(x, y)$ near (a, b).

The meaning of the above equation is that as we zoom in towards the surface $z = f(x, y)$ around $(x, y) = (a, b)$, the function is well approximated by a __________.

For a function $f(x, y, z)$, the same *linearisation* process yields the approximation near (a, b, c):

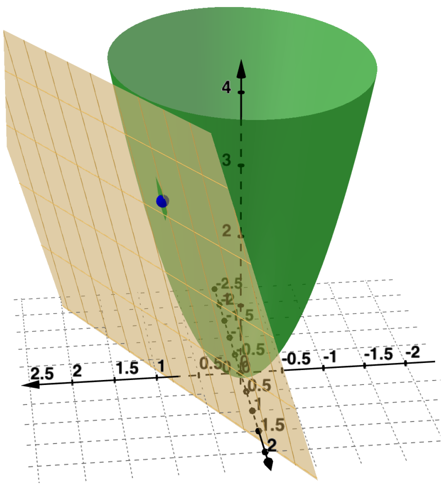
You can see that a similar technique will also yield the linear approximation of an n-variable function $f(\mathbf{x})$ near $\mathbf{x_0}$. Writing the linear approximation in vector notation, we have the following result.

Lemma 4.1. The linear approximation of a function $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(\mathbf{x})$ near $\mathbf{x_0}$ is given by

Example 1. Find the linear approximation of $f(x, y) = \sqrt{1 - x^2 - y^2}$ near $(x, y) = (0, 0)$. Interpret this result graphically.

When we obtain the tangent plane in the form $Ax + By + Cz = D$, as a bonus, we also get the vector (A, B, C) which is __________ to the plane (and also to the surface).

Example 2. Find the linear approximation of $f(x, y) = 2x^2 + y^2$ near $(x, y) = (1, 1)$. Hence, find the *outward-pointing* unit normal to the surface at $(x, y) = (1, 1)$.



Note that the normal to a surface is not a unique vector, so it's helpful to determine whether the normal we obtain points *upwards*, *downwards*, *outwards* or *into* a surface (and whether it's a unit vector). This is going to be an important skill later when we come to integration over a surface.

Lemma 4.2. Consider the surface defined by $z = f(x, y)$ (a __________). The vector $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1)$ is normal to each point (x_0, y_0, z_0) on the surface.
[*Do not memorise this Lemma! We'll see why later.*]

4.2 Normal to a surface

Lemma 4.2 gives us one way to find normal vectors to a given surface of the form $z = f(x, y)$. Let's study a more general method. Note that from now on I'm going to occasionally use the subscript notation for partial derivatives to reduce clutter.

Theorem 4.3. Consider the functions $F : \mathbb{R}^2 \to \mathbb{R}$ and $G : \mathbb{R}^3 \to \mathbb{R}$.
a) The vector $\nabla F = (F_x, F_y)$ is normal to the curve $F =$ constant.
b) The vector $\nabla G = (G_x, G_y, G_z)$ is normal to the surface $G =$ constant.

Proof: Consider a point (x_0, y_0) on the curve $F(x, y) =$ constant (a contour, or a level curve). Clearly the constant equals __________.

Suppose we use the variable t to parametrise this level curve so that

$$F\left(x(t), y(t)\right) = F(x_0, y_0),$$

and let (x_0, y_0) correspond to where $t = 0$. Differentiating the above equation with respect to t and using the Chain Rule, we find

$$\tag{4.3}$$

Now let's evaluate this expression at (x_0, y_0). Recall that the vector $\mathbf{t} = (x'(0), y'(0))$ is tangent to the curve at (x_0, y_0). Eq. 4.3 can then be expressed as the dot product:

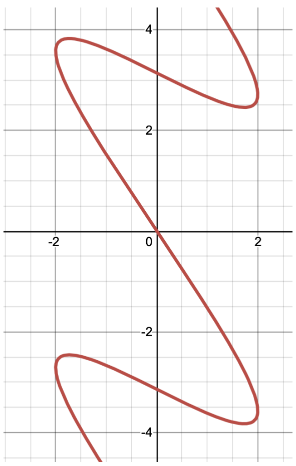
$$= \tag{4.4}$$

In other words, ∇F is perpendicular to the tangent of the curve $F =$ constant at (x_0, y_0). This proves (a).

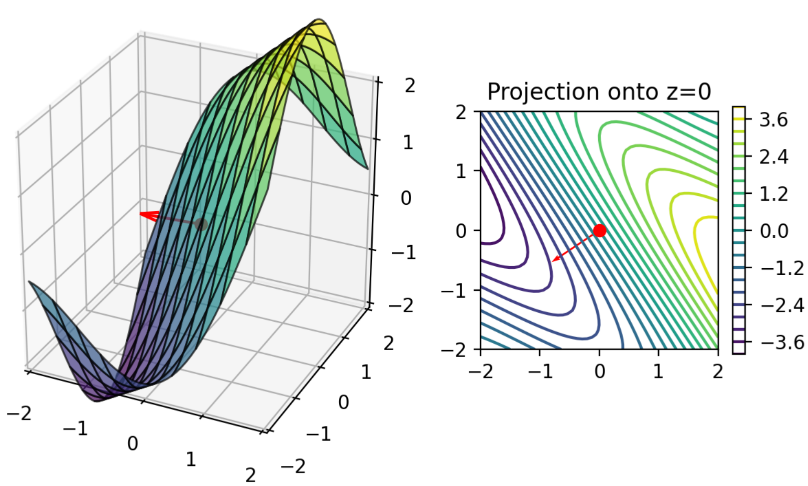
For part (b), we start by considering *any* parametrised curve on the isosurface $G =$ constant, passing through (x_0, y_0, z_0) where $t = 0$. Proceeding identically, we come to the conclusion that ∇G is perpendicular to *all possible tangent vectors* at (x_0, y_0, z_0). All such tangent vectors lie on the tangent *plane* at (x_0, y_0, z_0), hence we conclude that ∇G is normal to the surface $G =$ constant. This proves (b). $\square$

Example 3. Use Theorem 4.3(b) to deduce Lemma 4.2.

Example 4. Let $F(x, y) := x + 2\sin(x + y)$. Find the equation of the straight line normal to the (level) curve $F(x, y) = 0$ at the origin.



Example 5. Consider the surface defined by $z = x + 2\sin(x + y)$.
Find the *upward-pointing* unit normal to the surface at the origin.
Hence write down the equation of the tangent plane to the surface at the origin.
(More about the figure below on the next page.)



The visualisations of Examples 4 and 5 are shown at the bottom of the previous page. The right-hand panel shows colour-coded contour lines corresponding to $F(x, y) =$ constant, with the (red) dot at the origin lying on the contour $F(x, y) = 0$. The red vector points along the normal line found in Example 4.

The unit normal to the surface found in Example 5 is shown as the (red) arrow on the left panel. The projection of this arrow on the x-y plane is shown on the right panel. The `ipynb` file for making this figure (and also useful for plotting contours) is on Moodle.

Example 6. Consider the surface defined by the equation

$$x^2 + y^2 - z^2 = 1.$$

This surface is called a ____________________.

a) Sketch the surface.

b) Sketch the level curves on the x-y plane.

c) Find the equation of the tangent plane at the point $(1, 1, 1)$.

4.3 Critical points and their classification

Recall that the critical point of a function defined by $f(x)$ is where the derivative $\frac{\mathrm{d}f}{\mathrm{d}x} = 0$. Each critical point (say $x = a$) may be classified as a local maximum point or local minimum point using the *second-derivative* test. Can you remember how this test goes?

Important: Either say $x = a$ is a local maximum *point*, or $f(a)$ is a local maximum. These terms are **not** interchangeable!

Let's see how we can do something similar for a two-variable function.

Definition. A point (a, b) is said to be a *critical point* of $f : \mathbb{R}^2 \to \mathbb{R}$ if:

OR, equivalently,

Each critical point can be classified using the 2nd-derivative test for 2-variable functions.

Theorem 4.4. Suppose $f(x, y)$ has a critical point at (a, b). Let $D = \det H$ where

$$H =$$

then

(a) If __________ and __________ at (a, b), then (a, b) is a local minimum point.

(b) If __________ and __________ at (a, b), then (a, b) is a local maximum point.

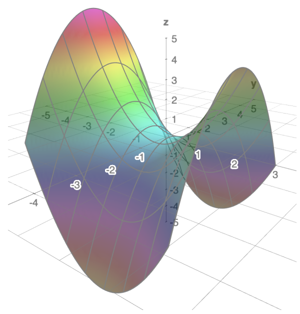
(c) If $D < 0$ at (a, b), then (a, b) is a ____________________.

(d) If $D = 0$ then ____________________________________

The (symmetric) matrix H is called the _______________ of f. It is named after the German mathematician *Otto Hesse* (1811-1874).

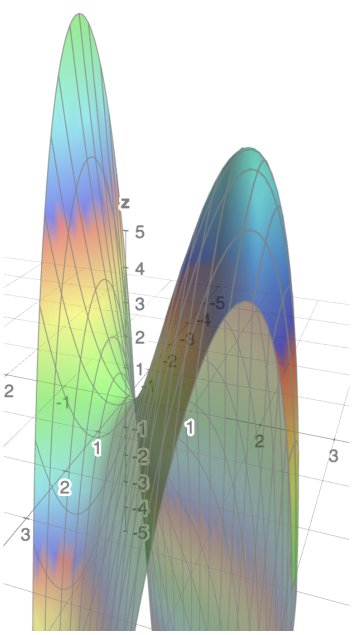
Question. Can we replace f_{xx} by f_{yy} in (a) and (b)?

A *saddle point* is a point on a surface $z = f(x, y)$ such that it is a local minimum point in one direction, but a local maximum point in another, as the picture below illustrates. In this sense, a saddle point is simply a 3D generalisation of a point of inflection for a 2D curve. Around a saddle point, the surface looks like a Pringles chip (see Discussions for a more precise definition). Note how we draw contour lines around a saddle point.



A nice proof of Theorem 4.4 uses quadratic forms which appear in more advanced *linear algebra*. You'll study this proof next year (for a non-linear algebra proof, see page 1016 of the Stewart textbook). For now, you need to be able to use this test confidently.

Example 7. Find and classify the critical points of $f(x, y) = 3y^2 - 2y^3 - 3x^2 + 6xy$. If $x, y > 0$, obtain the maximum possible value of $f(x, y)$.



4.4 Discussions

- **Definitions of maximum/minimum/saddle points** Let's state the precise definitions of these terms here. *Exercise*: Fill in each blank with either $\exists$ or $\forall$. Then try to interpret what these statements mean in words.

  Let $\mathbf{c}$ denote a critical point of a function $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}$, then:

  (a) $\mathbf{c}$ is a *local minimum* point if, _____$\delta > 0$, such that

  $$\_\_\_\_\_\mathbf{x} \in U \text{ such that } |\mathbf{x} - \mathbf{c}| < \delta \implies f(\mathbf{x}) \geq f(\mathbf{c}).$$

  (b) $\mathbf{c}$ is a *local maximum* point if, _____$\delta > 0$, such that

  $$\_\_\_\_\_\mathbf{x} \in U \text{ such that } |\mathbf{x} - \mathbf{c}| < \delta \implies f(\mathbf{x}) \leq f(\mathbf{c}).$$

  (c) $\mathbf{c}$ is a *saddle point* if, _____$\delta > 0$,

  $$\_\_\_\_\_\mathbf{x_1}, \mathbf{x_2} \in \text{the } \delta\text{-neighbourhood of } \mathbf{c} \text{ such that } f(\mathbf{x_1}) < f(\mathbf{c}) < f(\mathbf{x_2}).$$

- **Steepest descent** In real-world applications (*e.g.* physics, engineering, data science, machine learning), the local extrema of a multivariable function are not usually obtained by differentiation or the second derivative test, because the form of the function $f(x, y)$ may not be known explicitly. Instead, a numerical method called *steepest ascent/descent* is employed. We describe it briefly here.

  Take a function given by $f(x, y)$. Recall the definition of the directional derivative.

  $$D_{\hat{\mathbf{u}}} f = \nabla f \cdot \hat{\mathbf{u}} = |\nabla f| \cos\theta,$$

  where $\theta \in [0, \pi]$ is the angle between ∇f and the unit vector $\hat{\mathbf{u}}$. From this simple observation, we can deduce that

  – The directional derivative is maximised when $\theta = 0$, so f is increasing the fastest when we walk in the direction $\hat{\mathbf{u}} = \nabla f / |\nabla f|$ (normal to the contour line $f =$ constant). This is called the direction of *steepest ascent*.

  – Similarly, in the direction $\hat{\mathbf{u}} = -\nabla f / |\nabla f|$, f decreases the fastest. This is called the direction *steepest descent*.

Therefore, to quickly search for a local maximum/minimum point of $f(x, y)$, one simply keeps walking in the (x, y) plane along the direction of ∇f (perpendicular to the contour lines), until (x, y) converges to the top of the hill or the bottom of a valley. More about this in a topic called *optimisation*.

In summary:

Lemma 4.5. The direction of steepest descent for a function $f(\mathbf{x})$ is ________.

- **Normal vectors and computer graphics** Light reflects off objects at an angle which can be determined from the normal vectors on the surface. Thus, normal vectors play a key role in rendering realistic 3D graphics in computer games and cinematic animations. In such applications, paths of light rays are traced between the camera and light sources in a scene, reflecting off intervening surfaces where the normal ∇F is determined numerically. This is a somewhat crude description of the highly sophisticated technique of *raytracing*, which can now be done in real time on home computers and game consoles.

4.5 Can-do checklist

☐ Find the linear approximation of a function $f : \mathbb{R}^n \to \mathbb{R}$ at a given point $\mathbf{x_0}$.

☐ Find the equation of the tangent plane to a surface at a given point.

☐ Find the normal vectors to a surface at a given point (does your normal point up/down?). Derive and apply the mantra **"∇f is normal to $f =$ constant"**.

☐ Find and classify the nature of critical points of a surface $z = f(x, y)$ using the second derivative test.

☐ Find the direction of steepest descent for a function $f : \mathbb{R}^n \to \mathbb{R}$ at a given point $\mathbf{x_0}$.

CHAPTER 5

INTEGRATION I: CARTESIAN COORDINATES

5.1 Double integrals

Question. Find the area under the curve $y = f(x)$ from $x = a$ to b.

These steps are probably along the line of what you were taught in school.

(a) Divide the interval $[a, b]$ into n tiny intervals of size Δx.

(b) In each subinterval $[x_i, x_{i+1}]$, we sample a point x_i^* (say, the midpoint). Form a thin strip of height $f(x_i^*)$ over the interval.

(c) The area is the limit of the sum of the area of the strips as $\Delta x \to 0$, *i.e.*

$$= \lim \sum \tag{5.1}$$

Here's an alternative way to express the area under the graph. Instead of summing over strips, let's sum the area of tiny *rectangles*.

(a) Divide the area into tiny rectangles of area ΔA. Let each rectangle have width $\Delta x = x_{i+1} - x_i$ and height $\Delta y = y_{j+1} - y_j$, so that $\Delta A =$ __________

(b) The area is the limit of the *double* sum of the area of rectangles as both Δx and Δy go to 0, *i.e.*

$$= \lim_{\Delta x \to 0} \lim_{\Delta y \to 0} \sum_{i=0}^{n-1} \sum_{j=0}^{m_i - 1} \tag{5.2}$$

The integrals (5.1) and (5.2) give the same answer. You can see this by evaluating inner (dy) integral of (5.2), treating x as a constant.

This means that we can now express area in $\mathbb{R}^2$ as a double integral. The secret to success with multiple integration is to spend time carefully working out the integration limits. Use the recipe below to help you.

(a) Find the range of all possible x values for the tiny rectangles. The answer must be numbers and not functions.

(b) While a tiny rectangle is moving back and forth between the limits in step (a), *freeze it* at a random position (there should be nothing special about this random position at all). What y values must the rectangle then sweep out to generate the area?

(c) Use the previous answers to fill in the integration limits.
CHECK that the outermost limits are numbers.

Once you have the integrals, work from inside out (*i.e.* do the innermost integral first).

Example 1. Write down the area between the curve $y = x^2$ and $y = 2x$ as a double integral and evaluate it.

Tip: A sketch is always helpful when it comes to multiple integrals.

It's possible to swap the role of x and y in the recipe!

Example 2. Redo the previous example by switching the order or integration.

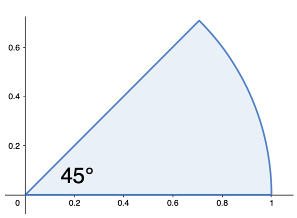
Important: The order of integration is determined by the ordering of your $\mathrm{d}x$ and $\mathrm{d}y$ at the end of the integral. If it helps, you can write something like $\int_{x=a}^{b}\int_{y=c}^{d} \mathrm{d}y\,\mathrm{d}x$.

Example 3. Sketch (and shade) the area of integration represented by these double integrals. Switch the integration order in each case.

a) $\displaystyle\int_0^2\int_{-1}^1 \mathrm{d}y\,\mathrm{d}x$ b) $\displaystyle\int_0^1\int_{3x}^3 \mathrm{d}y\,\mathrm{d}x$ c) $\displaystyle\int_0^1\int_y^{2-y} \mathrm{d}x\,\mathrm{d}y$

Tip: There is often one integration order which is easier than the other(s). Identifying a good order of integration comes with practice and experience.

Example 4. Write down the two double integrals representing the sector of a unit circle shown below.



Now that we can translate between the area of a region in $\mathbb{R}^2$ and a double integral, we can do some really powerful integrations. For area calculation, integrand equals 1, but it doesn't have to be! In fact, we can insert any scalar-valued function $f(x, y)$.

Example 5. Evaluate the following integrals.
a) $\int_1^2 \int_0^2 (3x^2y - xy^2)\,\mathrm{d}y\,\mathrm{d}x$ b) $\int_0^{\pi/2} \int_0^y y\cos x\,\mathrm{d}x\,\mathrm{d}y$
c) $\int_R e^{-y^2}\,\mathrm{d}A$, where $R = \{(x, y)\,|\,0 \le x \le 1,\ x \le y \le 1\}$.

Let's interpret the integrals above. Consider the surface $z = f(x, y)$ over $R \subseteq \mathbb{R}^2$.

Each tiny rectangle in R with area $\Delta A = \Delta x \Delta y$ can be associated with height $z^* = f(x^*, y^*)$, where (x^*, y^*) is a point sampled from the rectangle (say, the centroid).

Therefore, $z^*\Delta A$ represents the __________ of a tall pillar with rectangular base.

Thus, the integrand $\int\int_R z\,\mathrm{d}A = \lim_{\Delta x \to 0} \lim_{\Delta y \to 0} \sum\sum z^*\Delta x\Delta y$ represents the…

5.2 Triple integrals

Recap: The area under the curve $y = f(x)$ can be expressed as either:
a) A single integral $\int f(x)\,\mathrm{d}x$ (summing up area of thin ________).
b) A double integral $\iint \mathrm{d}A$ (summing up area of tiny ______________).

Similarly, the *volume* under the surface $z = f(x, y)$ can be expressed as either:
a) A double integral $\iint f(x, y)\,\mathrm{d}A$ (summing up area of thin ______________).
b) A *triple* integral $\iiint \mathrm{d}V$ (summing up area of tiny ______________).

Here's a recipe for working out the limits of triple integrals.

(a) Start by working out the limits of the double integral describing the region $R \subseteq \mathbb{R}^2$ over which the volume occupies (use the previous recipe.)

(b) Add one more integral in $\mathrm{d}z$ as the innermost integral. This (usually) has the most complicated limits which are typically functions of x and y.

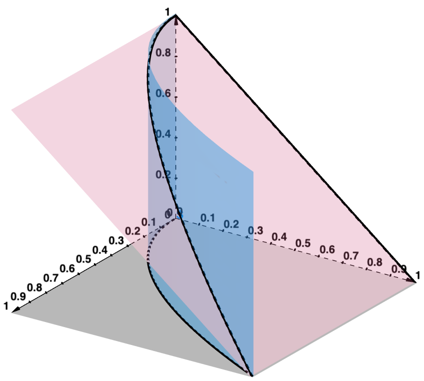
(c) To work out the limits in (b), *freeze* a cuboid at a random potion in R. What z values must the cuboid sweep out to generate the volume?

(d) **CHECK** that the outermost limits are always numbers. All variables must be integrated away into a number as you work from inside out.

As with the previous recipe, the roles of x, y, and z can change in the above recipe.

Example 6. Write down a triple integral representing the volume of a tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. How many ways are there to do this? Compute the volume.

Example 7. Calculate the volume of a solid in the first octant bounded by the surfaces $y + z = 1$ and $y = x^2$. Do this in two ways.



When the integrand $f(x, y, z)$ of the triple integral equals 1, we get a volume. In mechanics, we often need to evaluate triple integrals with integrands that are not unity,

Example 8. Evaluate $\iiint_\Omega xy \, \mathrm{d}V$ where Ω is the volume under the plane $z = 1 + x + y$ and above the region on the x-y plane bounded by the curves $y = \sqrt{x}$, $y = 0$ and $x = 1$.

[*NB: This integral arises in the calculation of the moment of inertia of a solid.*]

Example 9. Suppose that a solid occupying a region $\Omega \subset \mathbb{R}^3$ has density $\rho(x, y, z)$ which varies with position. Derive the expression for the total mass M of the solid.

Example 10. Consider a solid occupying a region $\Omega \subset \mathbb{R}^3$ with density $\rho(x, y, z)$ and total mass M. The coordinates of its *centre of mass* (also called *centroid*), $(\overline{x}, \overline{y}, \overline{z})$, are

$$\overline{x} = \frac{1}{M} \iiint_{\Omega} \rho x \, \mathrm{d}V, \qquad \overline{y} =$$

Calculate the z-coordinate of the centre of mass of the tetrahedron in Example 6, assuming that it has uniform density $\rho = 1$. (Maybe guess what the answer should be.)

[*Using the same formula, we can also find the centre of mass* $(\overline{x}, \overline{y})$ *of a thin lamina in 2D. For derivation, see §15.4 in Stewart.*]

So far we have done everything in Cartesian coordinates, which are good for squarish, boxy areas and volumes with some straight edges or pointy corners. What about these?

Example 11. Express each of the following as an integral in Cartesian coordinates (no need to evaluate). Do they seem easy to evaluate?
a) The area of the circle $x^2 + y^2 = R^2$.
b) The area in the first quadrant bounded by the polar curve $r = 1 + \cos\theta$.
c) The volume of a solid bounded by $z = 4 - x^2 - y^2$ and the x-y plane.
d) The volume of the sphere $x^2 + y^2 + z^2 = R^2$ in the first octant.

5.3 Discussions

- **Fubini's Theorem** In this course, we can assume that that we can switch the order of integration in $\iint_R f(x,y)\,\mathrm{d}A$ without penalty (just adjust the limits). This is allowed as long as the function $f(x,y)$ is continuous on the domain R (where continuity in $\mathbb{R}^n$ will be defined next year). The theorem that legitimises this switching is known as *Fubini's Theorem* after the Italian mathematician *Guido Fubini* (1879-1943).

- **An integration trick** (non-examinable) Let's calculate this seemingly impossible integral
$$I = \int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x.$$
I don't think we can make much progress with this integral using just the integration skills you've learnt in school. But here's a lovely trick which is a fun application of Fubini's Theorem. We turn it into a *double integral*!

  This trick is based on the following simple observation:
$$\int_0^\infty e^{-xy}\sin x\,\mathrm{d}y = \underline{\hspace{3cm}},$$
(where x is held constant in the integral). Therefore, we can write
$$I = \int_0^\infty \int_0^\infty e^{-xy}\sin x\,\mathrm{d}y\,\mathrm{d}x.$$
Now apply Fubini's theorem to switch the integration order. You would most likely have come across the integral $\int_0^\infty e^{-xy}\sin x\,\mathrm{d}x$ as an exercise in integration by parts with the use of a *reduction formula* (*i.e.* integrate by part twice and spot the original integral in the answer). Using this technique, you should verify that:
$$\int_0^\infty e^{-xy}\sin x\,\mathrm{d}x = \frac{1}{1+y^2}.$$
Returning to the original integral, we then find:
$$I = \int_0^\infty \frac{\mathrm{d}y}{1+y^2} = \underline{\hspace{3cm}}$$

5.4 Can-do checklist

- ☐ Express area of a region R in $\mathbb{R}^2$ as a double integral (in Cartesian coordinates). Conversely, given a double integral $\iint_R \mathrm{d}A$, sketch the area R.
- ☐ Express volume of a region Ω in $\mathbb{R}^3$ as a triple integral (in Cartesian coordinates). Conversely, given a triple integral $\iiint_\Omega \mathrm{d}V$, sketch the volume Ω.
- ☐ Switch the order of integration.
- ☐ Apply multiple integrals (in Cartesian coordinates) to calculate area, volume, mass and coordinates of centre of mass.

CHAPTER 6
INTEGRATION II: SPECIAL COORDINATES

Recall that a double integral in Cartesian coordinates is of the form $\iint_R f(x,y)\,\mathrm{d}A$ where

$$\mathrm{d}A = \underline{\hspace{3cm}}$$

And for a triple integral, we have $\iiint_\Omega f(x,y,z)\,\mathrm{d}V$ where

$$\mathrm{d}V = \underline{\hspace{3cm}}$$

Depending on the integrand and the symmetry of the domain of integration, Cartesian coordinates may not be the best choice. This chapter is all about how we can exploit the symmetry of the problem to facilitate calculations. In particular, we want to obtain new expressions for $\mathrm{d}A$ and $\mathrm{d}V$ in special coordinates that exploit these symmetries.

6.1 Polar coordinates

In $\mathbb{R}^2$, the Cartesian (x,y) and polar coordinates (r,θ) are related by:

$$x = \qquad\qquad\qquad y =$$

In the double integral $\iint_R f(x,y)\,\mathrm{d}A$, if the boundary of R can be described by a simple polar curve, or when the integrand can be expressed as a simple function of r and θ, it is natural to perform the integration using polar coordinates. Let's work out how $\mathrm{d}A$ looks like in this case.

In the figure above, on the left, we divide R into polar grids (r_i, θ_j) where $i = 0, 1 \ldots n$ and $j = 0, 1, \ldots m$. The grid lines are curves of constant r and θ, with uniform subdivisions of length Δr and $\Delta\theta$ respectively.

On the right, we zoom in on one such area element. If we sample a point (r^*, θ^*) within this area element, say, the midpoint where

$$r^* = \qquad\qquad\qquad \theta^* =$$

Polar coordinates: $\mathrm{d}A =$

Example 1. Find the area in the first quadrant bounded by the polar curve $r = 1 + \cos\theta$.

Tip: Most of the time we do $\int d\theta$ in the outermost integral.

Example 2. In A-level textbooks, one finds the formula for the area bounded by a polar curve written as

$$\text{Area} \ =$$

But where does this formula come from?

You will probably not be told explicitly to change coordinates. You must judge the symmetry of the integrand and the domain of integration, then take your own initiative.

Example 3. Calculate $\iint_R xy^2 \,\mathrm{d}x \,\mathrm{d}y$ where R is the region in the first quadrant bounded by concentric circles radius 1 and 2, centred at the origin.
Write down the answer if the phrase "the first quadrant" is replaced by $\mathbb{R}^2$.

6.2 Cylindrical coordinates

Now let's consider the triple integral $\iiint_\Omega f \, \mathrm{d}V$ where $\Omega \subseteq \mathbb{R}^3$. Suppose Ω has a rotational symmetry about the z axis (like a cylinder, or any solids with circular horizontal slices), then, instead of using Cartesian coordinates (x, y, z), we might consider using *cylindrical coordinates* (r, θ, z), where r and θ are the usual polar coordinates on the x-y plane. The figure below explains this setup.

$x =$ __________ $y =$ __________ $z =$ __________

Volume element: $\mathrm{d}V =$

In short: cylindrical coordinates =________________________

Example 4. Find the volume of

a) a cylinder base radius a, height h, b) a cone with base radius a, height h.

Cylindrical coordinates are particularly good for problems that have rotational symmetry about the z axis (*e.g.* when a 2D region is revolved around the z axis to form a solid). One way to spot this symmetry is when you see occurrences of ______________ in the integrand or in the equations defining the integration domain. Can you see what's special about this combination?

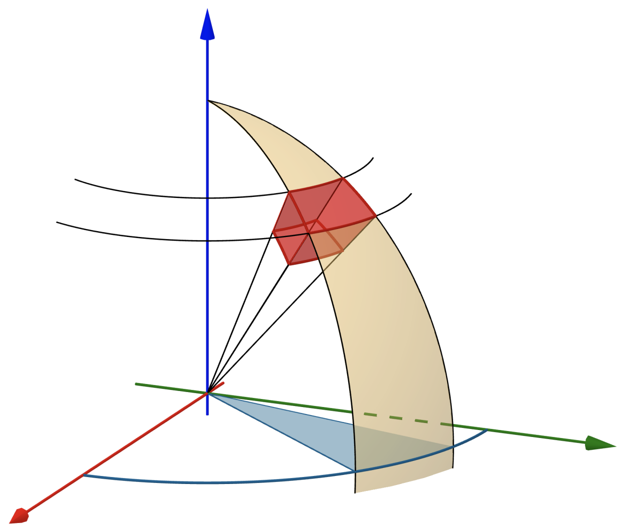
Example 5. Find the coordinates of the centre of mass of the solid bounded by the paraboloid $z = 1 - x^2 - y^2$ and the x-y plane. Assume that the density ρ is constant.

Example 6. Evaluate $\iiint_\Omega (x^2 + y^2)\,\mathrm{d}V$ where the domain Ω is the volume bounded by the surfaces $x^2 + y^2 = 1$, $z = 2 - x$, and $z = 0$.

6.3 Spherical coordinates

In $\mathbb{R}^3$, cylindrical coordinates do a good job of exploiting the rotational symmetry about the z axis. But what if we have even more rotational symmetries, say, about the x or y axis? (*e.g.* a sphere).

One idea is to replace the Cartesian z coordinate by a polar-like angle, _____, measured from the positive z axis (*i.e.* set $\phi = 0$ along the z axis). This gives us a new set of coordinates (r, θ, ϕ) called *spherical coordinates*. From the figure below, let's work out the relationship between spherical and Cartesian coordinates.



$$x =$$

$$y =$$

$$z =$$

Volume element: $\mathrm{d}V =$

Example 7. Find the volume of a sphere radius a.

Spherical coordinates are particularly good for problems that have sphere-like symmetries about the origin. One way to spot this symmetry is when you see occurrences of ______________ in the integrand or in the equations defining the integration domain. What's special about this combination?

Warning: a) Some books use ρ instead of r for spherical coordinates. Why?

b) Some people/books/modules swap the definitions of θ and ϕ (especially in physics).

Example 8. Evaluate $\iiint_\Omega z^2 \,\mathrm{d}V$ where Ω is the volume lying between the spheres radius 1 and 2, centred at the origin. Hence, write down $\iiint_\Omega x^2 \,\mathrm{d}V$ and $\iiint_\Omega y^2 \,\mathrm{d}V$.

Example 9. Evaluate $\iiint_B \exp{(x^2+y^2+z^2)^{3/2}} \,\mathrm{d}V$ where B is the volume within the surface $x^2+y^2+z^2=1$.

6.4 Discussions

- **A Gaussian integral and a trick** (non-examinable) Here is a neat integration trick to evaluate
$$I = \int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x.$$
Since we can also write $I = \int_{-\infty}^{\infty} e^{-y^2}\,\mathrm{d}y$, we can multiply the two expressions for I and, assuming that we can move terms around (appealing to Fubini's theorem), we have
$$I^2 = \int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x \int_{-\infty}^{\infty} e^{-y^2}\,\mathrm{d}y = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y.$$
Note that the domain in the final integral is the whole of $\mathbb{R}^2$. We can evaluate this integral in polar coordinates (r, θ). We have:
$$I^2 = \int_{\theta=0}^{2\pi} \int_{r=0}^{\infty} re^{-r^2}\,\mathrm{d}r\,\mathrm{d}\theta = \underline{\hspace{3cm}} \implies I = \underline{\hspace{3cm}}.$$
This kind of integral, known as a *Gaussian integral*, occurs frequently in university mathematics, especially in probability and statistics (you may recognise that the integrand is the ______________ distribution). However, note that if we change the integration limits to $\int_a^b e^{-x^2}\,\mathrm{d}x$ for arbitrary real numbers a, b, then there are no elementary expressions for the answer.

6.5 Can-do checklist

☐ Evaluate a double integral in polar coordinates.

☐ Define cylindrical coordinates and use them to evaluate triple integrals.

☐ Define spherical coordinates and use them to evaluate triple integrals.

☐ Judging from the domain of integration and the integrand, decide which coordinates are most suitable for evaluating a given double/triple integral.

Example 10 (Extra practice - adapted from Stewart Ex.15.8)**.** In each case, sketch the domain of integration in $\mathbb{R}^3$ and evaluate the integral.

a) $\displaystyle\int_0^{\pi/6}\int_0^{2\pi}\int_0^{3} r^2\sin\phi \; \mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$ b) $\displaystyle\int_0^{\pi/4}\int_0^{2\pi}\int_0^{\sec\phi} r^2\sin\phi \; \mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$

CHAPTER 7

CALCULUS OF MULTIVARIABLE FUNCTIONS

We will now revisit this calculation of $\mathrm{d}A$ and $\mathrm{d}V$ using a method that can be applied to coordinate transformations more generally.

From this point onwards, the distinction between scalar and vector quantities will be even more crucial, so please make sure you distinguish how you write x and ____.

7.1 Change of variable for multiple integrals

The coordinate transformation from (x, y) (Cartesian coordinates) to (r, θ) (polar coordinates) can be represented by a bijective *coordinate transformation* $\mathbf{F} : U \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ where $\mathbf{F}(r, \theta) = (x(r, \theta), y(r, \theta)) = (r \cos\theta, r \sin\theta)$. Let's sketch the domain and the image of $\mathbf{F}$.

We are interested in calculating the area element in the image plane. We used a fairly geometrical argument to derive $\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$. We will rederive this result in this section.

More generally, let $\mathbf{F}(u, v) = (x(u, v), y(u, v))$ be a bijective coordinate transformation from (u, v) to (x, y). Let's obtain an expression for the area element $\mathrm{d}A$ in the x-y plane.

Thus we have proved the following:

Theorem 7.1. Let $\mathbf{F} : U \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\mathbf{F}(u, v) = (x, y)$ be a bijection which represents a coordinate transformation $x = x(u, v)$ and $y = y(u, v)$. Then, the integral of $f : R \subseteq \mathbb{R}^2 \to \mathbb{R}$ can be expressed as

$$\iint_R f(x, y)\, \mathrm{d}x\, \mathrm{d}y = \iint_S f(u, v)\ |\det \mathbf{DF}(u, v)|\ \mathrm{d}u\, \mathrm{d}v.$$

where $S = \mathbf{F}^{-1}(R)$ is the corresponding region in the (u, v) plane. In this above equation, the matrix $\mathbf{DF}(u, v)$, called the ________________________, is defined by

The Jacobian matrix is named after the mathematician *Carl Gustav Jacob Jacobi* (1804-1851). Try to write down a triple-integral version of Theorem 7.1.

Theorem 7.1 (for double and triple integrals) essentially says:

$$\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y =$$
$$\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z =$$

Example 1 (Polar coordinates)**.** Calculate the Jacobian matrix and its determinant for the mapping $\mathbf{F}$ defined by $\mathbf{F}(r, \theta) = (r\cos\theta, r\sin\theta)$.

Example 2 (Cylindrical coordinates)**.** Calculate the Jacobian matrix and its determinant for the mapping $\mathbf{F}$ defined by $\mathbf{F}(r, \theta, z) =$

Example 3 (Spherical coordinates)**.** Calculate the Jacobian matrix and its determinant for the transformation $\mathbf{F}$ defined by

Example 4. Evaluate $I = \iint_R \frac{x-2y}{3x-y}\,\mathrm{d}A$ where R is the parallelogram enclosed by the lines $y = x/2$, $y = x/2 - 2$, $y = 3x - 1$ and $y = 3x - 8$.

Study the examples of coordinate transformations in Quiz 7 carefully. Further remarks:

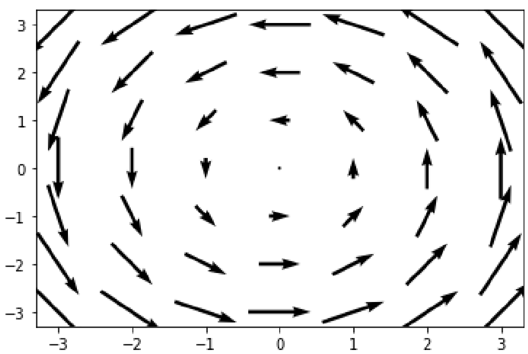
- Some modules/books (including the textbook) define the "*Jacobian*" as the determinant of $\mathbf{DF}$. To avoid this kind of confusion, always say "*Jacobian matrix*" or "*Jacobian determinant*".
- Some books define the Jacobian matrix as the transpose of our $\mathbf{DF}$.
- Alternative notations that are commonly used for $\mathbf{DF}(u, v)$ include __________ or

7.2 Vector fields

Recall from the previous chapter that that $f :\subseteq \mathbb{R}^n \to \mathbb{R}$ is called a scalar-valued function, also known as a *scalar field*. It assigns a *number* to every point in $\mathbb{R}^n$.

In the previous section, we saw functions of the form $\mathbf{F} : U \subseteq \mathbb{R}^2 \to \mathbb{R}^2$. Such a function is called a __________________. It assigns a *vector* in $\mathbb{R}^n$ to every point in $\mathbb{R}^n$.

Question. What are some examples of vector fields in real life?



We often represent vector fields by plotting some representative vectors (arrows) in $\mathbb{R}^2$ or $\mathbb{R}^3$. The above figure shows the vector field $\mathbf{F}(x, y) = (-y, x)$ produced in Python using the `quiver` function. We plot the arrows at integer grid points in $[-3, 3] \times [-3, 3]$. Longer arrows = larger magnitude.

Plotting a vector field in $\mathbb{R}^2$

	`import numpy as np` `import matplotlib.pyplot as plt`
Create array of 7 integer points in [-3,3] Build a 2D grid	`grid = np.linspace(-3,3,7)` `x, y = np.meshgrid(grid,grid)`
$u(x, y) = -y$ $v(x, y) = x$	`u = lambda x,y: -y` `v = lambda x,y: x`

Plot a vector field $\mathbf{F} = (u, v)$
Adjust arrow size (optional)
Sample points = arrow midpoints
	`fig, ax = plt.subplots()`
	`ax.quiver(x, y, u(x,y), v(x,y),`
	`          units='xy', scale=3,`
	`          pivot = 'mid')`
	`plt.show()`

See our Moodle page for a Python code for plotting vector fields in $\mathbb{R}^3$.

Example 5. Sketch the vector fields:
a) $\mathbf{F}(x, y) = (x, 0)$ b) $\mathbf{F}(x, y) = (x, y)$ c) $\mathbf{F}(x, y) = (y, x)$

Actually, there's a vector field we've met before, namely, $\mathbf{F}(x, y) = \nabla f(x, y)$ for some scalar-valued f. $\mathbf{F}$ is a vector field whose flow is ____________ to f= constant.

A vector field $\mathbf{F}$ which can be expressed as ∇f is called a _________________ field.

Question. Are the vector fields in Example 5 conservative?

7.3 Div and curl

Recall the grad operator ∇ which acts on a scalar/vector and returns a scalar/vector. Let's define two more differential operators.

Definition. Let $\mathbf{F} : U \subseteq \mathbb{R}^3 \to \mathbb{R}^3$. The _________________ of $\mathbf{F}$ is, denoted $\nabla \cdot \mathbf{F}$ or _______________ is defined by

$$\text{div } \mathbf{F} =$$

Definition. Let $\mathbf{F} : U \subseteq \mathbb{R}^3 \to \mathbb{R}^3$. The ____________ of $\mathbf{F}$ is, denoted $\nabla \times \mathbf{F}$ or _______________ is defined by

$$\text{curl } \mathbf{F} =$$

The div operator __________ takes a _____________ and returns a _____________.
The curl operator __________ takes a _____________ and returns a _____________.

Example 6. Calculate $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$ where $\mathbf{F}$ is defined by:
a) $(yz,\ xz,\ xy)$ b) $(2xz,\ z + 2\cos y,\ 2z^3)$.

Example 7. Find the div and curl of vector fields (x, y) and $(-y, x)$

Example 8. Prove that if $\mathbf{F}(x, y, z)$ is a conservative field, then $\operatorname{curl} \mathbf{F} = \mathbf{0}$.
Is this consistent with the result in Example 6(a)?

See Quiz 7 for further examples, plus a mega drill on grad, div and curl.

7.4 Discussions

- In multivariable calculus, the Jacobian determinant plays an analogous role to the derivative in one-variable calculus. In particular, we have the following multivariable version of a result from Analysis.

  Theorem 7.2. [*Inverse Function Theorem*] Let $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth function. If the Jacobian determinant is nonzero at a point $\mathbf{x_0}$ in the domain, then there exists a neighbourhood U of $\mathbf{F}(\mathbf{x_0})$ such that $\mathbf{F}^{-1}$ exists and is smooth on U.

  The proof of this result will be studied in *multivariable analysis.*

- Therefore, to ensure that the coordinate transformation in Theorem 7.1 is bijective, we need the Jacobian determinant to be nonzero, and so the domain of the map $\mathbf{F}$ has to be restricted to ensure this. It's not a major concern in this course, but this will be very important in future courses (*e.g.* differential geometry).

7.5 Can-do checklist

☐ Define the Jacobian matrix of a mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$ ($n = 2$ or 3). Explain the significance of the Jacabian determinant.

☐ Calculate the area or volume element ($\mathrm{d}A$ or $\mathrm{d}V$) for a new coordinate system defined by a transformation of another coordinate system.

☐ Compute the div and curl of a vector field. Use grad, div and curl in combination.

CHAPTER 8

GREEN'S AND STOKES' THEOREM

8.1 Line integrals

A *line integral* is of the form

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}, \tag{8.1}$$

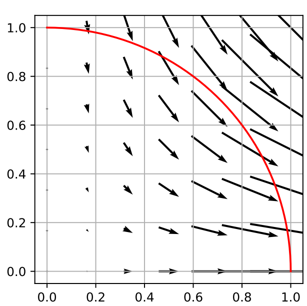
where $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field and C is a curve parametrised by $\mathbf{r}(t)$ with $t \in [a, b]$. In practice, we evaluate the line integral by writing it as

(8.2)

Note that since $\mathbf{r}'(t)$ is the tangent vector along C, the line integral essentially quantifies the tendency of the vector field $\mathbf{F}$ to point in the same direction as C.

Here is a physical interpretation of the line integral (8.2). If $\mathbf{F}$ represents a force acting on an object, then (8.2) represents the energy spent in moving it along the curve C from $t = a$ to $t = b$. In physics, this is called the ______________ on the object by $\mathbf{F}$.

Example 1. Calculate the line integral of $\mathbf{F}(x, y) = (x^2, -xy)$ along the quarter circle shown below, where the circle is traversed a) from $(0, 1)$ to $(1, 0)$, b) from $(1, 0)$ to $(0, 1)$.



It's clear from the expression $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ that the line integral does not depend on the specific parametrisation of the curve C.

Example 2. Find the work done by $\mathbf{F} = (2y^2, z^2, x^2)$ on a particle along the line segment from (0,0,1) to (2,1,0).

Example 3. Calculate the line integral of $\mathbf{F} = (y, x, 2)$ around the closed curve $\mathbf{r}(t) = (5\cos t, 5\sin t, 1)$, $t \in [0, 2\pi]$. Note: a line integral around a *closed* curve is denoted $\oint \mathrm{d}\mathbf{r}$.

Example 4. Prove that if $\mathbf{F}(x, y, z)$ is a conservative field, then the line integral of $\mathbf{F}$ is independent of the path joining the end points of C. (This explains the significance of the word *conservative*.) Is this consistent with the result in Example 3?

The proof in the last Example gives a good indication of what's to come in the rest of this course. We will study 3 integral theorems that look like the FTC:

n-dimensional integral of a differential quantity = __________ dimensional integral.

At our disposal are: line integrals (1D), area integrals (2D) and volume integrals (3D). The differential quantities will involve partial derivatives, grad, div and curl.

8.2 Green's Theorem

Our first theorem reduces an area integral to a line integral. This theorem works in $\mathbb{R}^2$. A curve C in $\mathbb{R}^2$ is said to be ______________________ if it is traversed anti-clockwise.

Green's Theorem in the plane Let C be a positively oriented, simple closed curve and D the region bounded by C. For any two-variable functions P, Q that have continuous partial derivatives on D, we have:

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y = \oint_C (P\,\mathrm{d}x + Q\,\mathrm{d}y) \tag{8.3}$$

The proof is in Quiz 8. More about the mathematician *George Green* in the Discussion.

The weird notation on the RHS is simply $\oint \mathbf{F}\cdot \mathrm{d}\mathbf{r}$ where $\mathbf{F} =$ __________ and $\mathbf{r} = (x, y)$. We can then insert our favourite parametrisation $\mathbf{r}(t)$ if necessary.

Example 5. Use Green's theorem to evaluate $\int_C(-y\,\mathrm{d}x + x\,\mathrm{d}y)$ where C is the circle $x^2 + y^2 = 9$ traversed anticlockwise.

Example 6. Let C be the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$. Using Green's Theorem with $P = -y$ and $Q = x$, calculate the area of the ellipse.

The previous example shows us a neat application of Green's Theorem in turning an area enclosed by C into a line integral along C. This is the principle behind the *planimeter* (a device that, when used to trace the boundary of a region, tells us its area).

Example 7. Evaluate $I = \oint_C (y^2 \, \mathrm{d}x + xy \, \mathrm{d}y)$ where C is the triangle OAB with vertices at $O(0,0), A(1,0), B(1,2)$. Do this a) directly, and b) using Green's Theorem.

Example 8. Use Green's Theorem to evaluate $\oint_C \left(-y^3 \, \mathrm{d}x + x^3 \, \mathrm{d}y\right)$, where C is the unit circle $x^2 + y^2 = 1$, traversed anti-clockwise.

8.3 Circulation

We introduced the *curl* operator __________ in Chapter 7. Here is a physical intuition of what it means. For mathematical details of the following paragraph, see Quiz 8.

Think of $\mathbf{F}(x, y, z)$ as the velocity of a fluid at a given point (x, y, z). We would like to quantify the *rotation* of the fluid at P. This 'pointwise' rotation can be characterised by 2 properties: the *speed* of the rotation, and the *orientation* of the rotation plane. It turns out that the vector $\nabla \times \mathbf{F}$ completely captures these characteristics. Its magnitude is proportional to the rotation speed, and its direction is normal to the rotation plane.

If P also lies on a surface S with unit normal $\hat{\mathbf{n}}$, we might also ask, how much is the fluid circulating *around* $\hat{\mathbf{n}}$? This quantity is called the pointwise ____________________

$$\text{Pointwise circulation} =$$

The circulation is curl $\mathbf{F}$ ____________ in the direction of $\hat{\mathbf{n}}$. It is a *number* whose sign tells us about the direction in which the fluid is locally rotating around $\hat{\mathbf{n}}$...

Make a thumbs-up gesture with your right hand, and align the thumb in the direction of $\hat{\mathbf{n}}$. If curl $\mathbf{F} \cdot \hat{\mathbf{n}} > 0$, then the direction in which the fluid locally rotates around $\hat{\mathbf{n}}$ is the direction in which your other fingers curl. If curl $\mathbf{F} \cdot \hat{\mathbf{n}} < 0$, the fluid rotates in the opposite direction to your fingers. This is sometimes called the *right-hand rule*.

As a rule of thumb (literally), we always choose $\hat{\mathbf{n}}$ (the thumb) to be the ____________________ This isn't strictly necessary, but this convention helps us avoid sign errors.

Since we now have a numerical measure of circulation at a point on a surface. It is natural to find the *net circulation* by summing the pointwise circulation over the entire surface S, *i.e.*

$$\text{Net circulation} \; =$$

Another weird notation: Many books use the symbol $\mathrm{d}\mathbf{S}$ to mean $\hat{\mathbf{n}}\,\mathrm{d}S$.

8.4 Stokes' Theorem

The next theorem is a generalisation of Green's Theorem to 3D curves and surfaces.

Theorem 8.1. (*Stokes' Theorem*) Let $\mathbf{F}(x, y, z)$ be a vector field. Let S be a surface with unit normal $\hat{\mathbf{n}}$ and boundary curve C oriented positively. Then,

The positive orientation of C is now determined by the outward-pointing unit normal $\hat{\mathbf{n}}$.

The theorem will be proved properly in next year's *multivariable analysis*. An accessible proof for a simplified setup (using Green's Theorem) can be found in Stewart §16.8. Here is a pictorial summary of Stokes' Theorem which also helps us work out the correct orientation of the boundary curve.

Example 9. Obtain Green's Theorem from Stokes' Theorem.

The amazing thing about Stokes' Theorem is that it holds for all smooth surfaces with boundary C. If we think of C as the rim of a ___________________, then Stokes' Theorem holds regardless of the shape of the net. Here is an example.

Example 10. Calculate the integral $\iint_S \nabla \times \mathbf{F} \cdot \mathrm{d}\mathbf{S}$ where $\mathbf{F} = (y,\, 2z,\, x^2)$ and S is the surface $z = 4 - x^2 - y^2$ where $0 \leq z \leq 4$.

Let's explore 2 methods: a) Use Stokes' Theorem , b) Evaluate the net circulation on a simpler surface with the same 'rim'.

8.5 Discussions

- **George Green** Stokes' Theorem takes its name from *George Stokes* (1819-1903), an Irish mathematician and physicist who made profound contributions particularly in fluid dynamics. Stokes' theorem, however, was not Stokes' own, but his name stuck because of his habit of setting it as an exam question at Cambridge. The theorem was probably first discovered by *George Green* (1793-1841).

  Green was a remarkable English mathematician who, having taught himself mathematics at a library in Nottingham in his 30s, entered Cambridge University as an undergraduate at almost 40 years old. His name lives on today most notably in *Green's Theorem* and *Green's function*, an indispensable tool in solving partial differential equations.

- **Stokes' Theorem in physics** There are many manifestations of Stokes' Theorem in physics, particularly in electromagnetism, e.g. *Faraday's law* and *Ampère's Law*, which are themselves part of the four *Maxwell's equations*. The other two are *Gauss's Laws* for electricity and magnetism, which are consequences of the *Divergence Theorem* (the subject of the next Chapter).

8.6 Can-do checklist

☐ Calculate line integral of a given vector field along an oriented curve (which you may have to parametrise yourself).

☐ State Green's Theorem (not given in exam, but see Example 9) and use it to evaluate a line integral or an area integral (proof not examinable).

☐ Explain the uses of the right-hand (grip) rule, especially for determining the positive orientation of the curve bounding a surface.

☐ Use Stokes' Theorem (given in exam) to evaluate a line integral or a surface integral on an appropriate open surface (bounded by the 'rim of the butterfly net').

CHAPTER 9

THE DIVERGENCE THEOREM

9.1 Parametric surfaces

Recall that we parametrised a curve $\mathcal{C}$ in $\mathbb{R}^3$ using a one-variable function $\mathbf{r} : [a, b] \to \mathbb{R}^3$, such that $\mathbf{r}(t) = (x(t), y(t), z(t))$. Similarly, a *surface* $\mathcal{S}$ in $\mathbb{R}^3$ can be parametrised using a two-variable function $\mathbf{r}$:

Example 1. a) Find a parametrisation $\mathbf{r}(u, v)$ for the unit disc in $\mathbb{R}^2$ defined by

$$\mathcal{D} = \{(x, y) : x^2 + y^2 \leq 1\}.$$

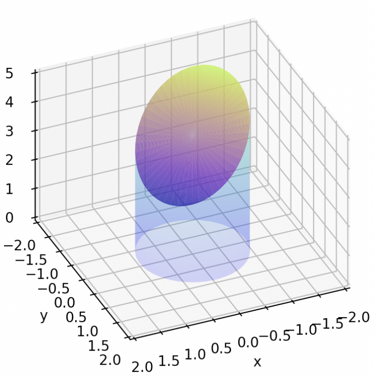
b) Sketch curves of constant u and v.

c) Is this parametrisation unique?

Example 2. a) Find a parametric equation of the plane Π with equation $x + y + z = 4$.
b) Find a parametrization for the surface S defined by

$$\mathcal{S} = \{(x, y, z) : x + y + z = 4 \text{ and } x^2 + y^2 \leq 1\}.$$

c) Write down a parametrization $\mathbf{r}(t)$ for the curve bounding the surface S.



Example 3. Write down a trigonometric parametrisation $\mathbf{r}(u, v)$ for each of the following surfaces in $\mathbb{R}^3$. Assume $a > 0$. a) $x^2 + y^2 = a^2$ b) $x^2 + y^2 + z^2 = a^2$.
Find the *outward-pointing unit normal* on each surface in terms of (u, v).

Example 4. a) Sketch the surfaces with equation

$$z = (x^2 + y^2 + k)^{1/2}, \qquad \text{where } k = 0, 1, -1.$$

b) Write down a vector parametrisation $\mathbf{r}(u, v)$ for each surface.

c) Write down the *outward-pointing unit normal* on each surface in terms of u and v.

To find the normal to a surface, we could use the mantra "∇f *is normal to the surface* f = *constant*". Another way is to note that at each point on the surface $\mathbf{r}(u,v)$ where $(u,v) = (u_0, v_0)$, the partial derivatives $\mathbf{r}_u(u_0, v_0)$ and $\mathbf{r}_v(u_0, v_0)$ are two distinct families of tangent vectors. The tangent plane at $\mathbf{r}(u,v)$ is ______________ by these vectors. Thus, the vector ______________, evaluated at (u_0, v_0),is normal to the tangent plane.

Lemma 9.1. Let $\mathbf{r}(u,v)$ be the parametrisation of a surface $\mathcal{S}$. The *unit normal* at the point P on S is given by

$$\hat{\mathbf{n}} = \qquad (9.1)$$

evaluated at the point P.

In most applications for this module, we will require $\hat{\mathbf{n}}$ to be *outward*-pointing, and you will have to deduce from the geometry yourself which sign to choose.

Example 5. Use formula (9.1) to find the outward pointing unit normal to the cone $z = \sqrt{x^2 + y^2}$.

9.2 Surface area

From the proof of Theorem 7.1, we have obtained the element $\mathrm{d}S$ on a surface $\mathcal{S}$ parametrised by $\mathbf{r}(u, v)$.

$$\mathrm{d}S =$$

Thus, the surface area of $\mathcal{S}$ for a given parameter domain $(u, v) \in \Omega$ is

$$\text{Surface area } = \iint_{\Omega}$$

Top Tip: In the integrand, note that we need the *length* of a cross product rather than the vector itself. As far as you can, avoid performing the cross product (which is a common source of error). The first check to do is to see if $\mathbf{r}_u \cdot \mathbf{r}_v = 0$. If so, $|\mathbf{r}_u \times \mathbf{r}_v| =$____________. This is so much simpler to calculate!

Example 6. Calculate the surface area of a sphere radius a.

It is useful to collect results about the sphere together in one convenient place. You can use these info in the exams (except when asked to derive volume or surface area).

Sphere radius a

Cartesian equation:

Paremetrisation of points within the volume:

$$\mathbf{r}(r,\theta,\phi) =$$

Volume =

Parametrisation of points on the surface:

$$\mathbf{r}(\theta,\phi) =$$

Surface area =

9.3 The Divergence Theorem

We will now study the *Divergence Theorem* - another FTC like theorem. It has wide ranging applications in physics, particularly in fluid mechanics and electromagnetism. The proof is very similar to that of Green's Theorem - see Stuart page 1201 for details. Below we give an intuitive discussion of why it holds.

Start with a closed container with volume V containing fluid that flows outward through its surface S. At each point $P(x,y,z)$ on the surface, suppose that the fluid velocity is $\mathbf{v}(x,y,z)$ with unit $\mathrm{m\,s^{-1}}$, and the density at P is $\rho(x,y,z)$ with unit $\mathrm{kg\,m^{-3}}$. Note that the vector $\mathbf{F} = \rho\mathbf{v}$ has unit kg $\mathrm{m^{-2}\,s^{-1}}$, meaning that it quantifies the rate at which the fluid flows through a small _______ element $\mathrm{d}S$ containing the point P.

The vector $\mathbf{F}$ is called the *flux*. Furthermore, we can work out the total mass of fluid emptying from V per unit time by integrating the flux over the entire surface S, *i.e.*

$$\text{Total flux across } S = \tag{9.2}$$

where $\hat{\mathbf{n}}$ is the outward-pointing unit normal at each point on S.

On the other hand, consider a volume element $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ within V. Let us write the flux vector as $\mathbf{F} = (F_1, F_2, F_3)$. In Quiz 9, you will show that the rate at which the fluid flows out of this volume element is given by

$$\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z =$$

Therefore, total mass of fluid emptying from the entire volume per unit time can be obtained by integrating over the entire volume:

$$\text{Rate of outflow from } V = \tag{9.3}$$

Eqs. 9.2 and 9.3 are the same quantity calculated in two ways. Therefore, we have the following grand conclusion.

Theorem 9.2. (The Divergence Theorem) Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a differentiable vector field. Let V be a finite volume in $\mathbb{R}^3$ and S its (closed) surface. Then

$$=$$

where $\hat{\mathbf{n}}$ is the outward-pointing unit normal to the surface S.

The name of the theorem refers to the div operator on the LHS. Discovered by Lagrange in 1764, the Divergence Theorem was proved decades later by Gauss and also by the Russian mathematician *Mikhail V. Ostrogradsky* (1801-1862). In some texts, the Divergence Theorem is called the Gauss or Gauss-Ostrogradsky theorem.

We see that the Divergence Theorem *looks* like the FTC. Like Stokes' Theorem, the dimensionality of a multiple integral is reduced thanks local cancellations, resulting in only the contribution from the boundary.

The flux integral (the RHS) is sometimes written $\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$ where

$$\begin{aligned} \mathrm{d}\mathbf{S} &= \hat{\mathbf{n}}\, \mathrm{d}S && (9.4) \\ &= \hat{\mathbf{n}} |\mathbf{r}_u \times \mathbf{r}_v|\, \mathrm{d}u\, \mathrm{d}v && (9.5) \\ &= \pm(\mathbf{r}_u \times \mathbf{r}_v)\, \mathrm{d}u\, \mathrm{d}v, && (9.6) \end{aligned}$$

where you have to choose the correct sign to ensure the normal points outwards. Use whichever expression for $\mathrm{d}\mathbf{S}$ that's the most convenient for the situation.

Another weird notation: The boundary surface of a volume V is often denoted ________.

Example 7. Let V be the volume bounded by the unit sphere $x^2 + y^2 + z^2 = 1$ and let $\mathbf{F}(x, y, z) = (z, y, x)$. Calculate the net flux $\iint_{\partial V} \mathbf{F} \cdot \hat{\mathbf{n}} \, \mathrm{dS}$. Let's do this in two ways:

a) Using the Divergence Theorem,

b) Direct calculation.

Example 8. Let $\mathbf{V} = (-2, -2x, -1)$. Calculate $\iint_S \mathbf{V} \cdot \mathrm{d}\mathbf{S}$ where S is the open surface $z = 4 - x^2 - y^2$, where $z \geq 0$. Do this in two ways.
(Note: This gives us two more ways to solve Example 10 in Chapter 8.)

9.4 Discussions

- **Divergence Theorem in physics** Here are some examples of physical laws that are manifestations of the Divergence Theorem: *Gauss's Law* in electromagnetism, *Poisson's Equation* in the theory of gravitation, the *Continuity Equation* in fluid mechanics. You're very likely to meet these results in physics options.

- **What is a surface?** The formal definition of surfaces is quite technical. Why? Well, as an analogy, we can easily understand what a *function* does, but it takes a bit of work to define them formally. With the formal definition, we were able to understand why, for instance, the circle is not a function, but is actually two functions that can be patched together. For completeness, I have given below the formal definition of a surface (non-examinable).

  Definition. A subset $S \subset \mathbb{R}^3$ is called a *surface* if, $\forall \mathbf{p} \in S$, there exists an open set $U \subseteq \mathbb{R}^2$, and an open set $V \subseteq \mathbb{R}^3$ containing $\mathbf{p}$, such that U is homeomorphic to $S \cap V$.

  A *homeomorphism* can be thought of as stretch and bending (as the saying goes, a coffee mug is homeomorphic to a ______________). The definition formalises the following idea: a surface is locally a piece of a 2D plane that has been stretched and bent.

- **Another way to avoid the cross product** In calculating the area element $|\mathbf{r}_u \times \mathbf{r}_v|$, one can also use the following identity

$$|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{EG - F^2} \tag{9.7}$$

  where $E = \mathbf{r}_u \cdot \mathbf{r}_u$, $F = \mathbf{r}_u \cdot \mathbf{r}_v$, $G = \mathbf{r}_v \cdot \mathbf{r}_v$. The RHS is free from cross-products and is generally easier to evaluate. You can prove this result using the vector identity $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$, and substituting $\mathbf{a} = \mathbf{c} = \mathbf{r}_u$ and $\mathbf{b} = \mathbf{d} = \mathbf{r}_v$.

- **Unifying Green's, Stokes' and Divergence Theorems** In year 3 or 4 (or graduate courses in *differential geometry* or *manifolds*), you would be pleasantly surprised to discover that Green's, Divergence and Stokes' Theorems can in fact be elegantly unified into a single equation. In this unified version, called *generalised Stokes' Theorem*, we have the equation

$$\int_{\Omega} \mathrm{d}\omega = \int_{\partial\Omega} \omega,$$

  where ω is a *differential form*, d is the *exterior derivative*, Ω is a *manifold* and $\partial\Omega$ its boundary. These technical terms are simply higher-dimensional generalisations of vector-calculus concepts such as partial derivatives, curves and surfaces. Not surprisingly, this equation also looks like the FTC.

9.5 Can-do checklist

- ☐ Identify standard quadric surfaces including the cylinder, cone, hyperboloid of 1 sheet and 2 sheets, paraboloid, hyperbolic paraboloid (saddle), ellipsoid and sphere.

- ☐ Obtain a vector parametrisation of a surface described by a Cartesian equation (and vice versa).

- ☐ Calculate the normal to a parametrised surface. Obtain the outward-pointing unit normal where required (the normal can simply be written down if obvious from the geometry).

- ☐ Given a parametrised surface, calculate the area element $\mathrm{d}S$ (scalar form) or $\mathrm{d}\mathbf{S}$ (vector form).

- ☐ Calculate the surface area of a parametrised surface.

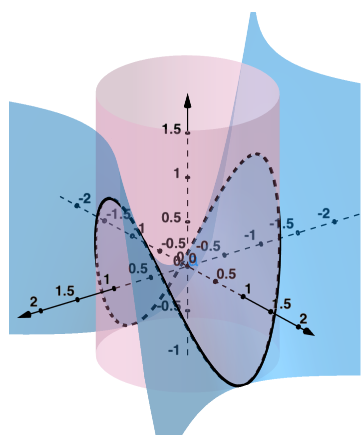
- ☐ Use the Divergence Theorem (given in exam) to evaluate a surface integral, or recast it as a volume integral.

REVISION ON INTEGRAL THEOREMS

1. Prove that the formula of the area of the surface $z = f(x, y)$ (where $(x, y) \in R \subseteq \mathbb{R}^2$ is given by

$$\text{Surface area } = \iint_R \sqrt{1 + f_x^2 + f_y^2} \, \mathrm{d}x \, \mathrm{d}y.$$

Hence calculate the area of the surface $z = x^2 - y^2$ contained within $x^2 + y^2 = 1$.

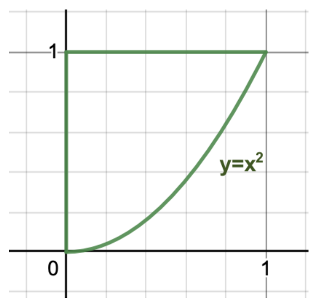


2. Find the surface area of the portion of the sphere $z = \sqrt{a^2 - x^2 - y^2}$ where
a) $z_1 \leq z \leq z_2$, b) $\theta_1 \leq \theta \leq \theta_2$ (where θ is the polar angle).
Repeat the calculations for the cylinder $x^2 + y^2 = a^2$, $0 \leq z \leq a$. Make a conjecture.

Questions on verifying Green's, Stokes' or Divergence Theorems are really useful for revision because they help us revise techniques in calculating line/surface/volume integrals. Plus, you get the same answer in 2 ways, which is good assurance.

Some of these questions are from the recommended textbooks. Where the answers are given in brackets, they are given at the back of the books. Great for practice!

3. (From Stewart) Let $\mathbf{F} = (x^2y^2,\ xy)$. Let C be the curve shown below (traversed anti-clockwise). Verify Green's Theorem.



4. (From Thomas' Calculus) Calculate $\iint_S \text{curl } \mathbf{F} \cdot \mathrm{d}\mathbf{S}$ where $\mathbf{F} = (3y, 5-2x, z^2-2)$ and S is the surface parametrised by $\mathbf{r}(u,v) = (\sqrt{3}\cos u \sin v, \sqrt{3}\sin u \sin v, \sqrt{3}\cos v)$, with $0 \leq u \leq 2\pi$ and $0 \leq v \leq \pi/2$.

   Do this in two ways: a) directly and b) using Stokes' Theorem. (Ans: -15π)

5. (2021 MA134 Exam) Let $R > 0$ and let $D = D_1 \cap D_2 \cap D_3$ where

$$\begin{aligned} D_1 &= \{(x,y,z) \in \mathbb{R}^3 \mid z \geq -R\}, \\ D_2 &= \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq R^2\}, \\ D_3 &= \{(x,y,z) \in \mathbb{R}^3 \mid z \leq \sqrt{R^2 - x^2 - y^2}\}. \end{aligned}$$

   Let $\mathbf{v}(x,y,z) = (2x, 2y, 0)$. Verify the Divergence Theorem. (12 marks)

> If the question doesn't say 'verify', the method that involves the fewest components is usually easier (but not always). Think about which integral is easier in each of the previous 3 questions.

6. (From Boas) Calculate $\int_C ((y^2 - x^2)\,\mathrm{d}x + (2xy + 3)\,\mathrm{d}y)$ where C comprises a line segment from $(0, 0)$ to $(\sqrt{5}, 0)$, and a circular arc from $(\sqrt{5}, 0)$ to $(1, 2)$. (Ans: 29/3)

7. (From Schaum's) Let $\mathbf{F} = (y + z,\, -xz,\, y^2)$. Let S be the surface above the x-y plane and bounded by $2x + z = 6$, $y = 2$, $y = 0$ and $x = 0$. Calculate $\iint_S \operatorname{curl} \mathbf{F} \cdot \mathrm{d}\mathbf{S}$. (Ans: -6)

8. (From Stewart) Let $\mathbf{F} = (y^2 z^3,\, 2yz,\, 4z^2)$. Let S be the surface bounded by $z = x^2 + y^2$ and $z = 4$. Calculate $\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$.