MA145 Mathematical Methods and Modelling 2

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What is this module about?

This module provides a first introduction to concepts in Multivariable Calculus, which is the mathematical study of functions which have inputs or outputs in spaces of several real variables. This topic goes back to the birth of calculus (and so modern Mathematics) in the 17th century, but thanks to the power of these ideas, is still very much in use throughout the Sciences and beyond. Various notions we will discuss are closely linked to lots of the Maths you saw in school, as well as the contents of Mathematical Analysis modules (and there are lot of connections with Linear Algebra too).

How I recommend approaching this module

The focus in this module is on mathematical methods and techniques for calculation, rather than proofs. These notes provide you with detail on some of the theory and definitions we will use, but to really understand them, you need to practice using them: there is no shortcut!

As we go through the module you will have to practice: exercise sheets and online quizzes will be provided. When it comes to any topic where you find the methods more challenging, and particularly to help with revision, you will find many good textbooks where you can find additional problems to test your understanding and get more practice. A list of books I recommend can be found linked from the module's moodle page.

Learning objectives

The following are statements that describe the intended outcome of studying this module, and you may find them useful particularly when revising.

At the end of this module, you should be able to:

- *Define what is meant by a scalar-valued function and a vector-valued function, and provide example uses for these functions.*
- *Construct parametric representations of curves, surfaces and functions, and interpret parametrisations and functions through sketching.*
- *Relate geometric properties of curves and surfaces to parametrisations of these objects, including calculating lengths, areas, tangents and normals.*
- *Perform coordinate transformations (including the use of polar coordinates, cylindrical coordinates and spherical coordinates) for multivariable functions, and use such transformations to calculate multiple integrals.*
- *Define and compute different notions of differentiation for functions from $\mathbb{R}^n$ to $\mathbb{R}^m$, including partial derivatives, the gradient, Jacobian matrix, directional derivative, divergence and curl.*
- *Find and classify the critical points of scalar-valued functions of two variables using the second derivative test.*

- *Compute integrals of multivariable functions over curves and areas in 2D; and over curves, surfaces and volumes 3D; and provide interpretations of such integrals where appropriate.*
- *Recall and use theorems which relate line, surface and volume integrals, including the Fundamental Theorem of Line Integrals, Green's Theorem, Stokes' Theorem, and the Divergence Theorem.*

Week 1: Parametric curves

Introduction

In the first third of the module, our main focus will be on *curves*, sometimes also called *paths*. These are one–dimensional sets of points in space. The word path shows that a curve can be thought of as reflecting the past history of a moving point.

One way to describe a curve you have seen many times already is to consider the graph of a function $f(x)$. The *graph* of a function is the set of points (x, y) where $y = f(x)$. On the other hand, not all graphs of functions match with our intuitive notion of a curve. We need f to be continuous and the domain of f to be an interval of real numbers.

Even worse, not all curves can be described as the graph of a function (consider for example a circle). This shows that *it is unnatural to treat x and y differently*. This brings us to vector functions and parametrisations.

1.1 Scalar– and vector–valued functions

You should be familiar with the basic idea of a vector (a basic review is given at the end of the chapter, and you will see much more in modules on Linear Algebra). The position of a point relative to a chosen origin can be represented as a vector (more precisely called a *position vector*) from the origin to the point. We typically use the vector $\mathbf{r}$ to denote position and write

$$\mathbf{r} = (x, y) \in \mathbb{R}^2 \quad \text{for a point in the plane}$$
$$\mathbf{r} = (x, y, z) \in \mathbb{R}^3 \quad \text{for a point in space}$$

so x, y, and possibly z are the components of the position vector. These are also the **Cartesian coordinates** for a point. At times we consider the general case where the dimension is some unspecified n. In these cases we label the components of position by $\mathbf{r} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

You already know what a real–valued function of a real variable is: $f : U \subseteq \mathbb{R} \to \mathbb{R}$. To each real number in U, a subset of $\mathbb{R}$, f assigns a value in $\mathbb{R}$. In this course, we will refer to single real numbers as *scalars*. This means that $f : U \to \mathbb{R}$ is a **scalar–valued function**.

A **vector–valued function**

$$\mathbf{r} : U \subseteq \mathbb{R} \to \mathbb{R}^n$$

assigns a point in $\mathbb{R}^n$, that is a vector, to each real number in U.

We usually denote the independent variable by t (often thought of as "time"). We frequently call $\mathbf{r}$ a **map**, and think of it as taking points in $\mathbb{R}$ and mapping them to points in $\mathbb{R}^n$. We sometime write $t \mapsto \mathbf{r}(t)$.

A vector–valued function can be viewed in terms of its component functions

$$\mathbf{r}(t) = \big(x_1(t), x_2(t), \cdots x_j(t), \cdots, x_n(t)\big),$$

where each component function $x_j(t)$ is a scalar–valued (or real–valued) function. You can think of a vector–valued function as just a collection of real–valued functions arranged as components of a vector.

In this module, we will primarily be interested in either the plane ($n = 2$) or the space ($n = 3$), and in these cases we usually use the canonical notation $x(t)$, $y(t)$, and possibly $z(t)$ for component functions rather than $x_1(t)$, $x_2(t)$, and $x_3(t)$. For example, in space we would typically write

$$\mathbf{r}(t) = \big(x(t), y(t), z(t)\big).$$

You should also be familiar with the unit vectors $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, and $\mathbf{k} = (0, 0, 1)$. Using these we can avoid outer parentheses and write

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

Things you know about vectors and functions can now be applied to vector functions in the obvious way, e.g. given vector functions

$$\mathbf{f}(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j}, \quad \mathbf{g}(t) = g_1(t)\mathbf{i} + g_2(t)\mathbf{j},$$

and a scalar α, we can define the vector function $\mathbf{r}$

$$\begin{aligned}\mathbf{r}(t) &= \mathbf{f}(t) + \alpha\mathbf{g}(t)\\ &= \big(f_1(t) + \alpha g_1(t)\big)\mathbf{i} + \big(f_2(t) + \alpha g_2(t)\big)\mathbf{j}\end{aligned}$$

Example: A linear vector function. Let

$$\mathbf{r}(t) = \mathbf{a}t + \mathbf{b},$$

where $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b}$ are constant vectors. This function traces a straight line as t varies. This representation is better than considering only $y = f(x) = mx + b$, since we can consider a line in any dimension and in any direction.

1.2 Curves

Our definition of what it means to be a curve will rely on the definition of a vector–valued function we have just seen.

Let $\mathbf{r}\colon I \to \mathbb{R}^n$ be a continuous vector function, where $I \subseteq \mathbb{R}$ is an interval. Then the set of points

$$\mathcal{C} = \{\mathbf{r}(t) \,|\, t \in I\}$$

is a **curve**. The mapping $\mathbf{r}$ is called a **parametrisation** of the curve $\mathcal{C}$.

Remarks

- The definitions requires $\mathbf{r}$ to be a *continuous* vector function, $\lim_{t\to t_0} \mathbf{r}(t) = \mathbf{r}(t_0)$ for any $t_0 \in I$, more on this next week. Continuity means what you think it does: there can be no 'jumps' or 'breaks' in the curve $\mathcal{C}$.
- Generically, we will write the interval $I = [a, b]$, but it should be understood that I could be infinite or semi-infinite. For example: $I = (-\infty, \infty)$, $I = (-\infty, 0]$, or $I = [1, \infty)$.
- You should be aware that there is some variation in the definition of a curve. In some texts a curve is defined as the map $\mathbf{r}$ not the set of points $\mathcal{C}$. Some authors require I to be a closed interval, others do not.
- There are infinitely–many possible parametrisations for a given curve $\mathcal{C}$. For example, if $\mathbf{r} : [a, b] \to \mathbb{R}^n$, then for any $\lambda > 0$, we can define a family of different parametrisations $\mathbf{r}_\lambda : [\lambda a, \lambda b] \to \mathbb{R}^n$ by
$$\mathbf{r}_\lambda(t) = \mathbf{r}(t/\lambda).$$
More generally, if $\mathbf{r} : I \to \mathbb{R}^n$ is a parametrisation of $\mathcal{C}$, $J \subseteq \mathbb{R}$ is an interval, and $h : J \to I$ is onto and continuous, then $\mathbf{r} \circ h : J \to \mathbb{R}^n$ is also a parametrisation of $\mathcal{C}$. Two parametrisations are called equivalent if they define the same curve. Equivalence classes of parametrisations are curves in the sense of our definition.
- The definition in terms of parametrisations gives us something in addition to the set of points $\mathcal{C}$: if $\mathbf{r} : I \to \mathcal{C}$ is injective, it defines an orientation - the direction of 'travel' along $\mathcal{C}$ corresponding to increasing t. There are two possible orientations as we can travel either from $\mathbf{r}(a)$ to $\mathbf{r}(b)$ or in the opposite direction. Injectivity is sufficient, but not necessary for an orientation of a curve to exist.
- If all we remember about $\mathbf{r}$ is its image, we refer to $\mathcal{C}$ simply as a *curve*; if we remember $\mathrm{Im}(\mathbf{r})$ and and the orientation, then $\mathcal{C}$ is called an *oriented curve*; if we remember the map $\mathbf{r}$ itself, then we are talking about a *parametrised curve*.
- From the way we motivated the subject, one sees a close connection between curves (purely geometrical objects) and particle paths (trajectories associated with motion in time). This is further reinforced by using t for the independent variable.

  It is generally not necessary to distinguish these cases. However, if a curve actually arises from the motion of a "particle", we shall say path rather than curve. When the independent parameter is actual time, then there are specific physical meanings to expressions defined later.

  It may be that a particle retraces the same points in space multiple times. If we are only interested in the curve as a set in itself, we would probably not consider a parametrisation covering the same points multiple times; on the other hand, if we are thinking of a particle path, then we do want to remember the number of times we have passed over the same point.
- Graphs of continuous functions are curves: the graph $\{(x, f(x)) \in \mathbb{R}^2 \mid x \in [a, b]\} = \mathrm{Im}(\mathbf{r})$, where $\mathbf{r} : [a, b] \to \mathbb{R}^2$ is such that $\mathbf{r}(x) = (x, f(x))$.

1.3 Working with Parametrisations

Examples

It is helpful to start with a few basic examples. Space has been provided alongside for you to practice sketching these curves below.

- A circle of radius R centred on the origin can be parametrised by

$$\mathbf{r}(t) = \big(R\cos t, R\sin t\big), \quad t \in [0, 2\pi].$$

Note $x^2(t) + y^2(t) = R^2(\cos^2 t + \sin^2 t) = R^2$.

- A circle of radius R centred on (a, b) can be parametrised by

$$\mathbf{r}(t) = (a + R\cos t, b + R\sin t), \quad t \in [0, 2\pi].$$

- A circle in three–dimensional space can be parametrised by

$$\mathbf{r}(t) = (0, b + R\cos t, c + R\sin t), \quad t \in [0, 2\pi].$$

It has radius R, lies in the (y, z) plane, and is centred at $(0, b, c)$.

- If $\mathbf{r}(t) = \big(x(t), y(t)\big)$ is a parametrisation for some curve, then the curve parametrised by $\mathbf{r}_a(t) = \big(ax(t), y(t)\big)$ is 'stretched' in the x direction if $a > 1$ and 'compressed' in the x direction if $0 < a < 1$. For example,

$$\mathbf{r}(t) = (b\cos t, c\sin t), \quad t \in [0, 2\pi]$$

is a parametrisation for an ellipse if $b \neq c$.

- A parabola can be parametrised by

$$\mathbf{r}(t) = (t, at^2), \quad t \in \mathbb{R},$$

since $x(t) = t$ and $y(t) = at^2 = ax(t)^2$. Equally, we could parametrise a parabola by

$$\mathbf{r}(t) = (t^3, at^6), \quad t \in \mathbb{R}.$$

This again demonstrates directly that parametrisations are not unique. Does $\mathbf{r}(t) = (\sin t, a \sin^2 t)$ also parametrise the same curve?

- At school, you may have seen a hyperbola defined as the graph of $y = a/x$. A rotated version of this curve is the set of all solutions to

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$$

Solving for y, we see this curve has two branches:

$$y = \pm\sqrt{a^2 + a^2x^2/b^2}.$$

A parametrisation of the branch with $y > 0$ is

$$\mathbf{r}(t) = (b\sinh(t), a\cosh(t)), \quad t \in \mathbb{R}$$

where a and b are both positive constants.

In the examples above, we have expressed the curves by prescribing their Cartesian coordinates (x, y) as functions, and we have seen that we can take the solutions of an equation like $x^2+y^2 = R^2$ and convert it into a parametrised curve.

Recall however that Cartesian coordinates are not the only way of describing points in the plane; we can also use polar coordinates. At school, you may have studied curves given as radius as a function of angle: $r = f(\theta)$. We shall call such curves **polar graphs** or **graphs of a polar function** $r = f(\theta)$. The following examples, the Archimedean and Logarithmic spirals, are easily expressed as polar functions, but we can still convert them into parametrisation in the same way by using the fact that $x = r\cos\theta$ and $y = r\sin\theta$. A brief review of polar coordinate is given at the end of the chapter.

- An *Archimedean spiral* is given in polar coordinates by $r = a\theta$ for some $a > 0$. This can be parametrised by

$$\mathbf{r}(t) = (at\cos t, at\sin t), \quad t \geq 0$$

- A *logarithmic spiral* can be parametrised as

$$\mathbf{r}(t) = (e^{at}\cos t, e^{at}\sin t), \quad t \in \mathbb{R}.$$

  In polar coordinates, this is $r = \mathrm{e}^{a\theta}$.

- So far, we have been parametrising curves by functions which output Cartesian coordinates (x, y). However, there is nothing to stop us considering which are parametrised in polar coordinates. For example

$$\big(r(t), \theta(t)\big) = \big(2R\sin t, t\big), \quad t \in [0, \pi]$$

  is a parametrisation for a circle of radius R centred on the Cartesian point $(0, R)$, and can be rewritten as a polar graph $r = f(\theta) = 2R\sin(\theta)$.

- A *helix* can in three–dimensional space can be parametrised by

$$\mathbf{r}(t) = \big(R\cos t, R\sin t, kt\big), \quad t \in \mathbb{R},$$

  where $k \neq 0$.

Sketching curves when given a parametrisation

With a computer, plotting curves is easy, so you should learn to use Geogebra, Desmos, Python, Julia or another programming language or application to sketch curves. It is best to use these applications to check your sketches, rather than immediately using them. This is because sketching curves by hand requires an appreciation of why a curve looks the way it does, and is an important skill you are expected to develop in this module. When sketching curves, it is always important to label points and explain your reasoning to convince yourself (and whoever is assessing you!) that you understand why the curve looks the way it does!

To help sketch a parametrisation that is given to you, here are some techniques to help:

- Eliminating the dependence of the coordinate function on t can help. For example, being able to write an equation you know how to plot like $y(t) = x(t)^\alpha$ or $x(t) = y(t)^\alpha$ will enable you to quickly work out how to draw a curve.
- Spot rotation around a central point in the form of $\cos$ and $\sin$ terms in two of the coordinates. For example, if $x(t) = b(t)\cos(t)$ and $y(t) = b(t)\sin(t)$, then in polar coordinates $\theta = t$ and $r = b(\theta)$.
- In some cases, it might be desirable to first shift the origin of the coordinates, and then use another technique.
- Look the behaviour for extreme values of t, for example as $t \to \pm\infty$, or more generally consider the behaviour at the endpoints of the interval over which the parametrisation is defined. Also consider evaluating the functions at special values, for example $t = 0$, integers, or standard multiples of π.
- Try to separate behaviour in the different coordinate directions. For example, consider the largest and smallest values of $x_1(t)$, $x_2(t)$, ..., and the general behaviour. In some cases, it might be possible to reduce the complexity by considering only a few coordinates at a time.

Finding parametrisations

Finding a parametrisation is the opposite of sketching. Given some description or specification of a curve, you need to produce a parametrisation for it, generally for the purpose of doing some further calculation involving the curve. This can be quite challenging, but you will not be asked to do anything too complicated.

The curves you will be expected to parametrised in this course include the conic sections, and curves made of straight segments. You should be able to extend these to combinations and cases where the radius of a circle varies, e.g. spirals. (See next section on curves with multiple segments). You should be prepared to work and think in polar coordinates and to switch between polar and Cartesian coordinates as necessary.

1.4 Curves in multiple segments

We carefully defined curves and parametrisations so that curves are continuous images of a single interval I. However, when it comes to many applications, this restriction can become quite annoying and unnecessary. Consider parametrising the plane curve $\mathcal{C}$ which starts at $(1,0)$ and proceed to $(-1,0)$ along the upper unit semicircle and then returns to $(1,0)$ along the x-axis. The annoyance is that curve is easily described as the union of two segments, each of which is easily parametrised:

$$\begin{aligned} \mathbf{r}_1(t) &= \cos(t)\mathbf{i} + \sin(t)\mathbf{j}, & t &\in [0,\pi] \\ \mathbf{r}_2(t) &= t\mathbf{i}, & t &\in [-1,1] \end{aligned}$$

Of course one can adjust things to produce a mapping over a single interval I, (probably you would adjust the second parametrisation). However, for most things we care about this is simply unnecessary. It is sufficient to work with a set of parametrisations, $\mathbf{r}_1(t), \cdots, \mathbf{r}_k(t)$ defined on intervals $I_1, \cdots, I_k$, if these naturally describe a curve composed of a union of k segments. *Unless you are explicitly asked to produce a single parametrisation over a single interval I, you may work with multiple parametrisations whenever it is natural to do so.*

1.5 Some further concepts

There are a few further concepts related to curves which are worth defining, even though these will not be essential in this module.

A curve $\mathcal{C}$ is **closed** if $I = [a,b] \to \mathbb{R}^2$: and $\mathbf{r}(a) = \mathbf{r}(b)$. A closed curve is also called a **loop**.

Warning: Some very 'non-loop-like' curves turn out to be closed under the above definition, for example, $\mathbf{r}(t) = (\sin(t), 0),\ t \in [0, 2\pi]$. The corresponding curve is the interval $\{(x,0) \mid |x| \leq 1\}$ in $\mathbb{R}^2$. It is also unsatisfactory to define the geometric notion of closedness for a parametrised curve, rather than a curve. To circumvent these difficulties, it turns out to be easier to define closedness for simple curves (see below) first and generalise from there, see the 'Curve' page on Wikipedia for details and references.

A curve is **simple** if it does not self intersect. In terms of parametrisations, a curve is simple if it can be parametrised by a mapping $\mathbf{r} : I \to \mathbb{R}^n$ that is injective except possibly at the end points of $I = [a,b]$. We need to allow the possibility that $\mathbf{r}(a) = \mathbf{r}(b)$ since this just means that the curve is closed and does not correspond to an intersection. From the definition of injective, a curve is simple if it can be parametrised by $\mathbf{r}(t)$ where $t_1 \neq t_2$ implies $\mathbf{r}(t_1) \neq \mathbf{r}(t_2)$, except if $t_1 = a$ and $t_2 = b$.

A curve is **regular** if there exists a parametrisation such that its derivative $\mathbf{r}'$ is defined and is non–zero at all points. We have not yet defined the derivative $\mathbf{r}'$, but we will next week. The important point is that regular curves have no corners or cusps and hence are nice to deal with. We will say more next week.

Review Material

Recap of vector basics

This is just a quick reminder of some basic things you already be familiar with. In Linear Algebra you will be spending a lot of time discussing vectors in much more detail.

Notation. Vectors will be denoted by boldface and sometimes using an 'over–arrow', e.g. $\mathbf{v}$ or $\overrightarrow{v}$. On the blackboard, underlining will be used instead of bold face, e.g. $\underline{\mathrm{v}}$. We only consider real vectors in this module, so $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$. The real numbers $v_1, v_2, \ldots, v_n$ are call the **components** of $\mathbf{v}$.

Scalars. We shall use the term **scalar** to refer to a real number, to contrast with vectors. You should know things such as how to multiply or divide a vector $\mathbf{v}$ by a scalar α:

$$\alpha\mathbf{v} = (\alpha v_1, \alpha v_2, \ldots, \alpha v_n), \qquad \frac{\mathbf{v}}{\alpha} = \frac{1}{\alpha}(v_1, v_2, \ldots, v_n) = \left(\frac{v_1}{\alpha}, \frac{v_2}{\alpha}, \ldots, \frac{v_n}{\alpha}\right)$$

Norms. The **Euclidean norm**, **length**, or **magnitude** of a vector $\mathbf{v}$ is defined as: $\|\mathbf{v}\| = \left(\sum_{i=1}^{n} v_i^2\right)^{1/2}$.

Dot Product. You should know properties of the dot product and how to compute the dot product, either as

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \, \cos\theta$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, or as

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} u_i v_i,$$

where u_i and v_i are the components of $\mathbf{u}$ and $\mathbf{v}$. The dot product of two vectors is a scalar, and $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$. Notice that $\|\mathbf{v}\| = (\mathbf{v} \cdot \mathbf{v})^{1/2}$.

Unit Vectors. A **unit vector** is any vector with length one. Often a generic unit vector is given a hat, for example $\hat{\mathbf{v}}$ where $\|\hat{\mathbf{v}}\| = 1$. We say $\hat{\mathbf{v}}$ has "unit norm".

Important examples of unit vectors are the standard basis vectors. For example, the basis vectors for Cartesian coordinates in three dimensions are (hats are not used for these unit vectors)

$$\mathbf{i} = (1, 0, 0) \qquad \mathbf{j} = (0, 1, 0) \qquad \mathbf{k} = (0, 0, 1)$$

Then we can write the position vector $\mathbf{r}$ as a sum of components times basis vectors:

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (x, y, z)$$

One sometimes uses the following notation for these bases vectors:

$$\mathbf{e}_1 = (1, 0, 0) \qquad \mathbf{e}_2 = (0, 1, 0) \qquad \mathbf{e}_3 = (0, 0, 1)$$

(See Linear Algebra for the details on basis vectors. In this module we will only be concerned with standard basis vectors for typical coordinate systems such as Cartesian and polar coordinates.)

Given a nonzero vector $\mathbf{v}$, that is $\mathbf{v} \neq (0, 0, \ldots, 0) = \mathbf{0}$, the **direction** of $\mathbf{v}$ is: $\hat{\mathbf{v}} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$

So each nonzero vector can be written $\mathbf{v} = \|\mathbf{v}\|\,\hat{\mathbf{v}} = \big(\text{magnitude}\big) \times \big(\text{direction}\big) = \big(\text{scalar}\big) \times \big(\text{unit vector}\big)$

Cross Product. The **cross product** (sometimes also called the **vector product**) is defined for two three dimensional vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, and in terms of the components of $\mathbf{u}$ and $\mathbf{v}$, is defined to be

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.$$

This product has various properties you should know, for example:

- It is anti-symmetric, i.e. $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$ for any two vectors.
- The cross product $\mathbf{u} \times \mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$, i.e. $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = 0$ and $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = 0$.
- The magnitude of the cross product is the area of the parallelogram which has sides $\mathbf{u}$ and $\mathbf{v}$, which can be expressed via the formula $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin\theta$, where θ is the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$.

Recap of polar coordinates

You should be familiar with polar coordinates, typically denoted (r, θ). The relationship between polar and Cartesian coordinate is:

$$x = r\cos\theta \qquad\qquad r^2 = x^2 + y^2$$
$$y = r\sin\theta \qquad\qquad \tan\theta = \frac{y}{x}$$

You should know how to go back and forth between Cartesian and polar coordinates and be able to use polar coordinates as needed. There is an issue to be careful of, which is that polar coordinates are not unique: (r, θ), $(r, \theta+2\pi)$, and $(-r, \theta+\pi)$ all correspond to the same point in the plane (and you can generate even more examples like these). One sometimes works with $r \geq 0$ and θ restricted to a fixed range to avoid the non-uniqueness. We will generally work with $\theta \in [0, 2\pi)$, but this will depend on the problem, and you should be clear what values of θ you accept when doing problems. Sometimes it is more convenient to work with $\theta \in (-\pi, \pi]$. Sometimes it is best to live with the non-uniqueness.

Week 2: Geometry of curves

Introduction

Last week, we defined what curves are. The next natural step to study them mathematically is to try to describe their properties. This week, we will ask how curves change along their lengths, leading us naturally to the idea of differentiating 'along' a curve.

2.1 The derivative of a vector–valued function

As we discussed last week, we can think of a vector–valued function as just being a collection of scalar–valued functions. This means that to differentiate the functions we have seen so far, we can simply apply the rules of limits and differential calculus to each coordinate function. For example, "taking the limit of $\mathbf{r}(t)$ as $t \to a$" means

$$\begin{aligned}\lim_{t\to a} \mathbf{r}(t) &= \lim_{t\to a} (x_1(t), \cdots, x_n(t)) \\ &= \Big(\lim_{t\to a} x_1(t), \cdots, \lim_{t\to a} x_n(t)\Big)\end{aligned}$$

Just as for scalar–valued functions, a vector–valued function is continuous at a point a if

$$\mathbf{r}(a) = \lim_{t\to a} \mathbf{r}(t).$$

Component by component this reads

$$\Big(x_1(a), \ldots, x_n(a)\Big) = \Big(\lim_{t\to a} x_1(t), \ldots, \lim_{t\to a} x_n(t)\Big)$$

Hence a vector function is continuous if and only if each of its component function is continuous. This is the formal meaning of 'being continuous' we used informally last week when defining curves.

The derivative of the vector function $\mathbf{r}$ is defined exactly as you would expect based on scalar–valued functions

$$\frac{d\mathbf{r}}{dt}(t) = \lim_{\triangle t\to 0} \frac{\mathbf{r}(t+\triangle t) - \mathbf{r}(t)}{\triangle t}$$

Again, component by component

$$\begin{aligned}&\lim_{\triangle t\to 0} \frac{\mathbf{r}(t+\triangle t) - \mathbf{r}(t)}{\triangle t} \\ =& \lim_{\triangle t\to 0} \left(\frac{x_1(t+\triangle t) - x_1(t)}{\triangle t}, \cdots, \frac{x_n(t+\triangle t) - x_n(t)}{\triangle t}\right) \\ =& \left(\lim_{\triangle t\to 0} \frac{x_1(t+\triangle t) - x_1(t)}{\triangle t}, \cdots, \lim_{\triangle t\to 0} \frac{x_n(t+\triangle t) - x_n(t)}{\triangle t}\right) \\ =& \left(\frac{dx_1}{dt}(t), \frac{dx_2}{dt}(t), \cdots, \frac{dx_n}{dt}(t)\right)\end{aligned}$$

Thus the derivative of a vector function is itself a vector and is computed by differentiating component by component. We often denote the derivative with a $'$, e.g. $\mathbf{r}'(t)$, so

$$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt}(t) = \left(\frac{dx_1}{dt}(t), \frac{dx_2}{dt}(t), \cdots, \frac{dx_n}{dt}(t)\right)$$

For example, in space we can write the derivative as

$$\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$$

2.2 Being tangent to a curve

If $t \mapsto \mathbf{r}(t)$ is a parametrisation of a curve $\mathcal{C}$, and if the derivative exists at t_0 and is non–zero, i.e.

$$\mathbf{r}'(t) \neq \mathbf{0},$$

then the vector $\mathbf{r}'(t_0)$ is said to be *tangent* to $\mathcal{C}$ at the point $\mathbf{r}(t_0)$. We will now argue why this definition makes sense. As you have seen earlier in your mathematical careers, for real–valued functions, we think of the derivative geometrically as being the 'slope' of or 'tangent' to a curve. In more formal mathematical terms, finding the slope of a function at some point is the same as finding the best possible approximation by a straight line (or a linear function) which passes through that point.

Returning to the parametrisation of our curve, and defining $\mathbf{a} = \mathbf{r}(t_0)$ and $\mathbf{b} = \mathbf{r}'(t_0)$, we indeed see that the curve parametrised by

$$t \mapsto \mathbf{a} + \mathbf{b}t, \quad t \in \mathbb{R},$$

is a straight line which passes through $\mathbf{a} = \mathbf{r}(t_0)$, and $\mathbf{b} = \mathbf{r}'(t_0)$ is clearly tangent to the line. When $\triangle t$ is small enough, the definition of a limit means that

$$\mathbf{r}'(t_0) \approx \frac{\mathbf{r}(t_0 + \triangle t) - \mathbf{r}(t_0)}{\triangle t}.$$

Rearranging, we see that

$$\mathbf{r}(t_0 + \triangle t) \approx \mathbf{r}(t_0) + \mathbf{r}'(t_0)\triangle t = \mathbf{a} + \mathbf{b}\triangle t,$$

and so the curve $\mathbf{r}(t)$ is well–approximated by this straight line for values of t close to t_0, justifying our choice to describe $\mathbf{r}'(t_0)$ as being 'tangent' to the curve.

This argument also explains why it's important that $\mathbf{r}'(t) \neq \mathbf{0}$: if $\mathbf{b} = \mathbf{0}$, then the line is not well–defined!

If $\mathbf{r}_1 : I_1 \to \mathbb{R}^n$ and $\mathbf{r}_2 : I_2 \to \mathbb{R}^n$ are two parametrisations of the curve such that $\mathbf{r}_1(t) = \mathbf{r}_2(s)$ for some t and s, $\mathbf{r}_1'(t) \neq \mathbf{0}$ and $\mathbf{r}_2'(s) \neq \mathbf{0}$, then $\mathbf{r}_1'(t)$ is parallel to $\mathbf{r}_2'(s)$.

Recall that last week we defined a curve to be regular if there is a parametrisation such that $\mathbf{r}'$ is defined and nonzero at all points. In other words, regular curves have a tangent vector at each point. Regular curves are nice in that they do not have corners or cusps and these are mainly the ones we want to do calculus on. We will commonly assume, without explicitly stating so, that curves we consider are regular, or at least piecewise regular (see below).

2.3 Differentiation Rules

Suppose $\mathbf{r}(t)$ and $\mathbf{s}(t)$ are differentiable vector functions, and $f(t)$ is a differentiable real function, and a is a scalar. Then the following rules hold for sums, products and compositions:

- $\dfrac{d}{dt}a\mathbf{r}(t) = a\mathbf{r}'(t)$
- $\dfrac{d}{dt}\big(\mathbf{r}(t) + \mathbf{s}(t)\big) = \mathbf{r}'(t) + \mathbf{s}'(t)$
- $\dfrac{d}{dt}f(t)\mathbf{r}(t) = f(t)\mathbf{r}'(t) + f'(t)\mathbf{r}(t)$
- $\dfrac{d}{dt}\big(\mathbf{r}(t) \cdot \mathbf{s}(t)\big) = \mathbf{r}'(t) \cdot \mathbf{s}(t) + \mathbf{r}(t) \cdot \mathbf{s}'(t)$
- $\dfrac{d}{dt}\mathbf{r}\big(f(t)\big) = \mathbf{r}'\big(f(t)\big)f'(t)$

These are obvious extensions of things you know for functions from $\mathbb{R} \to \mathbb{R}$, and all can be proved by expressing vector functions in terms of components. You should learn and be able to verify each of these.

2.4 Dynamics

We have thought so far about vector-valued functions as just parametrising curves in 2D or 3D, but when modelling, it can also be useful to think of a vector-valued function $\mathbf{r}(t)$ as being the location of a particle in space we are interested in at time t.

In this case, if $\mathbf{r}(t)$ models the path of a point particle and t is time, then the following terminology is used:

- $\mathbf{r}(t)$ is the **position** of the particle at time t.
- $\mathbf{r}'(t)$ is the **velocity** of the particle at time t. This is a vector quantity, often written

$$\text{Velocity} = \mathbf{v}(t) := \mathbf{r}'(t)$$

- The length or magnitude of the velocity vector is the **speed**. It is a positive scalar quantity:

$$\text{Speed} = c(t) = \|\mathbf{v}(t)\| = \left\|\mathbf{r}'(t)\right\|$$

- The derivative of velocity (and therefore the *second derivative* of the position) is the **acceleration**. This is a vector quantity, often written

$$\text{Acceleration} = \mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t).$$

  The acceleration is not, in general, tangent to $\mathcal{C}$.

Example. Consider a particle moving with a constant speed $c > 0$, so that

$$c^2 = \|\mathbf{v}(t)\|^2 = \mathbf{v}(t) \cdot \mathbf{v}(t).$$

Differentiating this expression, we find that

$$\begin{aligned} 0 &= \frac{d}{dt}\Big(\mathbf{v}(t) \cdot \mathbf{v}(t)\Big) \\ &= \mathbf{v}'(t) \cdot \mathbf{v}(t) + \mathbf{v}(t) \cdot \mathbf{v}'(t) = 2\mathbf{v}(t) \cdot \mathbf{a}(t), \end{aligned}$$

where we used the rule for differentiating scalar products stated above. We conclude that the acceleration vector is perpendicular to the velocity vector at every time t, and is equivalently perpendicular to the particle's path.

2.5 Arc length

Now that we have dealt with differentiation along curves, a second natural concept is to integrate along a curve. This is a particularly important concept both for describing curves and for physical modelling, as it allows us to answer questions such as

- How long is a curve?
- How much of some quantity is accumulated as a particle moves?

The first of these questions is the one we consider first.

In much the same way you thought of integration in school, our method for defining an integral will be to take $\mathcal{C}$ and divide it up into small segments, and sum up the lengths of these pieces. This provides us with an approximation to the length: taking the limit of the sum as the size of the pieces goes to zero while the number of pieces going to infinity then gives us the true length.

Let $\mathbf{r} : [a, b] \to \mathbb{R}^n$ be a parametrisation of $\mathcal{C}$. To divide $\mathcal{C}$ into small arcs, we split $[a, b]$ into N segments of equal size $\triangle t$. Let

$$t_0 = a, t_1 = a + \triangle t, \cdots, t_j = a + j\triangle t, \cdots, t_N = b.$$

These $N + 1$ t–values divide the interval I into N segments, and we set $\triangle t = \frac{b-a}{N}$.

Now, the j^{th} segment $[t_j, t_{j+1}]$ is mapped onto a small segment of $\mathcal{C}$ by $\mathbf{r}$, lying between $\mathbf{r}(t_j)$ and $\mathbf{r}(t_{j+1})$. The length of this segment, which we will call $\triangle s_j$, is approximately the distance between $\mathbf{r}(t_j)$ and $\mathbf{r}(t_{j+1})$, i.e.

$$\triangle s_j \approx \|\mathbf{r}(t_{j+1}) - \mathbf{r}(t_j)\| .$$

Multiplying and dividing by $\triangle t$, and remembering that $t_{j+1} = t_j + \triangle t$, we have

$$\triangle s_j \approx \left\| \frac{\mathbf{r}(t_j + \triangle t) - \mathbf{r}(t_j)}{\triangle t} \right\| \triangle t \approx \|\mathbf{r}'(t_j)\| \triangle t. \tag{1}$$

It follows that the total length of the curve s is then approximately

$$s = \sum_{j=0}^{N-1} \triangle s_j \approx \sum_{j=0}^{N-1} \|\mathbf{r}'(t_j)\| \triangle t$$

Taking the limit $N \to \infty$ (so that $\triangle t = \frac{b-a}{N} \to 0$) we obtain that

The **arc length** of a curve $\mathcal{C}$, denoted $\ell(\mathcal{C})$, is defined to be

$$s = \ell(\mathcal{C}) = \int_a^b \|\mathbf{r}'(t)\| \, dt$$

where $t \mapsto \mathbf{r}(t)$, $t \in [a, b]$ is a parametrisation of $\mathcal{C}$.

Example. If $f : [a, b] \to \mathbb{R}$ is a function, we can consider the curve $\mathcal{C}$ defined by its graph

$$\mathcal{C} = \big\{ \big(x, f(x)\big) \mid x \in [a, b] \big\}.$$

In this case, a sensible parametrisation is

$$\mathbf{r}(x) = \big(x, f(x)\big), \quad x \in [a, b],$$

and so

$$\mathbf{r}'(x) = \big(1, f'(x)\big).$$

From this expression, we deduce that

$$\ell(\mathcal{C}) = \int_a^b \sqrt{1 + f'(x)^2} \, dx.$$

Discussion

- In our considerations above, we implicitly assumed that the parametrisation $\mathbf{r}(t)$ does not repeat points of $\mathcal{C}$. In practice, this is not always obvious. For example, to compute the length of a circle (i.e. its circumference) one would not use $r(t) = (R\cos t, R\sin t)$, with $t \in [0, 4\pi]$.
- While we defined the length of a curve based on a parametrisation, the length of a curve is actually independent of the parametrisation. If $\mathbf{r}_1 \colon [a,b] \to \mathbb{R}^n$ and $\mathbf{r}_2 \colon [c,d] \to \mathbb{R}^n$ are two different parametrisations the same curve $\mathcal{C}$, the lengths will be the same. **Exercise.** Let $f : [c,d] \to [a,b]$ be a smooth bijection: $f' > 0$. Check that the arc lengths computed using the parametrisations $\mathbf{r} : [a,b] \to \mathbb{R}^n$ and $\mathbf{r} \circ f : [c,d] \to \mathbb{R}^n$ are equal to each other.
- Arc length is never negative. The limits of integration are *always* from the left endpoint (smallest value) of I to the right endpoint (largest value) of I.
- An infinitesimal increment dt in the parameter t corresponds to an infinitesimal increment of arc length ds along the curve $\mathcal{C}$. These are related by

  $$ds = \|\mathbf{r}'(t)\|dt$$

  This is the infinitesimal version of Eq. (1). It can be equivalently written as

  $$\frac{ds}{dt} = \|\mathbf{r}'(t)\| \tag{2}$$

- Eq. (2) and the formula for length have simple physical interpretations when $\mathbf{r}(t)$ is a particle path. Recall that we defined $\|\mathbf{r}'(t)\|$ to be the speed $c(t)$ of a particle. Eq. (2) states simply that the change in distance covered by the particle per change in time, $\frac{ds}{dt}$, is the speed of the particle. Likewise, the equation for arc length becomes

  $$\text{Distance covered} = \int_a^b c(t)dt$$

  i.e. distance travelled between times a and b is just the integral of speed over the interval of time.
- Recall that to differentiate a vector-valued function, we simply differentiate component by component, e.g. given

  $$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}.$$

  the derivative is

  $$\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}.$$

  On this basis, you might have expected to see a similar start to the integration of vector-valued functions, e.g.

  $$\int_a^b \mathbf{r}(t)dt = \int_a^b x(t)dt\,\mathbf{i} + \int_a^b y(t)dt\,\mathbf{j} + \int_a^b z(t)dt\,\mathbf{k}$$

  This formula is correct and it is the correct meaning of $\int_a^b \mathbf{r}(t)dt$. The reason we did not start with this is that such integration does not often arise in the course, and in any case it is easy. Integration along a curve, such as in computing arc length, is more important and we will see it again later in this course when we do line integrals. Be sure you understand the difference between

  $$\int_a^b \mathbf{r}(t)dt \quad \text{and} \quad \int_a^b \|\mathbf{r}'(t)\|dt.$$

Example. If $\mathcal{C}$ is a circle of radius R centred at the origin, and is parametrised by

$$\mathbf{r}(t) = \big(R\cos(t), R\sin(t)\big), \quad t \in [0, 2\pi],$$

then we have

$$\int_0^{2\pi} \mathbf{r}'(t)dt = 0, \text{ and } \int_0^{2\pi} ||\mathbf{r}'(t)||dt = 2\pi R.$$

2.6 Curves in multiple segments

Last week we noted that it is often convenient to work with curves composed of multiple segments. Let $\mathcal{C}$ be the union of k segments which meet end to end

$$\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \cdots \cup \mathcal{C}_k.$$

Assuming each segment $\mathcal{C}_j$ is regular, then the curve $\mathcal{C}$ is said to be **piecewise regular**. Such curves may fail to be regular where the curves join. For example if $\mathcal{C}$ is a square, then it is not regular because it has 4 corners. However it can clearly be composed of 4 regular curves and hence $\mathcal{C}$ is piecewise regular.

Assuming that no section of curve is repeated by the parametrisation given, the length of $\mathcal{C}$ is given by

$$\ell(\mathcal{C}) = \ell(\mathcal{C}_1) + \ell(\mathcal{C}_2) + \cdots + \ell(\mathcal{C}_k).$$

In practice one would typically use whatever parametrisation is most convenient for each segment to compute the length of that segment, and then add up the lengths to obtain the length of $\mathcal{C}$.

2.7 Arc length parametrisation

Imagine that the interval I is a straight piece of wire and that the mapping $\mathbf{r}$ corresponds to taking the wire and bending in it around into a curve $\mathcal{C}$ *without any stretching or compression*. In this case, there will be an exact correspondence between the distance between points t_1 and t_2 in I and the arc length of the curve between the corresponding point $\mathbf{r}(t_1)$ and $\mathbf{r}(t_2)$ on $\mathcal{C}$. For example, let $I = [0, 10]$ (think of a 10 centimetre long straight wire). Each subinterval $[0, 1], [2, 3], \ldots, [9, 10]$ of I gets mapped by $\mathbf{r}$ to a segment of $\mathcal{C}$ with arc length one: Each centimetre of straight wire gets bent around, but maintains its length of one centimetre.

Such a parametrisation of a curve $\mathcal{C}$ is called an **arc length parametrisation** and the curve is said to be parametrised by arc length. This is also sometimes called the **natural parametrisation**. Denoting such a parametrisation by $\mathbf{r}_{\text{arc}}$, then the arc length between $\mathbf{r}_{\text{arc}}(t_1)$ and $\mathbf{r}_{\text{arc}}(t_2)$ is precisely $|t_2 - t_1|$, the distance between t_1 and t_2.

If $\mathcal{C}$ has finite length $\ell(\mathcal{C}) < \infty$, then if $\mathcal{C}$ is parametrised by arc length, the interval I can be chosen to be $I = [0, \ell(\mathcal{C})]$.

Arc length parametrisations are important for two reasons. The first is conceptual: we know that there are infinitely many ways to parametrise a curve, yet these all give the same arc length. By choosing arc length along the curve *as the parameter* we select a parametrisation that is intrinsic to the curve rather than arbitrary. Apart from orientation and shifts of the starting point, the parametrisation by arc length is unique.

The second reason is more important in practice. Let $\mathbf{r}_{\text{arc}}(t)$ be an arc length parametrisation of $\mathcal{C}$. Then by definition

$$\int_0^s \|\mathbf{r}'_{\text{arc}}(t)\|\, dt = s$$

for all $s \in [0, \ell(\mathcal{C})]$. Differentiating this with respect to s gives (see Additional Material),

$$\frac{d}{ds}\int_0^s \|\mathbf{r}'_{\text{arc}}(t)\|\,dt = \frac{d}{ds}\,s$$
$$\|\mathbf{r}'_{\text{arc}}(s)\| = 1,$$

and so the derivative $\mathbf{r}'_{\text{arc}}$ always has unit length for an arc length parametrisation. One often says that an arc length parametrisation has speed one, whether or not one is thinking of particle paths. Many calculations for curves, such as those we will see next week, greatly simplify when considering an arc length parametrisation.

To obtain an arc length parametrisation of a curve $\mathcal{C}$, start with some parametrisation $\mathbf{r}(t) : [a, b] \to \mathbb{R}^n$ and compute the arc length from $\mathbf{r}(a)$ to $\mathbf{r}(t)$ for $t \in [a, b]$

$$s = \ell(t) = \int_a^t \|\mathbf{r}'(\tau)\|d\tau$$

Inverting this relationship, $t = \ell^{-1}(s)$, gives us the parameter value t corresponding to arc length s from the end point $\mathbf{r}(a)$. Using this to eliminate t in favour of s, we can (at least in principle!) obtain an arc length parametrisation

$$\mathbf{r}_{\text{arc}}(s) = \mathbf{r}\big(\ell^{-1}(s)\big)$$

Be warned, however, even though this parametrisation exists, obtaining an explicit arc length parametrisation in practice is generally impossible because no nice formula will exist for $\ell^{-1}(s)$.

It is common practice to use s rather than t for the independent variable in an arc length parametrisation, and so to write $\mathbf{r}_{\text{arc}}(s)$ as appose to $\mathbf{r}_{\text{arc}}(t)$. We will sometimes do this if we want to emphasise arc length parametrisation.

Additional Material

Partitions

In our derivation of the formula for arc length we partitioned the interval I into N equal segments of equal size $\triangle t = t_{j+1} - t_j$. It is not necessary that the segments be of equal size. All that matters is that in the limit all the segments of our partition shrink to zero. However, regular partitions are the most intuitive and they suit our purposes well. Hence we use regular partitions now and throughout this course when we derive integration formulas. In later analysis modules this will be treated more rigorously.

You probably are familiar with partitions from the integral calculus you already know. Have a look at the Wikipedia article *Riemann sum* for a nice discussion of different approximating sums converging to the the same limit. (While this does not fully address the issue of irregular partitions, it gives you some idea why in the end different approximations will converge to the same resulting integral.)

Differentiating limits of integrals

In the treatment of arc length parametrisation we used the following result:

$$\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$$

In other words, if you differentiate a definite integral with respect to the upper limit of integration, the result is the integrand.

This is called the **Fundamental Theorem of Calculus (FTC)**. You should be familiar with this result, but if not you can see that it follows immediately from the following method you use to evaluate definite integrals:

$$\int_a^b f(t)dt = F(b) - F(a), \quad \text{where} \quad F'(x) = f(x).$$

This is true for any b, so let b vary and call it x instead

$$\int_a^x f(t)dt = F(x) - F(a)$$

Differentiating with respect to x

$$\frac{d}{dx}\left(\int_a^x f(t)dt\right) = \frac{d}{dx}\left(F(x) - F(a)\right) = F'(x) - 0 = f(x)$$

In most treatments, one starts with the FTC and from this derives as a corollary the method for evaluating definite integrals.

Hyperbolic trigonometric functions

It is time you starting recalling or learning about hyperbolic trig function. The basic functions $\cosh$ and $\sinh$ are defined by

$$\cosh t = \frac{e^t + e^{-t}}{2} \quad \text{and} \quad \sinh t = \frac{e^t - e^{-t}}{2}, \quad t \in \mathbb{R}.$$

You should know what the graphs $\cosh t$ and $\sinh t$ and also of $\tanh t \equiv \frac{\sinh t}{\cosh t}$ look like. From the definitions you can verify the following properties:

(i) $\cosh^2 t - \sinh^2 t = 1$,

(ii) $\frac{d}{dt}\cosh t = \sinh t, \quad \frac{d}{dt}\sinh t = \cosh t$,

(iii) $\cosh 2t = \cosh^2 t + \sinh^2 t, \quad \sinh 2t = 2\cosh t \sinh t$,

(iv) $\frac{d}{dt}\cosh^{-1} t = \frac{1}{\sqrt{t^2-1}}, \quad \frac{d}{dt}\sinh^{-1} t = \frac{1}{\sqrt{t^2+1}}$,

(v) $\sinh^{-1} t = \log\left(t + \sqrt{t^2+1}\right)$.

Formulae (iv) and (v) are easily verified by letting $t = \cosh y$ or $t = \sinh y$ as needed. In (iv) it is understood that $\cosh^{-1}$ means the positive branch of the inverse of $\cosh$ (note that $\cosh$ is not one–to–one because $\cosh(t) = \cosh(-t)$).

The reason we care particularly about the hyperbolic trig functions now is that they occur frequently in arc length integrals. In particular, the following integral is common.

$$I = \int \sqrt{t^2+1}\, dt$$

This can be integrated by parts to give

$$I = t\sqrt{t^2+1} - \int \frac{t^2}{\sqrt{t^2+1}}\, dt = t\sqrt{t^2+1} - \int \frac{t^2+1}{\sqrt{t^2+1}}\, dt + \int \frac{1}{\sqrt{t^2+1}}\, dt = t\sqrt{t^2+1} - I + \int \frac{1}{\sqrt{t^2+1}}\, dt$$

Therefore

$$2I = t\sqrt{t^2+1} + \int \frac{1}{\sqrt{t^2+1}}\, dt$$

and so

$$\int \sqrt{t^2+1}\, dt = \frac{t}{2}\sqrt{t^2+1} + \frac{1}{2}\sinh^{-1} t = \frac{t}{2}\sqrt{t^2+1} + \frac{1}{2}\log\left(t + \sqrt{t^2+1}\right).$$

You should be able to redo this calculation for the more general case $\int \sqrt{t^2+a^2}\, dt$.

Using integral tables

As you have no doubt noticed, integration is more tricky than differentiation and historically this has been dealt with through *Integral Tables* - collections of integrals organised by category. The most famous is by Gradshteyn and Ryzhik which has over 1000 pages of integrals.

You may use an Integral Tables and web tools such as Wolfram Alpha for completing the assignment sheets in this module. However, you should known how to derive the integrals you use and you should practice your substitutions regularly or you will get rusty!

Week 3: Differentiating Functions of Several Variables

Introduction

A scalar-valued function of several variables

$$f : U \subseteq \mathbb{R}^n \to \mathbb{R}$$

is a function that assigns a real number to each point in U, a subset of $\mathbb{R}^n$.

Over the course of the next few weeks we are going to study the differential and integral calculus of such functions.

Letting $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ denote a point in U and w the corresponding value in $\mathbb{R}$, we write

$$w = f(x_1, x_2, \cdots, x_n) \qquad \text{or} \qquad w = f(\mathbf{x})$$

We sometimes call f a function of n variables, or say f is a function on $\mathbb{R}^n$ (meaning perhaps a subset of $\mathbb{R}^n$).

When we focus specifically on $n = 2$ or $n = 3$ we commonly write

$$w = f(x, y) \qquad \text{or} \qquad w = f(x, y, z)$$

In fact when we consider the *graph* of a function when $n = 2$ (see below) we frequently use z for the dependent variable, e.g. $z = f(x, y)$.

Many physical systems are expressed as functions of several variables and the governing laws are expressed in the calculus of such functions. For example, consider the gas in a room. The density (the mass per unit volume) is a real number that will be a function of both position and time. This function is usually called ρ (the Greek letter rho), so $\rho(x, y, z, t)$ is the density at position (x, y, z) and time t. Under certain assumptions the physical law governing the evolution of temperature is:

$$\begin{aligned}\frac{\partial \rho}{\partial t}(x, y, z, t) &= \mathbf{v}(x, y, z, t) \cdot \nabla \rho(x, y, z, t), \\ &= v_1(x, y, z, t)\frac{\partial \rho}{\partial x}(x, y, z, t) + v_2(x, y, z, t)\frac{\partial \rho}{\partial y}(x, y, z, t) + v_3(x, y, z, t)\frac{\partial \rho}{\partial z}(x, y, z, t).\end{aligned}$$

Here $\mathbf{v}(x, y, z, t)$ is the velocity of the fluid at position (x, y, z) and time t, and its components are $\mathbf{v} = (v_1, v_2, v_3)$. You are familiar with ordinary differential equations. This is a first example of a **partial differential equation**. By the end of this week you will understand what these symbols mean, and given a function $T(x, y, z, t)$, you will be able to verify whether it satisfies the equation. Discussions about finding solutions to partial differential equations will come in future modules.

Visualising functions on $\mathbb{R}^n$

There are two primary ways to visualise scalar-valued functions of several variables: plotting graphs for functions of two variables, and plotting level sets of functions of two or three variables. Other techniques are possible, for example making movies of graphs or level sets to visualise functions of up to four variables. For larger n, visualisation is challenging.

Graphs

For $n = 2$, f can be visualised as the graph

$$G_f = \{(x, y, z) | \, (x, y) \in U, z = f(x, y)\}$$

The function is seen as a sheet of height $f(x, y)$ above or below each point (x, y).

This idea extends the idea you are already very familiar with of the graph of a function $f : \mathbb{R} \to \mathbb{R}$, where you plot $y = f(x)$.

To think about: Why is this method limited to functions of 2 variables?

Level sets

Level sets (sometimes also called contours in $\mathbb{R}^2$ or isosurfaces in $\mathbb{R}^3$) are subsets of U which are all mapped to the same value by f. Formally, the definition is as follows.

The **level sets** of a function $f : U \subset \mathbb{R}^n \to \mathbb{R}$ are sets of points

$$\mathcal{L}_k = \{\mathbf{x} \in U | \, f(\mathbf{x}) = k\}$$

for each constant k in the range of f.

The intuition is easy for function on $\mathbb{R}^2$. Plot the graph $z = f(x, y)$ then cut the graph horizontally with the plane $z = k$ for some constant k. Project the intersection points down onto $\mathbb{R}^2$ and these will make up the contour for this value of k. Typically, contours will be curves in $\mathbb{R}^2$. To represent a function using contours one typically plots several contours with the corresponding values of k labelled in some way. These are call **contour maps** or **contour plots**. The concept is familiar from topographic maps (where contours indicate points at the same height) and weather maps (where isobars reflect points where the air pressure is the same).

Example: Let $f : \mathbb{R}^2 \to \mathbb{R}$ with $f(x, y) = \sqrt{x^2 + y^2}$. Then for any $r \geq 0$, the level corresponding to the value $f(x, y) = r$ is

$$\mathcal{L}_r = \{(x, y) \in \mathbb{R}^2 \mid \sqrt{x^2 + y^2} = r\}.$$

These level sets are circles of radius $|r|$ centred on the origin.

The level sets of a function of three variables $f(x, y, z)$ are typically surfaces in $\mathbb{R}^3$ called **isosurfaces** (iso- coming from the Greek word meaning equal, so a surface of equal, i.e. constant, value of f). Visualisation programs use transparency or clipping to make several isosurfaces visible simultaneously, allowing us to develop an understanding of the underlying function f.

3.1 Partial Derivatives

When you know how to differentiate functions of one variable, partial derivatives are straightforward. For simplicity we initially restrict to the case $f : \mathbb{R}^2 \to \mathbb{R}$ and define

The **partial derivatives** of function f with respect to x and y at the point (a, b) are

$$\frac{\partial f}{\partial x}(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}$$

$$\frac{\partial f}{\partial y}(a, b) = \lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h},$$

(when these limits exist).

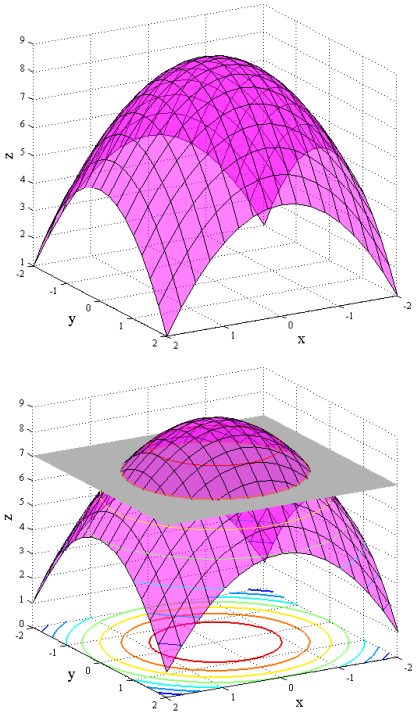


Figure 1: The function $f(x, y) = 9 - x^2 - y^2$ visualised as a graph (top) and as a contour plot (bottom) by slicing the graph at constant heights.

There are a lot of different variants of notation for partial derivatives in use. Sometimes partial derivatives are indicated by subscripts, e.g. $f_x(a, b)$ and $f_y(a, b)$, or $f_1(a, b)$ and $f_2(a, b)$. Sometimes a comma precedes the variable, e.g. $f_{,x}(a, b)$ or $f_{,y}(a, b)$ (this is more relevant for vector-valued functions which we discuss in the final part of the course). Sometimes upper case D is used, e.g. $D_x f(a, b)$ and $D_y f(a, b)$. Sometimes we use ∂, writing $\partial_x f(a, b)$ and $\partial_y f(a, b)$. In this module, we will always try to use the full $\frac{\partial f}{\partial x}$ notation, but you can use whichever notation you like, as long as you make it clear.

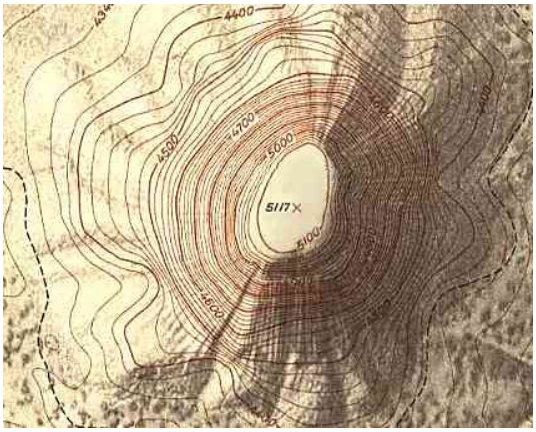
Interpretation

What this definition says is that *the partial derivative of f with respect to x is just the ordinary one-dimensional derivative treating y as a fixed constant*. Concretely, define $g(x) = f(x, b)$ and then compute the ordinary derivative $\frac{dg}{dx}$ at a. This is the partial derivative of f with respect to x. The partial derivative with respect to y is analogous. So the pedantic view of partial differentiation is: Given $f(x, y)$,

$$\text{if } g(x) = f(x, b), \text{ then } \frac{\partial f}{\partial x}(a, b) = \frac{dg}{dx}(a), \text{ and}$$
$$\text{if } h(y) = f(a, y), \text{ then } \frac{\partial f}{\partial y}(a, b) = \frac{dh}{dy}(b).$$

This view is illustrated in Figure 3 below. The function $g(x)$ is obtained by slicing f with a plane $y = b$, and similarly for $h(y)$.

While defining the auxiliary functions $g(x)$ and $h(y)$ is useful to explain partial derivatives, in practice it is unnecessary to explicitly form these functions. You will quickly master computing partial derivatives by working through some examples.



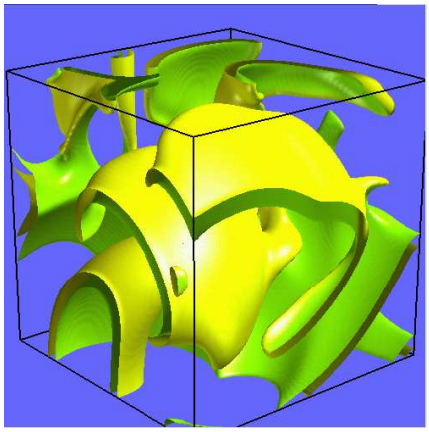


Figure 2: Example of a topographic map (contour plot) and an isosurface. (Map reproduced from http://mail.colonial.net/∼hkaiter/topographic_maps)

Partial derivatives are functions

In the above definition we defined the partial derivatives $\frac{\partial f}{\partial x}(a,b)$ and $\frac{\partial f}{\partial y}(a,b)$ at a point (a,b). If we allow this point to vary, then each partial derivatives will itself be a function of (x,y). In which case we would write

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}$$

to denote the functions and

$$\frac{\partial f}{\partial x}(x,y), \quad \frac{\partial f}{\partial y}(x,y)$$

to denote the values of these functions at the point (x,y). Sometimes vertical bars are used to indicate this evaluation, e.g.

$$\frac{\partial f}{\partial x}(a,b) = \left.\frac{\partial f}{\partial x}\right|_{(a,b)}$$

You are already familiar with everything just stated from functions of one variable. The derivative, f', is itself a function of x. The argument x is often supressed by writing just $\frac{df}{dx}$ to denote $f'(x)$, and we will do the same for partial derivatives when it is convenient. To compute the derivative at a point one differentiates and then evaluates the derivative function at the required point, e.g. $f(x) = \sin(x)$, gives $f'(x) = \cos(x)$, from which $f'(0) = 1$.

Functions of n variables

The definition of partial derivative generalises to functions of n variables

The **partial derivative** of $f(x_1, x_2, \cdots, x_n)$ with respect to x_i, $1 \le i \le n$, is

$$\frac{\partial f}{\partial x_i}(x_1, \cdots, x_n) =$$
$$\lim_{h\to 0} \frac{f(x_1, \cdots, x_i + h, \cdots, x_n) - f(x_1, \cdots, x_i, \cdots, x_n)}{h}$$

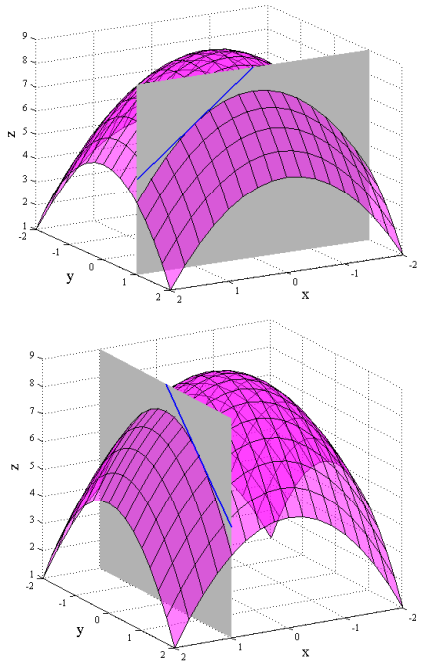


Figure 3: The x partial derivative (top) and y partial derivative (bottom) of the function $f(x, y) = 9 - x^2 - y^2$.

The most common cases we will focus on in this module will be functions of two and three variables: $f(x, y)$ and $f(x, y, z)$.

3.2 Gradient

The gradient plays a fundamental role in the differential calculus of functions of several variables, and it will appear in many subsequent courses.

Let f be a functions of n variables. The **gradient vector**, denoted by ∇f, is the vector-valued function formed from the n partial derivatives

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_i}, \cdots, \frac{\partial f}{\partial x_n} \right)$$

The gradient is a vector quantity. For a function defined on some subset of $\mathbb{R}^n$, it has n components, and it is a function of the coordinates $(x_1, \cdots, x_n)$. We are particularly interested in functions of two and three variable, for which we can write explicitly

$$\begin{aligned} \nabla f(x, y) &= \left(\frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y) \right) \\ &= \frac{\partial f}{\partial x}(x, y)\mathbf{i} + \frac{\partial f}{\partial y}(x, y)\mathbf{j} \end{aligned}$$

and

$$\begin{aligned} \nabla f(x, y, z) &= \left(\frac{\partial f}{\partial x}(x, y, z), \frac{\partial f}{\partial y}(x, y, z), \frac{\partial f}{\partial z}(x, y, z) \right) \\ &= \frac{\partial f}{\partial x}(x, y, z)\mathbf{i} + \frac{\partial f}{\partial y}(x, y, z)\mathbf{j} + \frac{\partial f}{\partial z}(x, y, z)\mathbf{k} \end{aligned}$$

3.3 Chain Rule

Almost all of the differentiation rules you know for functions of one variable carry over to rules for partial derivative exactly as you would expect. In fact, we do not usually even state them as rules for partial differentiation. For example, given $f(x, y)$ and $g(x, y)$ a partial derivative of their product is

$$\frac{\partial}{\partial x}(f\,g) = g\frac{\partial f}{\partial x} + f\frac{\partial g}{\partial x}.$$

This is an obvious consequence of the product rule for differentiation of functions of one variable since taking the x partial derivative means treating y as a constant. The product rule for partial derivatives really is just the product rule from ordinary differentiation.

The Chain Rule also generalises to the multivariable setting in a natural way. Unfortunately, because of the notational complexity of multivariable calculus, it can sometimes be difficult to recognise the chain rule in its generalised form. It is important to master it however, because understanding how it works is particularly important for the treatment of functions of several variables.

Recall the Chain Rule for functions of one variable: if $g(t) = f\,(h(t))$ (also written $g = f \circ h$, which is read 'g is f composed with h') then we have

$$g'(t) = h'(t)f'\big(h(t)\big).$$

That is, the derivative of a composition is the product of the derivatives of the functions that are being composed. For example, taking $f(t) = \sin t$ and $h(t) = t^2$, we have $g(t) = \sin t^2$, so you can verify that

$$g'(t) = 2t\cos t^2 = h'(t)f'\big(h(t)\big)$$

There are a range of variants of this result we can consider for functions of multiple variables, but we start by considering the basic case of the multivariable Chain Rule where we have a scalar-valued function of several variables.

For simplicity, consider a function of just two variables f depending on (x, y). Let both x and y be functions of a third variable t. We name these functions with the variable names and write $x(t)$ and $y(t)$. Using composition we can construct a function $g : \mathbb{R} \to \mathbb{R}$, with

$$g(t) = f\big(x(t), y(t)\big).$$

The Chain Rule for this case is

$$\frac{dg}{dt}(t) = \frac{\partial f}{\partial x}\,(x(t), y(t))\,\frac{dx}{dt}(t) + \frac{\partial f}{\partial y}\,(x(t), y(t))\,\frac{dy}{dt}(t)$$

which is often written simply as

$$\frac{dg}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

In the general case of $f : \mathbb{R}^n \to \mathbb{R}$ where $f(x_1, \cdots, x_n)$ and where each x_i is itself a function of a single variable t, we have

The **Chain Rule**. Let $g(t) = f(x_1(t), \cdots, x_n(t))$, then

$$\frac{dg}{dt}(t) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\,(x_1(t), \cdots, x_n(t))\,\frac{dx_i}{dt}(t) \tag{3}$$

or

$$\frac{dg}{dt} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\frac{dx_i}{dt} \tag{4}$$

In words, the derivative is computed are follows: starting at the left, compute the partial derivative of f with respect to its first argument (i.e. $\frac{\partial f}{\partial x_1}$), evaluate this partial derivative at $(x_1(t), \ldots, x_n(t))$ and multiply by the ordinary derivative taken with respect to t of the function which gives this first argument (i.e. $\frac{dx_1}{dt}$). Now do the same for the next argument of f and add that on. Continue until you have iterated over all components of the argument of f.

Warning: Most aspects of partial differentiation are straightforward extensions of what you know from functions of one variable. However, the Chain Rule has a tendency to cause trouble. The reason is compact notation that is used, as in Eq. (4). It is assumed you understand where the functions are being evaluated, so be sure you are clear on this point. If necessary, write out the arguments in full as in Eq. (3) to make it clear to yourself.

3.4 The Chain Rule revisited

Above, we wrote out all of the functions in full with all components, but it's often quicker to write things down using vector notation. Lets therefore look again at the Chain Rule using vector notation. Given a function of n variables $f : \mathbb{R}^n \to \mathbb{R}$ and a vector function $\mathbf{r} : \mathbb{R} \to \mathbb{R}^n$, into same n-dimensions, we can compose these to obtain

$$g = f \circ \mathbf{r} : \mathbb{R} \to \mathbb{R}, \quad \text{with} \quad g(t) = f(\mathbf{r}(t)).$$

Notice that we can only compose functions if the domain of the 'inner' function matches with the range of the 'outer' function. Here, the range of $\mathbf{r}$ is $\mathbb{R}^n$, and the domain of f is $\mathbb{R}^n$, so things match up. This is a general requirement for composition to make sense.

The function g is just the same composition considered in previous section, written in a different way. We have just notationally replaced all of the component functions $x_i(t)$ with a single vector function $\mathbf{r}(t)$ and used the $\circ$ notation for function composition.

Now re-write the Chain Rule using the gradient vector and the fact that the derivatives $\dfrac{dx_i}{dt}$ are just the components of $\dfrac{d\mathbf{r}}{dt}$. Then

$$\frac{dg}{dt} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\frac{dx_i}{dt} = \nabla f \cdot \frac{d\mathbf{r}}{dt} = \nabla f \cdot \mathbf{r}'$$

The Chain Rule reduces to the dot product between the gradient vector and the derivative vector $\mathbf{r}'$.

The **Chain Rule** (again). If we have functions $f : \mathbb{R}^n \to \mathbb{R}$, $\mathbf{r} : \mathbb{R} \to \mathbb{R}^n$ and $g : \mathbb{R} \to \mathbb{R}$ is defined to be the composition $g(t) = f(\mathbf{r}(t))$, then

$$\frac{dg}{dt}(t) = \nabla f(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt}(t) = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$$

You should see clearly that this is simply the previous Chain Rule written using different notation.

3.5 Directional Derivatives

Directional derivatives are a generalisation of partial derivatives. Where partial derivatives are taken in the directions of the Cartesian coordinate vectors, directional derivatives can be taken in any direction. We'll motivate and explore their definition below.

Lets first restrict to the case $f : \mathbb{R}^2 \to \mathbb{R}$ for simplicity. Using vector notation, we can write the definitions of partial derivatives as

$$\frac{\partial f}{\partial x}(\mathbf{x}) = \lim_{h\to 0} \frac{f(\mathbf{x}+h\mathbf{i}) - f(\mathbf{x})}{h}$$
$$\frac{\partial f}{\partial y}(\mathbf{x}) = \lim_{h\to 0} \frac{f(\mathbf{x}+h\mathbf{j}) - f(\mathbf{x})}{h}$$

where $\mathbf{x} = (x, y)$.

As you may have guessed, there is nothing special about the unit vectors $\mathbf{i}$ and $\mathbf{j}$ and the derivative can be generalised to any direction $\mathbf{u}$, where $\mathbf{u} \in \mathbb{R}^2$. To be a 'direction vector', we will require that $\mathbf{u}$ is a unit vector, which means that $\|\mathbf{u}\| = 1$. Under these conditions, the **directional derivative** of $f(x, y)$ in the direction $\mathbf{u}$ is denoted by $D_{\mathbf{u}}f$. Specifically, the definition of the directional derivative is

$$D_{\mathbf{u}}f(\mathbf{x}) = \lim_{h\to 0} \frac{f(\mathbf{x}+h\mathbf{u}) - f(\mathbf{x})}{h}$$

This makes the partial derivatives particular examples of directional derivatives.

In the general case we have

The **directional derivative** of $f : \mathbb{R}^n \to \mathbb{R}$ in the direction $\mathbf{u}$ is

$$D_{\mathbf{u}}f(\mathbf{x}) = \lim_{h\to 0} \frac{f(\mathbf{x}+h\mathbf{u}) - f(\mathbf{x})}{h}$$

where $\mathbf{u} \in \mathbb{R}^n$.

While one can compute the directional derivative from the definition, it is more common to use the gradient vector as follows. Let

$$g(t) = f(\mathbf{r}(t)), \quad \text{where} \quad \mathbf{r}(t) = \mathbf{x} + t\mathbf{u}$$

with $\mathbf{x}$ and $\mathbf{u}$ fixed vectors. These have the same meaning as above: $\mathbf{x}$ will be the point where we evaluate the directional derivative and $\mathbf{u}$ is the direction. Note

$$\mathbf{r}(0) = \mathbf{x} \qquad \mathbf{r}'(t) = \mathbf{u}$$

Now we compute $\dfrac{dg}{dt}(0)$ two ways. By definition:

$$\begin{aligned}\frac{dg}{dt}(0) &= \lim_{h\to 0} \frac{g(h) - g(0)}{h} = \lim_{h\to 0} \frac{f(\mathbf{r}(h)) - f(\mathbf{r}(0))}{h} \\ &= \lim_{h\to 0} \frac{f(\mathbf{x}+h\mathbf{u}) - f(\mathbf{x})}{h} = D_{\mathbf{u}}f(\mathbf{x})\end{aligned}$$

By the Chain Rule as discussed above:

$$\frac{dg}{dt}(0) = \nabla f(\mathbf{r}(0)) \cdot \mathbf{r}'(0) = \nabla f(\mathbf{x}) \cdot \mathbf{u}.$$

Equating these two expressions, we obtain

The directional derivative of $f : \mathbb{R}^n \to \mathbb{R}$ in the direction $\mathbf{u}$ can be obtained as the dot product of the gradient vector ∇f and the direction vector $\mathbf{u}$:

$$D_{\mathbf{u}}f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{u}$$

3.6 Differentiation as linear approximation

This expression allows us to highlight a general fact which we also saw when differentiating a vector-valued function $\mathbf{r} : I \subset \mathbb{R} \to \mathbb{R}^n$. In any context in which they are defined, first-order derivatives *always* provide an accurate *linear approximation* to a function. When h is close to zero, the definition of the directional derivative means that

$$D_{\mathbf{u}} f(\mathbf{x}) \approx \frac{f(\mathbf{x} + h\mathbf{u}) - f(\mathbf{x})}{h}$$

which can be rearranged to give

$$f(\mathbf{x} + h\mathbf{u}) \approx f(\mathbf{x}) + hD_{\mathbf{u}} f(\mathbf{x}).$$

Using the expression of $D_{\mathbf{u}} f(\mathbf{x})$ in terms of the gradient, this becomes

$$f(\mathbf{x} + h\mathbf{u}) \approx f(\mathbf{x}) + \nabla f(\mathbf{x}) \cdot h\mathbf{u}.$$

Here, the first-order derivative at $\mathbf{x}$ allows us to provide the *best linear approximation* to the function near to $\mathbf{x}$. You should compare this with the view we took in Week 2 when discussing $\mathbf{r}'$. In that case, $\mathbf{r}'$ allowed us to provide the best approximation to the parametrisation as a linear function (in that case, a parametrisation of a line).

3.7 Higher-order derivatives

Just as for functions of a single variable, it is often possible to differentiate a partial derivative and obtain what are termed *second-order* partial derivatives. In the case of functions of several variables, there are potentially many of these second-order erivatives. For example, $f(x, y)$ has the following second-order partial derivatives:

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \\
\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \\
\frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \\
\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)
\end{aligned}$$

In general, the number of differentiation operations which have been performed is called the order, so $\frac{\partial f}{\partial x_i}$ as we defined it previously is a *first-order* partial derivative, or sometimes a partial derivative 'of order 1'. We can also consider *third-order*, *fourth-order* and *higher-order* partial derivatives.

You can seen that there are many possibilities for higher-order derivatives of functions of several variables. One thing that you will learn is that for 'nice' functions, the sequence of differentiation does not matter for **mixed partial derivatives**, so

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

A theorem which we will not formally state or prove here says that if $\dfrac{\partial^2 f}{\partial x \partial y}$ and $\dfrac{\partial^2 f}{\partial y \partial x}$ are both continuous functions, then they are equal. This will normally be the case for functions we consider in this course, but be careful about always assuming it to be true more generally.

3.8 Caution on the extension to $\mathbb{R}^n$

A final word of warning: Extending the analysis of functions from the case $f : \mathbb{R} \to \mathbb{R}$ to functions $f : \mathbb{R}^n \to \mathbb{R}$ is more involved than it might at first appear. This can be seen from the example function

$$f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^4}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

What is the limit of $f(x,y)$ as $(x,y) \to (0,0)$? Since $f = 0$ when either $x = 0$ or $y = 0$ the limit $f(x,y)$ approaching the origin along either the x or y axis is 0. This is also the limit approaching the origin along any line $y = mx$. Hence is might seem that

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

However, if we approach $(0,0)$ along the curve $x = y^2$, we have

$$f(y^2,y) = \frac{y^4}{2y^4} = \frac{1}{2}.$$

Hence, if the origin is approached along this curve the limit is $\frac{1}{2}$. Since one obtains different values depending on how $(0,0)$ is approached, the limit does not, in fact, exist. The fact that there is no consistent limit means that f is *discontinuous* at $(0,0)$.

This illustrates that limits and continuity for functions on $\mathbb{R}^n$ cannot be view from a one-dimensional perspective, but must be properly generalised using regions (called *neighbourhoods*) in $\mathbb{R}^n$. This will be covered in later modules. While we will not define these things here, we will sometimes state properties that hold for continuous functions. You will just have to take this on the basis of faith for the present. Fortunately, several of the most important aspects of multivariable calculus are "one dimensional" and follow easily from things you know. Nothing stops us from being able to define and do calculations using these quantities.

Additional Material

Quadric surfaces

A **quadric surface** is the set of points in $\mathbb{R}^3$ that satisfy a second-degree equation three variables x, y, z. The most general form of such an equation is:

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

for constants $A, \cdots, J$. In most cases (non-degenerate cases), by translation and rotation of coordinates it is possible to bring the equation into standard form of

$$Ax^2 + By^2 + Cz^2 + J = 0 \qquad \text{or} \qquad Ax^2 + By^2 + Iz = 0$$

Quadric surfaces are the generalisation to three dimensions of conic sections in two dimensions.

You should see that points satisfying an equation in three variables, e.g. $f(x, y, z) = 0$, is no different than the zero level set, or isosurface, of a function of three variables $f(x, y, z)$.

We will potentially be interested in the following surfaces and will use them as examples throughout the remainder of the module.

Ellipsoid:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Horizontal and vertical cuts are ellipses. For $a = b \neq c$ this is a **spheroid**. For $a = b = c$ this is a **sphere**.

Note that here, and below, we follow common practice and write the equation with the constant, or lower-order terms, on the right hand side of the equal sign. In this form it is evident that the ellipsoid is the $k = 1$ isosurface of the second-degree polynomial

$$f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$$

Elliptic Paraboloid:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}.$$

Horizontal cuts are ellipses and vertical cuts are parabolas.

Hyperbolic Paraboloid:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}.$$

Horizontal cuts are hyperbolas and vertical cuts are parabolas.

Hyperboloid of One Sheet:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

Horizontal cuts are ellipses. Vertical cuts are hyperbolas.

Hyperboloid of Two Sheets:

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Horizontal cuts are ellipses (if they intersect the surface). Vertical cuts are hyperbolas.

We also will want to consider the degenerate case of circular cylinders.

Circular Cylinder:

$$x^2 + y^2 = a^2$$

In practice we will consider cylinders whose axis is parallel to the x or y coordinate axes.

Making contour plots

For simple functions you should be able to sketch contours and thus produce an approximate contour map. You should understand the relationship between graph of f and its contour map and you should be able to describe a function given a contour map.

In practice one often uses software to generate of contours. Algorithms for generating contours are non-trivial. Think a little about what you might do to numerically generate all *curves* at a given level for a function f. While commonly the contour levels correspond to an equal spacing in k, at times it might be more appropriate to choose a different spacing, e.g. powers of 10, $k = 1, 10, 100, \ldots$. It is also common to plot contours using colour or grey scale values. Here each set $\mathcal{L}_k$ is assigned a specific colour or grey value depending on k.

Another approach (far easier algorithmically) is to generate colour contour plots where one only need consider a grid of values (x_i, y_j) covering the region of interest. For each grid point one computes $f(x_i, y_j)$ and assigns a corresponding colour or grey level. It is not necessary to generate any curves in the plane – your eye will do that for you.

Linear approximation and the derivative

The following is a brief introduction to differentiation for functions of several variables, and will be covered in more detail in the second-year module MA259 Multivariable Calculus.

Recall from functions of one variable $f : \mathbb{R} \to \mathbb{R}$ that the derivative provides a linear approximation to a function near any point (assuming the derivative f' exists). There are different ways of writing this approximation, for example

$$(A) \quad f(x) \approx f(a) + f'(a)(x - a)$$
$$(B) \quad f(x + h) \approx f(x) + f'(x)h$$

It is essential that you understand the difference in notation for these two ways for writing the same thing. In (A), a is the fixed (but arbitrary) point at which the derivative is evaluated and x is varying. In (B), x is the fixed (but arbitrary) point at which the derivative is evaluated and h is varying. Depending of the context, one expression is more convenient than the other.

Let us focus on the (B) form where x is fixed (but arbitrary) and h is the variable. The essential issue is that while the left hand side is a general function of h, the right hand sides is linear in h. The graphical view is that the function f, (generally not linear), is approximated by the tangent line to the graph at any point where f is differentiable. You should think of it as

$$f(x + h) \approx f(x) + T(h)$$

where T is a linear map from $T : \mathbb{R} \to \mathbb{R}$. *The derivative of a function is a linear map.* The linear map, $T(h) = f'(x)h$ will depend on which x we are considering, but for each x it is a linear map on h.

Now let us generalise this to functions of several variables by letting $x \to \mathbf{x} \in \mathbb{R}^n$ and $h \to \mathbf{h} \in \mathbb{R}^n$.

$$f(\mathbf{x} + \mathbf{h}) \approx f(\mathbf{x}) + T(\mathbf{h})$$

where T will be a linear map $T : \mathbb{R}^n \to \mathbb{R}$. If it exists, this linear approximation to f will be the derivative of f. The derivative of a function $f : \mathbb{R}^n \to \mathbb{R}$ at a point $\mathbf{x} \in \mathbb{R}^n$ is a linear map $T : \mathbb{R}^n \to \mathbb{R}$. The linear map will depend on the point $\mathbf{x}$, but for each $\mathbf{x}$, T is a linear map.

We need to work out what this linear map is and we need to say briefly what $\approx$ means in this equation. It is a short calculation to deduce

$$f(\mathbf{x} + \mathbf{h}) \approx f(\mathbf{x}) + \nabla f(\mathbf{x}) \cdot \mathbf{h}$$

Hence $T(\mathbf{h}) = \nabla f(\mathbf{x}) \cdot \mathbf{h}$. This is a mapping between $\mathbb{R}^n$ and $\mathbb{R}$, and it is linear: $\nabla f(\mathbf{x}) \cdot (\alpha \mathbf{h}) = \alpha \nabla f(\mathbf{x}) \cdot \mathbf{h}$ and $\nabla f(\mathbf{x}) \cdot (\mathbf{h}_1 + \mathbf{h}_2) = \nabla f(\mathbf{x}) \cdot \mathbf{h}_1 + \nabla f(\mathbf{x}) \cdot \mathbf{h}_2$. Dotting $\nabla f(\mathbf{x})$ into $\mathbf{h}$ is the same as multiplying $\mathbf{h}$ as a column vector by a $1 \times n$ matrix. You should readily understand these things from Linear Algebra.

The derivative of a function $f : \mathbb{R}^n \to \mathbb{R}$ at a point $\mathbf{x}$ is a linear map $T : \mathbb{R}^n \to \mathbb{R}$ given by the gradient vector $\nabla f(\mathbf{x})$.

Finally, we give a meaning to $\approx$ in the above expressions. $\approx$ represents the following equality

$$f(\mathbf{x} + \mathbf{h}) = f(\mathbf{x}) + \nabla f(\mathbf{x}) \cdot \mathbf{h} + \epsilon(\mathbf{h})\|\mathbf{h}\|$$

where ϵ is a function such that $\epsilon(\mathbf{h}) \to 0$ as $\|\mathbf{h}\| \to 0$. Think of $\epsilon(\mathbf{h})\|\mathbf{h}\|$ as the error in the linear approximation. Since $\epsilon(\mathbf{h}) \to 0$ as $\|\mathbf{h}\| \to 0$, the error $\epsilon(\mathbf{h})\|\mathbf{h}\| \to 0$ faster than $\|\mathbf{h}\|$. You will later learn that this is the definition of differentiability for functions of several variables. You should understand this key idea: *a function is differentiable at a point if it can be approximated by a linear map to within an error that goes to zero faster than $\|\mathbf{h}\|$.*

Week 4: Surfaces in 3D space

Introduction

Now that we have some tools from differentiation, we can study aspects of the geometry of surfaces in 3D space, which can be parametrised by functions of 2 variables.

Much of the emphasis will be on a new tool at our disposal: the level sets of functions of several variables. Not only are level sets valuable for visualising and understanding functions, they also provide a powerful technique in applications, particularly where one is interested in geometrical structures that evolve over time.

You may want to look now at the *Recap of Curves and Surfaces* at the end of the chapter and return to it again after you have finished the chapter.

4.1 Linear Approximation and tangent planes

Recall from functions of one variable $f : \mathbb{R} \to \mathbb{R}$ that the derivative provides a linear approximation to a function near a point (assuming the derivative f' exists)

$$f(x) \approx f(a) + f'(a)(x - a) \tag{5}$$

In this expression a is the fixed (but arbitrary) point at which the derivative is evaluated.

The graphical view is that the graph $y = f(x)$, (which is in general nonlinear), is approximated by the tangent line to the graph at any point where f is differentiable. Equivalently, if one zooms in on the graph of any function around a point where it is differentiable, eventually the graph looks linear.

As we discussed last week, the generalisation of (5) to the case where we consider a scalar–valued multivariable function $f : \mathbb{R}^2 \to \mathbb{R}$ is

$$f(a, b) \approx f(a, b) + \frac{\partial f}{\partial x_1}(a, b)(x - a) + \frac{\partial f}{\partial x_2}(a, b)(y - b).$$

Even more generally, for $f : \mathbb{R}^n \to \mathbb{R}$ evaluated at $\mathbf{x} \in \mathbb{R}^n$ which is close to $\mathbf{a} \in \mathbb{R}^n$, this can be succinctly written

$$f(\mathbf{x}) \approx f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}).$$

You may have already seen exactly this approach in other modules.

To approximate the graph $z = f(x, y)$ in the multivariable case, there are many more directions we can travel in near to (x, y), rather than just backwards and forward in the case of a function of one variable. Approximating in each of these directions results in a different line which is tangent to the graph. Together, these lines sweep out a plane which is again tangent to the surface.

You can easily show that the equation for the tangent plane can be written in standard form as

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0,$$
$$\text{where } A = \frac{\partial f}{\partial x_1}(a, b),\ B = \frac{\partial f}{\partial x_2}(a, b),\ C = -1$$
$$x_0 = a,\ y_0 = b \text{ and } z_0 = f(a, b).$$

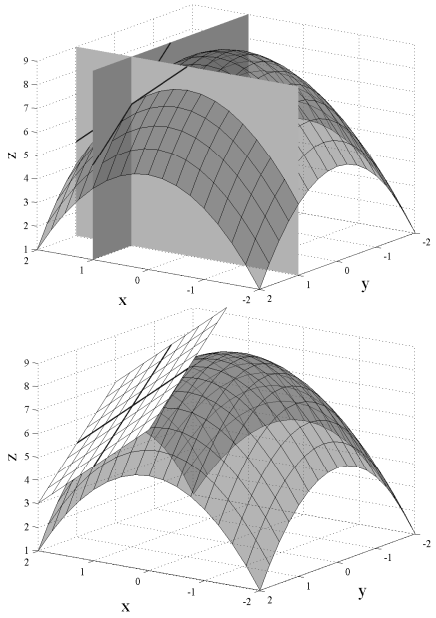


Figure 4: Both the x and y partial derivatives (top) and tangent plane to the graph (bottom) of the function $f(x, y) = 9 - x^2 - y^2$.

For a general function $f : \mathbb{R}^n \to \mathbb{R}$ for $n \geq 2$, we have the more general expression for what is sometimes called a **tangent hyperplane** to the graph $x_{n+1} = f(x_1, \ldots, x_n) = f(\mathbf{x})$ near the point

$$\mathbf{r}_0 = \big(\mathbf{a}, f(\mathbf{a})\big) = \big(a_1, \ldots, a_n, f(a_1, \ldots, a_n)\big).$$

This hyperplane is the set

$$\{\mathbf{r} \in \mathbb{R}^{n+1} : \mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0\} \subset \mathbb{R}^{n+1},$$
$$\text{and } \mathbf{n} = \big(\nabla f(\mathbf{a}), -1\big).$$

In the above expressions, note that $\mathbf{n}, \mathbf{r}, \mathbf{r}_0 \in \mathbb{R}^{n+1}$, while $\mathbf{x}, \mathbf{a}, \nabla f(\mathbf{a}) \in \mathbb{R}^n$.

4.2 Level Sets and the Gradient Vector

Contours

We begin with the case where $f : \mathbb{R}^2 \to \mathbb{R}$ and consider a contour $\mathcal{L}_k$ in the plane. Lets assume that $\mathcal{L}_k$ can be parametrised by a single function $\mathbf{r}(t)$; in general, we may need several curves to capture the full level set. Since $\mathbf{r}(t)$ is a contour we have

$$f\big(\mathbf{r}(t)\big) = k$$

Differentiating this equation with respect to t and using the Chain Rule

$$\frac{d}{dt} f(\mathbf{r}(t)) = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = \frac{d}{dt} k = 0.$$

In other words,

$$\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 0.$$

Recall that since $\mathbf{r}$ parametrises $\mathcal{L}_k$, $\mathbf{r}'(t)$ is tangent to the curve at $\mathbf{r}(t)$, and hence the gradient vector always lies perpendicular to the contour.

Here, we have implicitly assumed that neither the vector $\nabla f(\mathbf{r}(t))$ nor $\mathbf{r}'(t)$ is zero. The points $\mathbf{r}$ where $\nabla f(\mathbf{r}) = \mathbf{0}$ are critical points of the function f. You have seen critical points in Differential Equations and we will review them later. Close to critical points, contours may consist of isolated points or they may be complicated. For now, we will just consider the case where $\nabla f \neq \mathbf{0}$.

Steepest ascent/descent

Now consider a closely related, but more general calculation. Fix a point $\mathbf{a}$ in the domain of f and compute the directional derivative of f in any possible direction $\mathbf{u}$:

$$D_{\mathbf{u}} f(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{u} = \|\nabla f(\mathbf{a})\| \|\mathbf{u}\| \cos\theta = \|\nabla f(\mathbf{a})\| \cos\theta.$$

Here, θ is the angle between $\nabla f(\mathbf{x}_0)$ and $\mathbf{u}$. We can maximise $D_{\mathbf{u}} f(\mathbf{a})$ over all directions $\mathbf{u}$. The maximum occurs when $\theta = 0$ so that $D_{\mathbf{u}} f(\mathbf{a}) = \|\nabla f(\mathbf{x}_0)\|$, and in this case $\mathbf{u} = \nabla f(\mathbf{a})/\|\nabla f(\mathbf{a})\|$. This tells us that $\nabla f(\mathbf{a})$ points in the direction of greatest increase, or the direction of **steepest ascent**, of the function f at the point $\mathbf{a}$.

One can further read off from the above equation that the most negative value of $D_{\mathbf{u}} f(\mathbf{a})$ occurs for $\theta = \pi$, which means that $\mathbf{u} = -\nabla f(\mathbf{a})/\|\nabla f(\mathbf{a})\|$. In other words, $-\nabla f(\mathbf{a})$ points in the direction of the greatest decrease, or the direction of **steepest descent**, of the function f at the point $\mathbf{a}$. For physical reasons the direction of steepest descent, $-\nabla f(\mathbf{a})$, is often more important than steepest ascent. A common situation is that $f : \mathbb{R}^n \to \mathbb{R}$ represents an energy, and physical systems usually wish to reduce their energy. Think of gravitational potential energy: things want to roll down hill reducing it, rather than rolling up hill, which would cause it to increase.

Finally, $D_{\mathbf{u}} f(\mathbf{a}) = 0$ when $\mathbf{u}$ is perpendicular to $\nabla f(\mathbf{a})$. This is consistent with the fact that the contour through $\mathbf{a}$ is perpendicular to $\nabla f(\mathbf{a})$ and the function values do not change along a contour. The equation for tangent line to the contour at point $\mathbf{a}$ can be expressed as $\nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) = 0$.

Extension to functions of three variables

The basic calculations we have just performed are independent of dimension. Fixing a point $\mathbf{a} \in \mathbb{R}^n$ and computing $D_{\mathbf{u}} f(\mathbf{a})$ in any possible direction $\mathbf{u}$ with $\|\mathbf{u}\| = 1$ always gives

$$D_{\mathbf{u}} f(\mathbf{a}) = \|\nabla f(\mathbf{a})\| \cos\theta.$$

This means $D_{\mathbf{u}} f(\mathbf{a})$ will always assume its maximum of $\|\nabla f(\mathbf{a})\|$ for $\theta = 0$ and its minimum of $-\|\nabla f(\mathbf{a})\|$ for $\theta = \pi$. This means that $\nabla f(\mathbf{a})$ and $-\nabla f(\mathbf{a})$ are always the steepest ascent and descent directions of f at the point $\mathbf{a} \in \mathbb{R}^n$, independently of the dimension n.

The aspect which does vary with dimension is the number of directions $\mathbf{u}$ for which $D_{\mathbf{u}} f(\mathbf{a}) = 0$. When $n = 2$, there are only two directions perpendicular to $\nabla f(\mathbf{a})$, but for larger n, there will be more.

For example, for a function of three variables these direction vector all lie on a plane with $\nabla f(\mathbf{a})$ as its normal. The direction vectors for which $\mathbf{u}$ satisfies $D_{\mathbf{u}} f(\mathbf{a}) = 0$ form a circle in this plane. In higher dimensions, these $\mathbf{u}$ lie in an $(n-1)$–dimensional hyperplane.

The situation is that at each point $\mathbf{a} \in \mathbb{R}^n$, there is a plane passing through $\mathbf{a}$ with $\nabla f(\mathbf{a}) \in \mathbb{R}^n$ as its normal. The plane is tangent to the isosurface passing through $\mathbf{a}$, and the gradient vector is said to be **normal** to the isosurface at each point $\mathbf{a}$. The equation for the tangent plane at $\mathbf{a}$ is

$$\nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) = 0$$

or

$$A(x-a)+B(y-b)+C(z-c)=0$$

with $\mathbf{a}=(a,b,c)$ and

$$A=\frac{\partial f}{\partial x}(\mathbf{a}),\quad B=\frac{\partial f}{\partial y}(\mathbf{a}),\quad C=\frac{\partial f}{\partial z}(\mathbf{a})$$

One can also examine the tangent plane to the isosurface by considering parametrised curve $\mathbf{r}(t)$ on the surface passing through $\mathbf{a}$ at $t=0$. Exactly as we argued for contours, since $f(\mathbf{r}(t))=k$, differentiating gives

$$\nabla f(\mathbf{x}_0)\cdot\mathbf{r}'(0)=0$$

Thus $\nabla f(\mathbf{a})$ is perpendicular to the tangent vector of every curve on the surface passing through $\mathbf{a}$, hence $\nabla f(\mathbf{a})$ is normal to the surface, or equivalently, every tangent vector is in the tangent plane with $\nabla f(\mathbf{a})$ as its normal.

Important: Our previous results for tangent planes to graphs of functions is a particular case of the discussion in this section. If $f:\mathbb{R}^2\to\mathbb{R}$ and we define $g:\mathbb{R}^3\to\mathbb{R}$ to be

$$g(x,y,z)=z-f(x,y),$$

then the graph

$$\mathcal{G}_f=\Big\{\big(x,y,f(x,y)\big):(x,y)\in\mathbb{R}^2\Big\}\subset\mathbb{R}^3$$

is just the zero level set of g:

$$\mathcal{L}_{g=0}=\Big\{g(x,y,z)=0:(x,y,z)\in\mathbb{R}^3\Big\}.$$

Just as when we considered parametrised curves generalising graphs, so we see that level sets (or contours or isosurfaces) provide a more general way to represent surfaces in higher dimensions.

One of the reasons to care about normal vectors and tangent planes is that they are used extensively in computer graphics. The way light reflects from objects depends on the direction of incoming light relative to the normal vector to a surface. This means that knowing the normal vector to a surface is essential to realistically render objects and make them look as we expect.

4.3 Critical Points

The final topic we study this week is critical points for functions of two variables.

A point $\mathbf{r}=(a,b)$ is called a **critical point** of a scalar–valued function $f:\mathbb{R}^2\to\mathbb{R}$ if

$$\frac{\partial f}{\partial x}(a,b)=0\quad\text{and}\quad\frac{\partial f}{\partial y}(a,b)=0,\quad\text{or equivalently}\quad\nabla f(\mathbf{r})=\mathbf{0}.$$

We are interested in three generic critical points for functions $f(x,y)$ and it worth discussing these using prototypical functions.

- $f(x,y) = x^2 + y^2$ is a canonical example of a function with a **local minimum** at the critical point $(a,b) = (0,0)$. The graph of this function is a circular paraboloid. The contours are circles for $k > 0$ and a single point for $k = 0$. The ODEs are

  $$\dot{x} = -2x \quad \dot{y} = -2y$$

  with solutions

  $$x(t) = x(0)e^{-2t}, \quad y(t) = y(0)e^{-2t},$$

  so the fixed point at $(0,0)$ is a sink (because the system evolves down to the bottom of the function f).

- $f(x,y) = -x^2 - y^2$ is a canonical example of a function with a **local maximum** at the critical point $(a,b) = (0,0)$. The graph of this function is again a circular paraboloid. The contours are circles for $k < 0$ and a single point for $k = 0$. The ODEs are

  $$\dot{x} = 2x \quad \dot{y} = 2y$$

  with solutions

  $$x(t) = x(0)e^{2t}, \quad y(t) = y(0)e^{2t},$$

  so the fixed point at $(0,0)$ is a source (because the system evolves away from the peak at $(0,0)$).

- $f(x,y) = x^2 - y^2$ is a canonical example of a function with neither a local maximum nor a local minimum at the critical point $(a,b) = (0,0)$. Such a point is called a **saddle point**. The graph of this function is a hyperbolic paraboloid. The contours are hyperbolas for $k \neq 0$ and crossed lines for $k = 0$. The ODEs are

  $$\dot{x} = -2x \quad \dot{y} = 2y$$

  The fixed point at $(0,0)$ is a called a saddle fixed point.

Continuing with the definitions, $f(x,y)$ has a **local maximum** at (a,b) if $f(x,y) \leq f(a,b)$ for all (x,y) near (a,b). The value $f(a,b)$ is called the **local maximum value**. $f(x,y)$ has a **local minimum** at (a,b) if $f(x,y) \geq f(a,b)$ for all (x,y) near (a,b). The value $f(a,b)$ is called the **local minimum value**. Here "(x,y) near (a,b)" means that

$$\|(x,y) - (a,b)\| = \sqrt{(x-a)^2 + (y-b)^2} < \delta$$

for some $\delta > 0$. We do not care how small δ is as long as it is positive. It is easily shown, using arguments from functions of one variable, that if $f(x,y)$ has a local maximum or a local minimum at (a,b), then (a,b) is a critical point of f.

It is frequently possible to decide whether a critical point is a local maximum, local minimum, or a saddle point using a test based on the second-order partial derivatives.

We first state the test and then discuss it.

Second Derivative Test: Suppose $f(x, y)$ has a critical point at (a, b). Let

$$D = \frac{\partial^2 f}{\partial x^2}(a,b)\frac{\partial^2 f}{\partial y^2}(a,b) - \left[\frac{\partial^2 f}{\partial x \partial y}(a,b)\right]^2$$
$$= \det \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(a,b) & \frac{\partial^2 f}{\partial x \partial y}(a,b) \\ \frac{\partial^2 f}{\partial y \partial x}(a,b) & \frac{\partial^2 f}{\partial y^2}(a,b) \end{pmatrix}$$

then

- if $D > 0$ and $\frac{\partial^2 f}{\partial x^2}(a, b) > 0$, then $f(a, b)$ is a local minimum.
- if $D > 0$ and $\frac{\partial^2 f}{\partial x^2}(a, b) < 0$, then $f(a, b)$ is a local maximum.
- if $D < 0$, then $f(a, b)$ is a saddle point.
- if $D = 0$, then the test in inconclusive.

The tests assumes that the second-order partial derivatives exist, and moreover they must be continuous in the vicinity of (a, b).

We are not going to derive this test. It is not difficult but requires Taylor's Theorem for functions of several variables, which we have not yet covered. You should verify that the test works for the three prototype examples above.

You should also recognise that in terms of the ODEs

$$\frac{\partial^2 f}{\partial x^2} = -\frac{\partial f_1}{\partial x}, \quad \frac{\partial^2 f}{\partial y^2} = -\frac{\partial f_2}{\partial y},$$
$$\frac{\partial^2 f}{\partial x \partial y} = -\frac{\partial f_2}{\partial x} = \frac{\partial^2 f}{\partial y \partial x} = -\frac{\partial f_1}{\partial y}$$

Hence D in the test is just the determinant of the Jacobian matrix at the fixed point, and hence D is the product of the eigenvalues at the fixed point. This should give you some intuition into the test.

Finally, one can construct examples for which the test fails. For example,

$$f(x, y) = x^4 + y^4, \quad f(x, y) = -x^4 - y^4$$

$$f(x, y) = x^4 - y^4$$

showing that when the tests fails the function may have a local minimum, local maximum, or a saddle point.

Additional Material

Recap of curves and surfaces

This may be helpful in orienting you on the various approaches we have considered, and will consider, in the description of curves and now surfaces. You may also want return and re-read it after Week 9.

Curves

If you think about it, you now know three potential ways to describe a curve in the plane.

- Graph of a function of one variable $f : I \subseteq \mathbb{R} \to \mathbb{R}$,

$$G_f = \{(x, y) |\, x \in I, y = f(x)\}$$

- Parametrised curve $\mathbf{r} \colon I \to \mathbb{R}^2$,

$$\mathcal{C} = \{\mathbf{r}(t) | t \in I\}$$

- The level set, or contour, of a function of two variables $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}$

$$\mathcal{L}_k = \{(x, y) \in U |\, f(x, y) = k\}$$

  A level set is itself equivalent to the set of points satisfying an equation in two variables, $f(x, y) = k$.

As you know, not all curves can be described by graphs. When a curve can be represented as a graph $y = f(x)$, it is easy to also write it as a parametrised curve or a level set. A simple parametrisation is $\mathbf{r}(t) = (t, f(t))$. For a level set, take the zero set, $\mathcal{L}_0$, of the function $F(x, y) = f(x) - y$. It is useful to keep in mind this interconnection between the various concepts.

As we emphasised in our study of parametrised curves, tangents and normals to curves are important. You know how to compute the tangent to the graph of a function from the derivative f'. You may not have computed the normal to a graph, but you could without much trouble. We have yet to address tangents and normals to level sets, but will this week.

Surfaces

The same three approaches exist for describing surfaces in space.

- Graph of a function of two variables $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}$, i.e.

$$G_f = \{(x, y, z) |\, (x, y) \in U, z = f(x, y)\}$$

- Parametrised surface $\mathbf{r} \colon U \in \mathbb{R}^2 \to \mathbb{R}^3$, which we treat in Week 9.
- The level set, or isosurface, of a function of three variables $f : U \subseteq \mathbb{R}^3 \to \mathbb{R}$

$$\mathcal{L}_k = \{(x, y, z) \in U |\, f(x, y, z) = k\}$$

As in the case of curves, not all surfaces can be represented as graphs, but if the surface can be represented as a graph $z = f(x, y)$, then it can also be parametrised and obtained as a level set. The parametrised surface can wait, but the simple level set description is again

the zero level set of $F(x, y, z) = f(x, y) - z$. Again, it will help you if you keep in mind the interconnection between these approaches.

As with curves, tangents and normals to surfaces are important. In this chapter we describe the tangent plane to a surface from two points of view, the tangent plane to a graph of $f(x, y)$ and the tangent plane to a level set of a function of three variables $F(x, y, z)$. These will be the same surface if we take the zero level set of $F(x, y, z) = f(x, y) - z$. We know from the section on isosurfaces that the gradient vector ∇F is normal to the tangent plane to the isosurface. At a point $(a, b, f(a, b))$ on the surface, the normal vector is

$$\nabla F\big(a, b, f(a, b)\big) = \big(\frac{\partial f}{\partial x}(a, b), \frac{\partial f}{\partial y}(a, b), -1\big)$$

You should satisfy yourself that this gives exactly the same plane and as tangent plane to the graph of $f(x, y)$.

Discussion

There is room for confusion. Functions $\mathbb{R} \to \mathbb{R}$, $\mathbb{R} \to \mathbb{R}^2$, or $\mathbb{R}^2 \to \mathbb{R}$, could all potentially be used in describing a curve in the plane. Functions $\mathbb{R}^2 \to \mathbb{R}$, $\mathbb{R}^2 \to \mathbb{R}^3$, or $\mathbb{R}^3 \to \mathbb{R}$, could all potentially be used in describing a surface in the space. Note in particular that $\mathbb{R}^2 \to \mathbb{R}$ appears is both lists: its levels sets are curves in the plane and its graph is a surface in space. Yes, much room for confusion.

Generally speaking, graphs are a tool for understanding and visualising functions, as opposed to describing the geometry of curves or surfaces. In most cases, parametrisations are the best approach to study the geometry of curves and surfaces.

Levels sets play multiple roles. On the one hand they are very useful is visualising functions. A notable example, in this module will be contour maps for functions of two variables (known as potential landscapes) that govern motion. On the other hand, an individual level sets can at times be the ideal way to characterise a particular curve or surface. This is particularly the case if the curve or surface is simply described by an equation. See for example the quadric surfaces at the end of Week 4. Again, set of points satisfying equations are nothing other than level sets of a particular function.

Week 5: Integration in Cartesian coordinates

Introduction

Given a continuous function $g(x) > 0$, you know that the integral $\int_a^b g(x)\,dx$ can be interpreted as the area under the curve $g(x)$ from $x = a$ to $x = b$. This interpretation as area under the curve is intimately connected with the definition of integration.

As discussed over the last two weeks, the graph of a function of two variables $z = f(x, y) > 0$ is a surface above the (x, y)-plane. In the same way as we ask about the area under the graph of g, it is natural to ask about the *volume* under f. This volume will be expressed as a *multiple integral*.

A big aspect of integration in two and three dimensions is the new freedom in geometry. For a function $f(x, y)$ one can ask about the volume above many different regions in the (x, y) plane; for the function of one variable g, we can only integrate over intervals.

For example, we might ask:

- What is the volume beneath the graph of $f(x, y)$ and above the square
$$\{(x, y) \in \mathbb{R}^2 |\ |x| + |y| \leq 1\}?$$
- Or, what is the volume beneath the graph of $f(x, y)$ lying above the annulus
$$\{(x, y) \in \mathbb{R}^2 |\ 1 \leq x^2 + y^2 \leq 2\}?$$

A second aspect is that we have freedom to change and choose coordinate systems to exploit symmetries and simplify our approach to integration. This will be the focus next week.

5.1 Multiple Integration

In this section, we give a quick, heuristic development of multiple integration. Consider a function f of two variables which is defined, and is positive, on the closed rectangle, i.e. $f : R \subset \mathbb{R}^2 \to \mathbb{R}^+$, where

$$R = [a, b] \times [c, d] = \{(x, y) |\ a \leq x \leq b, c \leq y \leq d\}$$

Our goal is to compute the volume V of the three dimensional region $\Omega \subset \mathbb{R}^3$ generated by the graph of f above this rectangle

$$\Omega = \{(x, y, z) |\ 0 \leq z \leq f(x, y), (x, y) \in R\}.$$

To compute the volume we partition the rectangle R into smaller rectangles by splitting up each of the intervals $[a, b]$ and $[c, d]$ as follows. Split $[a, b]$ into N subintervals $[x_i, x_{i+1}]$ of equal length by letting

$$x_i = a + i\triangle x \quad \text{where } \triangle x = \tfrac{b-a}{N}$$

and $i = 0, 1, \ldots, N$.

Similarly, partition $[c, d]$ into M equal subintervals $[y_j, y_{j+1}]$ of length $\triangle y$ by letting

$$y_j = c + j\triangle y \quad \text{where } \triangle y = \tfrac{d-c}{M}$$

and $j = 0, \ldots, M$. Using these partitions, form rectangles R_{ij} of R by taking the Cartesian product of the x and y subintervals:

$$R_{ij} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$$

Each of these rectangles has area $\triangle A = \triangle x \triangle y$.

We can now approximate the volume of the solid over each subrectangle as $f(x_i, y_j)\triangle A$. Summing these we get an approximation for the volume of Ω

$$V \approx \sum_{j=0}^{M-1} \sum_{i=0}^{N-1} f(x_i, y_j)\triangle A$$

For N and M large, the size of each subrectangle is small and f does not vary much over the subrectangle. Each term in the sum then corresponds to a subvolume that is very tall compared with its base dimensions. Now take the limit as the number of elements N and M goes to infinity, with the sizes $\triangle x$ and $\triangle y$ going to zero, or equivalently, $\triangle A \to 0$. In the limit, we get the exact volume

$$V = \lim_{M,N\to\infty} \sum_{j=0}^{M-1} \sum_{i=0}^{N-1} f(x_i, y_j)\triangle A$$

The volume under a function of two variables is just one of numerous quantities one wants to compute in this way. In general there is no reason to restrict functions taking on positive values, so we now drop this restriction and define

The **double integral** of f over the rectangle R is

$$\iint_R f\, dA = \lim_{M,N\to\infty} \sum_{j=0}^{M-1} \sum_{i=0}^{N-1} f(x_i, y_j)\triangle A$$

Comments

- We have assumed here that the limit exists. If the limit exists, f is said to be **integrable** and in later courses you will learn that f is continuous (in an appropriate sense which we have not defined), then it is integrable.
- It is unnecessary to require that R be partitioned into equal rectangles, all with the same area. In fact it is unnecessary that the partition of R use rectangles at all; we simply did this here for convenience. The key point is that the area of the largest region we have used to partition R has to tend to zero. You will see further discussion of this point when you study integration in MA244 Analysis 3.
- We chose to approximate the subvolumes using f evaluated at (x_i, y_j), but we could have used any sample point within R_{ij}. In the limit, it does not matter where we evaluate f within each element since in the limit their size goes to zero.

  For some applications, we may not be able to conveniently and explicitly compute an integral. In such cases, choosing a good partition of R and a good formula to approximate the height of f over the partition *without taking the limit* is the subject of a whole area of maths called numerical quadrature. We won't pursue this here, but see MA261 Differential Equations: Modelling and Numerics for more details.

5.2 Iterated integration over rectangles

One does not generally compute explicit integrals by forming the sums and taking the limits in the above definition (although, as mentioned, you might do this to approximate an integral on a computer). Rather one relies on iterated or repeated one-variable integration as we now explain.

It is instructive to again think in terms of volume under a surface. We know that the volume is given by a double integral which itself is expressed as a limits of a double sum

$$V = \iint_R f dA = \lim_{M,N\to\infty} \sum_{j=0}^{M-1} \sum_{i=0}^{N-1} f(x_i, y_j) \triangle A.$$

We can separate and reorder the limits and sums as follows

$$\begin{aligned} V = \iint_R f dA &= \lim_{M\to\infty} \lim_{N\to\infty} \sum_{j=0}^{M-1} \sum_{i=0}^{N-1} f(x_i, y_j) \triangle x \triangle y \\ &= \lim_{M\to\infty} \sum_{j=0}^{M-1} \lim_{N\to\infty} \sum_{i=0}^{N-1} f(x_i, y_j) \triangle x \triangle y \\ &= \lim_{M\to\infty} \sum_{j=0}^{M-1} \left[\lim_{N\to\infty} \sum_{i=0}^{N-1} f(x_i, y_j) \triangle x \right] \triangle y \end{aligned}$$

The expression in brackets is just the usual definite integral of $f(x, y)$ over x with y fixed at y_j. Computing such an integral is called **partial integration** – integration of a function of several variables with respect to one variable while treating any others as constants. You will recognise the analogy with partial differentiation. Let us denote this definite integral, which will depend on y, by $A(y)$,

$$A(y) = \int_a^b f(x, y)\, dx$$

A for area since $A(y)$ is the area under the curve $f(x, y)$ from $x = a$ to $x = b$ with fixed y. Then our multiple integral becomes

$$V = \iint_R f\, dA = \lim_{M\to\infty} \sum_{j=0}^{M-1} A(y_j) \triangle y$$

The expression on the right is just the usual definite integral of $A(y)$, so

$$V = \iint_R f\, dA = \int_c^d A(y)\, dy = \int_c^d \left[\int_a^b f(x, y)\, dx \right] dy$$

It is important to understand the interpretation of this. The inner integral gives the area under a curve at constant y. The outer integral integrates these areas over the range of y to give the volume.

We did not have to re-order such that the x-integral appeared on the inside. One could have equally moved the y sum and limit to the inside. You should satisfy yourself that this is true.

We now drop the specific interpretation of volume under a function and arrive at the following relationship between double integrals and iterated integrals.

The double integral of f over the rectangle $R = [a, b] \times [c, d]$ is given by iterated, also known as repeated, integration

$$\iint_R f dA = \int_c^d \int_a^b f(x, y) dx dy = \int_a^b \int_c^d f(x, y) dy dx$$

Discussion

- *Notation:* You will notice that we dropped the square brackets in boxed expression above. This is common. All of the following are used to denoted iterated integration

$$\int_c^d \int_a^b f(x, y) dx dy, \int_c^d \left[\int_a^b f(x, y) dx \right] dy,$$
$$\int_{y=c}^{y=d} \int_{x=a}^{x=b} f(x, y) dx dy, \int_{y=c}^{d} dy \int_{x=a}^{b} dx f(x, y)$$

  and other combinations of these forms.

- The boxed expression is known as *Fubini's Theorem*, or would be if we stated is as a theorem. Equality is guaranteed if f is a continuous function, and also holds even under weaker conditions on f.

- Understanding the following is crucial to iterated integration: Integrals are nested, and are performed from the inside outwards. In the examples we have just seen there were 2 levels, but in higher dimensions there will be more nesting. You work with a single variable at a time treating any variables outside the current level as constants. Make sure you understand this now: as we generalise to other integration domains this will be key to your success. If you are ever confused by an iterated integral, explicitly include the square brackets showing the nesting.

- Sometimes the function $f(x, y)$ separates into the product of a function of x only and a function of y only. Letting $f(x, y) = g(x)h(y)$,

$$\int_c^d \int_a^b f(x, y) dx dy = \int_c^d \left[\int_a^b g(x)h(y) dx \right] dy$$

  but $h(y)$ is a constant as far as the inner x integration is concerned, so it can be pulled out

$$\int_c^d h(y) \left[\int_a^b g(x) dx \right] dy$$

  and now the whole x integral is a constant as far as the y integration is concerned to it can be pulled out

$$\left[\int_a^b g(x) dx \right] \int_c^d h(y) dy$$

  Thus the double integral is said to **separate**

$$\int_c^d \int_a^b f(x, y) dx dy = \int_a^b g(x) dx \int_c^d h(y) dy.$$

  In this case, we can simply compute each integral separately, and multiply the results together.

5.3 Integration over boxes in three or more variables

The extension of the above to functions of more than two variables is straightforward. We will only explicitly consider functions of three variables, but functions of four, five, etc variables are essentially the same.

The following treatment is a natural extension of the above. Please fill in the missing details. Rather than consider volume under a surface, a useful physical example would be the total mass of a solid whose density (mass per unit volume) is given by $f(x, y, z)$. f is necessarily positive in this example. Assume the solid is in the shape of a rectangular box. We then have $f : B \subset \mathbb{R}^3 \to \mathbb{R}^+$, where B is the rectangular box

$$\begin{aligned} B &= [a, b] \times [c, d] \times [r, s] \\ &= \{(x, y, z) |\, a \le x \le b, c \le y \le d, r \le z \le s\}. \end{aligned}$$

By partitioning each of the x, y, and z directions, B can be partitioned into sub-boxes

$$B_{ijk} = [x_i, x_{i+1}] \times [y_j, y_{j+1}] \times [z_k, z_{k+1}]$$

each with volume $\triangle V = \triangle x \triangle y \triangle z$. The mass of each sub-box B_{ijk} will be approximately the density f evaluate at (x_i, y_j, z_k) multiplied by the box volume: $f(x_i, y_j, z_k)\, \triangle V$. Then the approximate total mass will be the sum of all the sub-boxes. Taking the limit as the partition becomes finer we have the total mass of the solid given as a triple integral

$$\begin{aligned} \text{Total Mass} &= \iiint_B f\, dV \\ &= \lim_{K,M,N\to\infty} \sum_{k=0}^{K-1} \sum_{j=0}^{M-1} \sum_{i=0}^{N-1} f(x_i, y_j, z_k) \triangle V \end{aligned}$$

Now forgetting the particular physical example, and the requirement that f be positive, we define

The **triple integral** of f over the rectangular box B is

$$\iiint_B f\, dV = \lim_{K,M,N\to\infty} \sum_{k=0}^{K-1} \sum_{j=0}^{M-1} \sum_{i=0}^{N-1} f(x_i, y_j, z_k) \triangle V$$

More important than the definition in terms of limits is evaluations by iterated integrals. Analogously to the two-variable case

The triple integral of f over the rectangular box $B = [a, b] \times [c, d] \times [r, s]$ is given by iterated integration

$$\iiint_B f\, dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\, dx dy dz$$

or other orderings of the x, y, and z integrals.

The same comments from Sec. 5.1 and 5.2 apply here as well.

5.4 Integration over more general domains

Now the fun starts. We are going to consider integration of functions over more general domains, first in 2D then in 3D. Often, doing the integrals is not going to be the issue, but getting the geometry right is.

Two variables

Let Ω denote the region or domain in $\mathbb{R}^2$ on which f is defined and over which we wish to integrate.

A **Type I** region can be expressed

$$\Omega = \{(x, y) \in \mathbb{R}^2 | \, a \leq x \leq b, \, g_1(x) \leq y \leq g_2(x)\}$$

where g_1 and g_2 are continuous functions on $[a, b]$. We evaluate the double integral of f over Ω as an iterated integral in the following way

$$\iint_\Omega f dA = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x, y) dy \right] dx$$

Let us elaborate by examining the nesting from inside to outside. Within the brackets, x is treated as a constant. Its actual value is not known, but everywhere it appears it acts like a constant. Hence the limits of the y-integral are the "constants" $g_1(x)$ and $g_2(x)$. Assuming one can partially integrate $f(x, y)$ with respect to y, then one can evaluate the inner brackets resulting in a function of x only. Then the outer integral is a standard integral over the interval $[a, b]$.

A **Type II** region or domain is one that can be expressed

$$\Omega = \{(x, y) \in \mathbb{R}^2 | \, h_1(y) \leq x \leq h_2(y), \, c \leq y \leq d\}$$

where h_1 and h_2 are continuous functions on $[c, d]$. In this case we evaluate the double integral of f over Ω as iterated integral in the following way

$$\iint_\Omega f dA = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x, y) dx \right] dy$$

Comments

- There are two aspects to iterated integrals over these types of domains. First setting up the integration correctly and second carrying out the integration correctly. To understand how this works, one simply needs to practice a lot.
- Given some specification of the domain Ω, first try writing it explicitly as a finite union of non-intersecting domains of Type I or Type II. Then your integral will be expressed as a sum of iterated integrals due to the property of additivity, see below.
- Many domains can be written in either Type I or Type II format. You should try to choose the one which simplifies the calculations, but the right choice is not always obvious when you start. Again, practice will help.

  Example: Consider the region

$$D = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq \pi; x \leq y \leq \pi\}.$$

Then

$$\iint_D \frac{\sin(y)}{y} dA = \int_0^\pi \int_x^\pi \frac{\sin(y)}{y} dy\, dx.$$

The inner integral cannot be expressed in terms of elementary functions and we seem to be stuck. However, by drawing a picture, we can see that the domain of integration D can be rewritten as a Type II domain:

$$D = \{(x, y) \in \mathbb{R}^2 \mid 0 \le y \le \pi; 0 \le x \le y\}.$$

Therefore,

$$\begin{aligned}\iint_D \frac{\sin(y)}{y} dA &= \int_0^\pi \int_0^y \frac{\sin(y)}{y} dx\, dy \\ &= \int_0^\pi y \frac{\sin(y)}{y} dy = -\cos(y)\big|_0^\pi = 2.\end{aligned}$$

- Just like single integrals, multiple integrals obey linearity

$$\iint_\Omega (cf + g) dA = c \iint_\Omega f dA + \iint_\Omega g dA$$

for functions f and g and constants $c \in \mathbb{R}$.

- The following fact is useful: If $\Omega = \Omega_1 \cup \Omega_2$, and Ω_1 and Ω_2 do not overlap except possibly on their boundaries, then

$$\iint_\Omega f dA = \iint_{\Omega_1} f dA + \iint_{\Omega_2} f dA$$

This not only helpful for computing the integral over Ω from integrals over Ω_1 and Ω_1, but it is also helpful for computing the integral over Ω_1 from the integrals over Ω and Ω_2.

Three variables

Now let Ω be a region or domain in $\mathbb{R}^3$ on which f is continuous and over which we wish to integrate. In analogy with the 2D case we could define 6 types of regions. We won't do this, but instead look at a one case and let you deduce the others.

Suppose the set Ω can be written

$$\begin{aligned}\Omega = \{(x, y, z) \in \mathbb{R}^3 \mid a \le x \le b,\ g_1(x) \le y \le g_2(x),\\ u_1(x, y) \le z \le u_2(x, y)\}\end{aligned}$$

Then

$$\begin{aligned}&\iiint_\Omega f dV \\ &\quad = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z) dz \right] dy \right] dx\end{aligned}$$

The inner most integral is a partial integral over z. Here x and y, and hence integration limits $u_1(x, y)$ and $u_2(x, y)$, are viewed as constants. The result will be a function of (x, y). The middle integral is a partial integral over y with x treated as constant, followed by the outer definite integral over x with limits a and b.

5.5 Applications

There are numerous applications of multiple integration. The point here is not to give a long list but rather to state a few general concepts from applications which you are expected to learn and know. Specifically, you should know the formulas below that appear in boxes.

Area and Volume

This is not so much an application as some special cases you should be aware of. Suppose you want to compute the area of a region $\Omega \subset \mathbb{R}^2$. Using the same formalism we used to derived double integrals, you can derive the formula for area as a double integral. The idea is simply to set the function $f(x, y) = 1$. That is, rather than summing up $f(x_i, y_j)\triangle A$ to get the volume of f over Ω, just sum up $\triangle A$ to get the area of Ω. For rectangular region R

$$\text{Area of R} = \lim_{M,N\to\infty} \sum_{j=0}^{M-1} \sum_{i=0}^{N-1} \triangle A = \iint_R dA$$

This extends to more general regions Ω in the plane. Similarly one can obtain the volume of a region Ω in space as a triple integral – just set $f(x, y, z) = 1$ in triple integration.

Denoting the area of a planar region Ω by $A(\Omega)$ and the volume of a space region by $V(\Omega)$, we have

Area of a region Ω in the plane

$$A(\Omega) = \iint_\Omega dA$$

Volume of a region Ω in space

$$V(\Omega) = \iiint_\Omega dV$$

Density and centre of mass

Let the shape of a solid be represented by the domain Ω. Let the density (mass per unit volume) at point (x, y, z) now be denoted $\rho(x, y, z)$, (ρ is a common symbol for a density). Physically, we require that $\rho(x, y, z) \geq 0$ everywhere on Ω.

The mass of the solid is

$$M = \iiint_\Omega \rho \, dV$$

There are a number of interesting quantities besides the total mass. One is the centre of mass, a point in space, i.e. a vector. Coordinates of the centre of mass of solid Ω with density ρ is

The centre of mass is $(\bar{x}, \bar{y}, \bar{z})$ where

$$\bar{x} = \frac{1}{M} \iiint_\Omega x\rho \, dV \quad \bar{y} = \frac{1}{M} \iiint_\Omega y\rho \, dV$$

$$\bar{z} = \frac{1}{M} \iiint_\Omega z\rho \, dV$$

The concept of the centre of mass is important in mathematical modelling for applications because the centre of mass is where we can place a point particle to represent an extended body.

Along with mass density, there are many other types of densities, for example charge density, energy density, etc. and many other quantities that can be computed. The concept of density applies in various dimensions. For example in the two-dimensional setting one could consider a population density or a charge density on a surface, in which case the density will be expressed in units of something per unit area.

A particularly important example of density is probability density. In this case the analog of total mass is total probability which will be 1 by definition. The analog of centre of mass will be expectation values (or mean values) of the random variables described by the probability density.

Week 6: Special coordinates for integration

Introduction

Many problems of interest naturally involve symmetry. Where symmetry occurs, it should always be exploited if possible, and this often means using coordinate systems other than Cartesian coordinates.

6.1 Overview

We start with a quick reminder of our derivation of multiple integration in Cartesian coordinates. We initially considered a rectangular domain R and partitioned it into sub-rectangles R_{ij} based on a partitioning each of the coordinates x and y separately. Essentially we divided up R using an equally spaced grid with increments $\triangle x$ and $\triangle y$. We then constructed the sum $f(x_i, y_i)\triangle A$, where $\triangle A$ is the area of R_{ij}. In this case $\triangle A = \triangle x \triangle y$. Taking the limit gives the relationship between differential of area dA and the differentials dx and dy of the Cartesian coordinates.

$$\triangle A = \triangle x \triangle y \longrightarrow dA = dx\, dy$$

Similarly for integration in 3D,

$$\triangle V = \triangle x \triangle y \triangle z \longrightarrow dV = dx\, dy\, dz$$

The way you should interpret these relationships is that if one makes a small (infinitesimal) change dx in the x coordinate, then the area or volume element will be proportional to that change, and similarly for y and z.

We are going to consider important non-Cartesian coordinate systems and address the question: what are dA and dV in terms of coordinate increments in these coordinate systems? Next week will will take a more general and systematic approach. However, the intuition in the current approach is correct and valuable.

6.2 Polar Coordinates

You know the relationship between polar coordinates (r, θ) and Cartesian coordinates (x, y), which we reproduce here for consistency with subsequent coordinate systems.

$$x = r\cos\theta \qquad\qquad r^2 = x^2 + y^2$$
$$y = r\sin\theta \qquad\qquad \tan\theta = \frac{y}{x}$$

It is useful to view the polar coordinate system in terms of a polar grid consisting of curves of constant r-coordinate – circles centred on the origin, and curves of constant θ-coordinate – radial lines.

Now consider the equivalent of a rectangular region R in polar coordinates.

$$R = \{(r, \theta) |\, a \leq r \leq b, \alpha \leq \theta \leq \beta\}$$

This is referred to as a **polar rectangle**. Of course this is not a rectangle in the plane, but typically a wedge (although these include disks and annuli). If we want to integrate a function

of polar coordinates $f(r,\theta)$ over this region, we partition R using an equally spaced grid in polar coordinates, with spacing $\triangle r$ and $\triangle\theta$

$$r_i = a + i\triangle r,\ i = 0,\ldots,N,\ \triangle r = \frac{b-a}{N},$$
$$\theta_j = \alpha + j\triangle\theta,\ j = 0,\ldots,M,\ \triangle\theta = \frac{\beta-\alpha}{M}.$$

This gives sub-regions R_{ij} in the form of small wedges

$$R_{ij} = \{(r,\theta)|\, r_i \le r \le r_i + \triangle r, \theta_j \le \theta \le \theta_j + \triangle\theta\}$$

The double integral of f over this region is obtained by sampling the function somewhere within each wedge R_{ij}, multiply by the area of R_{ij}, summing over all sub-regions, and taking the limit $M, N \to \infty$

$$\iint_R f\, dA = \lim_{M,N\to\infty} \sum_{j=0}^{M-1}\sum_{i=0}^{N-1} f(r_i^*,\theta_j^*)\triangle A_{ij}$$

where (r_i^*,θ_j^*) is some point in R_{ij} and $\triangle A_{ij}$ is the area of R_{ij}.

To derive the relationship between the double integral of f and its expression as an iterated integral over r and θ, it will be convenient to sample the function at the the mid-point of each R_{ij},

$$r_i^* = \frac{r_i + r_{i+1}}{2} \quad \theta_j^* = \frac{\theta_j + \theta_{j+1}}{2}$$

As before, in the limit as $M, N \to \infty$ it does not matter where we choose to sample f, but with this we can write the area of $\triangle A_{ij}$ compactly as

$$\triangle A_{ij} = \triangle r\left(\frac{r_i + r_{i+1}}{2}\triangle\theta\right) = \triangle r\,(r_i^*\triangle\theta) = r_i^*\triangle r\triangle\theta$$

Then the expression for double integral gives

$$\begin{aligned}\iint_R f\, dA &= \lim_{M\to\infty}\lim_{N\to\infty}\sum_{j=0}^{M-1}\sum_{i=0}^{N-1} f(r_i^*,\theta_j^*)\, r_i^*\triangle r\triangle\theta \\ &= \lim_{M\to\infty}\sum_{j=0}^{M-1}\lim_{N\to\infty}\sum_{i=0}^{N-1} f(r_i^*,\theta_j^*)\, r_i^*\triangle r\triangle\theta \\ &= \int_\alpha^\beta\int_a^b f(r,\theta)\, r\, dr\, d\theta\end{aligned}$$

Hence we have the desired relationship between the double integral and iterated integrals

In polar coordinates:

$$\iint_R f\, dA = \int_\alpha^\beta\int_a^b f(r,\theta)\, r\, dr\, d\theta$$

where

$$R = \{(r,\theta)|\, a \le r \le b, \alpha \le \theta \le \beta\}$$

The key feature that distinguishes integration in polar coordinates from integration in Cartesian coordinates is that now the area element is no longer independent of its coordinates. It depends on coordinate r because, for given increments $\triangle r$ and $\triangle\theta$, the area of the little wedges R_{ij} depend on their distance from the origin.

Area element for polar coordinates:

$$dA = r\,dr\,d\theta$$

This is the relationship between the infinitesimal area in polar coordinates and the infinitesimal changes in the coordinates dr and $d\theta$. You should understand this as follows. If one makes a small (infinitesimal) change dr in the r coordinate, then the area element will be proportional to that change, while if one makes an small (infinitesimal) change $d\theta$ in the θ coordinate, then the area element will be proportional $rd\theta$.

We will continue with integration in polar coordinates after we treat the other two special coordinate systems.

6.3 Iterated Integration in Cylindrical Coordinates

Cylindrical coordinates (r, θ, z) are a three dimensional coordinate systems composed of polar coordinates (r, θ) in the plane and a Cartesian coordinate z in the third direction, generally thought of as the vertical direction. The relationship between cylindrical and Cartesian coordinates is

$$\begin{aligned} x &= r\cos\theta & r &= \left(x^2+y^2\right)^{1/2} \\ y &= r\sin\theta & \tan\theta &= \frac{y}{x} \\ z &= z & z &= z \end{aligned}$$

Technically, one should give a different symbol ξ to the vertical coordinate in cylindrical coordinates and write relationship between this cylindrical coordinate and the Cartesian coordinate is $\xi = z$, but this is not really necessary.

It is useful to view the cylindrical coordinate system in terms of a grid consisting of surfaces of constant r-coordinate – cylinders centred on the z-axis, surfaces of constant θ-coordinate – radial half-planes, and surfaces of constant z-coordinate – horizontal planes.

We suppose have a function of cylindrical coordinates $f(r, \theta, z)$ defined on a cylindrical wedge given by

$$\Omega = \{(r, \theta, z) \,|\, a \le r \le b, \alpha \le \theta \le \beta, c \le z \le d\}.$$

and we want to compute the triple of f over Ω.

For integration in cylindrical coordinates we partition Ω with increments $\triangle r$, $\triangle\theta$, and $\triangle z$ equally spaced in each of the coordinate directions. The volume of an element of the partition is

$$\triangle V_{ijk} = (\triangle r)(r_i^*\triangle\theta)(\triangle z) = r_i^*\triangle r\triangle\theta\triangle z$$

where r_i^* is again midway between r_i and r_{i+1}. Note that $\triangle V = \triangle A\triangle z$ where $\triangle A$ is the corresponding area in the polar coordinates. The remainder of the derivation is analogous to previous section and so we skip the details and just state the result

In cylindrical coordinates:

$$\iiint_\Omega f\,dV = \int_c^d \int_\alpha^\beta \int_a^b f(r,\theta,z)\,r\,dr\,d\theta\,dz$$

where

$$\Omega = \{(r,\theta,z)\,|\,a \le r \le b, \alpha \le \theta \le \beta, c \le z \le d\}.$$

Key point is the expression of the infinitesimal volume element dV in terms of infinitesimal changes in the coordinates.

Volume element in cylindrical coordinates:

$$dV = rdrd\theta dz$$

6.4 Iterated Integration in Spherical Coordinates

Spherical coordinates (r,θ,ϕ) are a three dimensional coordinate systems where r is the distance from the origin (three-dimensional radial coordinate), θ is same angle as in cylindrical coordinates (corresponds to longitude where the x-axis is zero longitude), and ϕ is the angle from the vertical (angle from the north pole or co-latitude). The relationship between spherical and Cartesian coordinates is

$$\begin{aligned} x &= r\sin\phi\cos\theta & r &= \sqrt{x^2+y^2+z^2} \\ y &= r\sin\phi\sin\theta & \tan\phi &= \frac{\sqrt{x^2+y^2}}{z} \\ z &= r\cos\phi & \tan\theta &= \frac{y}{x} \end{aligned}$$

Some authors give a different symbol ρ to the radial coordinate in spherical coordinates, and other authors use θ and ϕ in exactly the reverse of the roles here. We will be consistent in our usage here however. The ranges of spherical coordinates are

$$r \ge 0 \qquad 0 \le \theta \le 2\pi \qquad 0 \le \phi \le \pi$$

It is useful to view spherical coordinate system in terms of a grid consisting of surfaces of constant r-coordinate – spheres centred on the origin, surfaces of constant θ-coordinate – radial planes, and surfaces of constant ϕ-coordinate – cones.

We suppose have a function of spherical coordinates $f(r,\theta,\phi)$ defined on a spherical wedge

$$\Omega = \{(r,\theta,\phi)\,|\,a \le r \le b, \alpha \le \theta \le \beta, \gamma \le \phi \le \delta\}.$$

and we want to compute the triple of f over Ω.

For integration in spherical coordinates we partition Ω with increments $\triangle r$, $\triangle\theta$, and $\triangle\phi$ equally spaced in each of the coordinate directions. The key is again the relationship between the volume of an element of the partition and the increments $\triangle r$, $\triangle\theta$, and $\triangle\phi$. Here the relationship is slightly more complicated. At a point (r_i,θ_j,ϕ_k) we have the approximation

$$\begin{aligned} \triangle V_{ijk} &\approx (\triangle r)(r_i\triangle\phi)(r_i\sin\phi_k\triangle\theta) \\ &= r_i^2\sin\phi_k\triangle r\triangle\theta\triangle\phi \end{aligned}$$

It can be shown (using the Mean Value Theorem) that there is a point $(r_i^*, \theta_j^*, \phi_k^*)$ within the element such that

$$\begin{aligned}\triangle V_{ijk} &= (\triangle r)(r_i^* \triangle \phi)(r_i^* \sin \phi_k^* \triangle \theta) \\ &= r_i^{*2} \sin \phi^* \triangle r \triangle \theta \triangle \phi\end{aligned}$$

The remainder of the derivation is analogous to previous sections and so we skip the details and just state the result

In spherical coordinates:

$$\iiint_\Omega f \, dV = \int_\gamma^\delta \int_\alpha^\beta \int_a^b f(r, \theta, \phi) \, r^2 \, \sin \phi \, dr \, d\theta \, d\phi$$

where

$$\Omega = \{(r, \theta, \phi) \,|\, a \leq r \leq b, \alpha \leq \theta \leq \beta, \gamma \leq \phi \leq \delta\}.$$

The key point is the expression of the infinitesimal volume element dV in terms of infinitesimal changes in the coordinates.

Volume element in spherical coordinates:

$$dV = r^2 \, \sin \phi \, dr \, d\theta \, d\phi$$

Discussion

- Sometimes we use special coordinates because of geometry, that is the region of integration suggest special coordinates, but equally often we use special coordinates because the integrand simplifies in these coordinates. Another way to say this is that in many cases the function we wish to integrate arises because a problem depends only on distance from the origin (polar coordinates in 2D and spherical coordinates in 3D) or because a problem depends only on distance from an axis (cylindrical coordinates).
- In problems with symmetry, the integral separates in the corresponding coordinate, or coordinates, and the integrals simplify.
- In our derivation of formulas for iterated integration we took the regions to be simple and hence the limits of integration were constants. However, just as with integration in Cartesian coordinates, the regions need not be this simple. Recall Type I and Type II regions for double integral for example. In general, in iterated integration the limits of any inner integrals may depend on coordinates of any of the outer integrals.

6.5 Applications

Some interesting application areas of integration in special coordinates are:

- Spherical geometries. Here a good example would be the earth's atmosphere.
- Interaction kernels or interaction potentials. Here good examples would be the interaction of atoms, the flocking and herding of animals or the spread of infectious diseases.
- Probability densities. An important example here is computing the Gaussian integral to obtain the normal distribution.

- Central forces. A good example here is electron orbitals about nuclei.

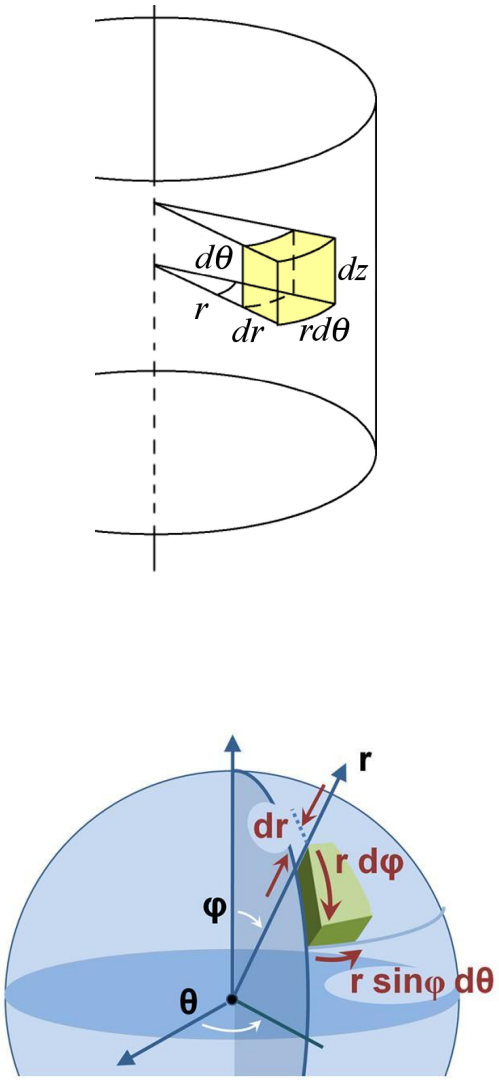


Figure 5: Pictures of the volume elements in cylindrical and spherical coordinate systems. (Cylindrical case taken from MIT Physics 8.01 course notes. Spherical case from Wikipedia.)

Week 7: Calculus for functions from $\mathbb{R}^n$ to $\mathbb{R}^m$

Introduction

At the beginning of this module we considered vector-valued functions of one variable:

$$\mathbf{r} : U \subseteq \mathbb{R} \to \mathbb{R}^n.$$

Next, we considered scalar-valued functions of several variables:

$$f : U \subseteq \mathbb{R}^n \to \mathbb{R}.$$

We will now broaden our study further to consider the more general situation where

$$\mathbf{F} : U \subseteq \mathbb{R}^n \to \mathbb{R}^m.$$

In particular, when $n \geq 2$ and $m \geq 2$:

A **vector-valued function of several variables**

$$\mathbf{F} : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$$

is a function that assigns a vector in $\mathbb{R}^m$ to each point in U, which is a subset of $\mathbb{R}^n$.

We shall focus primarily on the following important cases:

- If $n = m$ and the function $\mathbf{F}$ is a bijection, we can view $\mathbf{F}$ as a coordinate transformation (or change of coordinates) on $\mathbb{R}^n$. Such functions can be used to simplify integration in multiple variables.
- Another view of the case $n = m$ which is particularly relevant for applications is to view the function $\mathbf{F}$ as assigning an n-dimensional vector to points in $\mathbb{R}^n$. In this case, $\mathbf{F}$ is usually referred to as vector field on $\mathbb{R}^n$.
- In the case $n = 2$ and $m = 3$ the function provides a parametrisation of a two-dimensional surface in $\mathbb{R}^3$, in analogy with the cases $n = 1$ and $m = 2$ and $n = 1$ and $m = 2$ we studied in Part I, where such functions represented parametrisations of curves.

This week we will focus on the first case, and begin discussing the second.

7.1 Differentiating general vector-valued multivariable functions

The basic idea of differentiating a general vector-valued function echoes our definition for $\mathbf{r} : \mathbb{R} \to \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$. To compute the derivative, we must provide a linear approximation for the function $\mathbf{F}$ near to a point $\mathbf{x}$:

$$\mathbf{F}(\mathbf{x} + \mathbf{u}) \approx \mathbf{F}(\mathbf{x}) + \mathbf{J}\,\mathbf{u}$$

for $\mathbf{u} \in \mathbb{R}^n$ with a matrix $\mathbf{J} \in \mathbb{R}^{m \times n}$ so that $\mathbf{J}\,\mathbf{u} \in \mathbb{R}^m$:

$$\mathbf{J}\,\mathbf{u} = \begin{pmatrix} J_{11} & J_{12} & \dots & J_{1n} \\ J_{21} & J_{22} & \dots & J_{2n} \\ \vdots & & \ddots & \vdots \\ J_{m1} & J_{m2} & \dots & J_{mn} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}.$$

If $\mathbf{F}(\mathbf{x}) = (F_1(\mathbf{x}), \ldots, F_m(\mathbf{x}))$, then equating components, we find that

$$\begin{aligned} F_j(x_1 + u_1, \ldots, x_n + u_n) - F_j(x_1, \ldots, x_n) \\ \approx J_{j1}u_1 + \cdots + J_{jn}u_n \end{aligned}$$

for $j = 1, \ldots, m$. In particular, if all the components of $\mathbf{u}$ are zero except u_i, then this becomes

$$\begin{aligned} F_j(x_1, \ldots, x_{i-1}, x_i + u_u, x_{i+1}, \ldots, x_n) - F_j(x_1, \ldots, x_n) \\ \approx J_{ji}u_i. \end{aligned}$$

Recalling our computations in Part II, this just says that we should set

$$J_{ji} = \frac{\partial F_j}{\partial x_i}(x_1, \ldots, x_n).$$

The resulting matrix of partial derivatives is called the **Jacobian matrix**, which we denote $\mathbf{DF}(\mathbf{x})$:

$$\mathbf{DF}(\mathbf{x}) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1}(x_1, \ldots, x_n) & \cdots & \frac{\partial F_1}{\partial x_n}(x_1, \ldots, x_n) \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x_1}(x_1, \ldots, x_n) & \cdots & \frac{\partial F_m}{\partial x_n}(x_1, \ldots, x_n) \end{pmatrix}.$$

This is a notion of derivative for general functions $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^m$, and generalises all of the previous notions you have seen before, including the case of scalar- and vector-valued functions of a single variable (the cases $n = 1, m = 1$ and $n = 1, m \geq 2$) and the gradient (the case $n \geq 2, m = 1$)!

A useful thing to note is that we can view the Jacobian matrix in two ways: its rows are the gradients of the coordinate functions, so

$$\mathbf{DF}(\mathbf{x}) = \begin{pmatrix} \nabla F_1(\mathbf{x}) \\ \vdots \\ \nabla F_m(\mathbf{x}) \end{pmatrix}.$$

We can also view the columns as the partial derivatives of $\mathbf{F}$ with respect to each coordinatate, i.e.

$$\mathbf{DF}(\mathbf{x}) = \begin{pmatrix} \frac{\partial \mathbf{F}}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial \mathbf{F}}{\partial x_n}(\mathbf{x}) \end{pmatrix}.$$

Example: The function $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\mathbf{F}(x, y) = \left(x^2 y,\ y \tan x,\ 3y^{-1}\right)$$

has Jacobian matrix

$$\mathbf{DF}(x, y) = \begin{pmatrix} 2xy & x^2 \\ y \sec^2 x & \tan x \\ 0 & -3y^{-2} \end{pmatrix}.$$

7.2 Integrating general vector-valued multivariable functions

Once you know how to integrate scalar functions of multiple variables, it is straightforward to integrate vector-valued functions. In essence, all we do is integrate each of the coordinate

functions, treating each one as a scalar function. For example if we have $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ with $\mathbf{F}(\mathbf{x}) = (F_1(\mathbf{x}), F_2(\mathbf{x}))$, and R is the rectangle $[a, b] \times [c, d]$, then

$$\begin{aligned}\iint_R \mathbf{F}\, dV &= \left(\iint_R F_1\, dV, \iint_R F_2\, dV \right) \\ &= \left(\int_c^d \int_a^b F_1(x, y) dx dy, \int_c^d \int_a^b F_2(x, y) dx dy \right).\end{aligned}$$

Similar results hold for multivariable functions expressed in polar, cylindrical or spherical coordinate system. The general result is the following.

Integrating vector-valued functions

If $\mathbf{F} : \mathbb{R}^m \to \mathbb{R}^n$ with $\mathbf{F} = (F_1, \ldots, F_n)$ and $\Omega \subseteq \mathbb{R}^m$, then we have

$$\iint_\Omega \mathbf{F}\, dV = \left(\iint_\Omega F_1\, dV, \cdots, \iint_\Omega F_n\, dV \right).$$

Example: If we wish to integrate $\mathbf{F}(x, y) = x\mathbf{i} + y^2\mathbf{j}$ over the region $\Omega = [0, 1] \times [0, 2]$, we compute

$$\begin{aligned}\iint_\Omega \mathbf{F}\, dV &= \left(\int_0^2 \int_0^1 x\, dx dy, \int_0^2 \int_0^1 y^2\, dx dy \right) \\ &= \left(\int_0^2 \left[\tfrac{1}{2} x^2 \right]_{x=0}^{x=1} dy, \int_0^2 y^2 dy \right) \\ &= \left(\int_0^2 \tfrac{1}{2}\, dy, \tfrac{1}{3} y^3 \Big|_{y=0}^{y=2} \right) = \left(1, \tfrac{8}{3}\right)\end{aligned}$$

7.3 Coordinate Transformations

Another way in which functions $\mathbf{F} : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ arise is when considering coordinate transformations. To be a coordinate transformation, the function $\mathbf{F}$ should be a bijection, and the transformation from polar coordinates to Cartesian coordinates in two dimensions is an example. In this case, $\mathbf{F} : [0, +\infty) \times [0, 2\pi) \to \mathbb{R}^2$, and

$$\mathbf{F}(r, \theta) = \big(r \cos\theta, r \sin\theta\big).$$

The Jacobian matrix is then

$$\mathbf{DF}(r, \theta) = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

If we compute the determinant of this matrix, we find

$$\det \mathbf{DF} = r\cos^2\theta + r\sin^2\theta = r.$$

This is exactly as the factor in the area element we introduced for integral in polar coordinates we introduced last week!

The same holds true for cylindrical and spherical coordinates. In the cylindrical case we can consider $\mathbf{G} : [0, +\infty) \times [0, 2\pi) \times \mathbb{R} \to \mathbb{R}^3$ defined via

$$\mathbf{G}(r, \theta, z) = \big(r \cos\theta, r \sin\theta, z\big)$$

and for spherical coordinates, $\mathbf{H} : [0, +\infty) \times [0, 2\pi) \times [0, \pi) \to \mathbb{R}^3$ defined via

$$\mathbf{H}(r, \theta, \phi) = \big(r \cos\theta \sin\phi, r \sin\theta \cos\phi, r \cos\phi\big).$$

In these cases, you can verify that

$$\det \mathbf{DG}(r, \theta, z) = r$$
$$\text{and} \quad \det \mathbf{DH}(r, \theta, \phi) = -r^2 \sin \phi.$$

This is more than just a coincidence, it's actually the signature of a more general result:

Change of variables for integrals.

Suppose that $\mathbf{F} : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is a bijection. Then the integral of $f : \mathbb{R}^n \to \mathbb{R}$ over a subset $\Omega \subset \mathbb{R}^n$ with $\Omega = \mathbf{F}(K)$ for some $K \subset \mathbb{R}^n$ can be expressed as

$$\iiint_{\Omega} f(\mathbf{x})\, dx_1 \cdots dx_n = \iiint_{K} (f \circ \mathbf{F})(\mathbf{u}) \left| \det \mathbf{DF}(\mathbf{u}) \right| du_1 \cdots du_n,$$

where $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{u} = (u_1, \ldots, u_n)$.

To use this in practice, we need to take several steps:

1. Choose a change of variables $\mathbf{x} = \mathbf{F}(\mathbf{u})$.
2. Write $f(\mathbf{x})$ in terms of the new variables $\mathbf{u}$, i.e. find an expression for $f \circ \mathbf{F}(\mathbf{u})$.
3. Express the region of integration Ω in terms of the new variables, i.e. find $K = \mathbf{F}^{-1}(\Omega)$.
4. Compute (or recall) $\det \mathbf{DF}(\mathbf{u})$.
5. Through repeated coordinate integration, compute the resulting integral with respect to the variables u_1 to u_n.

You should compare this formula with the version for one-dimensional integrals: there, we have

$$\int_a^b f(x)\, dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f\big(g(u)\big) \left| \frac{dg}{du}(u) \right| du,$$

so $\frac{dg}{du}$ takes the place of $\det \mathbf{DF}$, and the region of integration transforms in the same way because $K = \mathbf{F}^{-1}(\Omega)$, just as $g^{-1}([a, b]) = [g{-}1(a), g^{-1}(b)]$. We will explore the reasoning behind this result in more detail after doing some examples.

An example: Polar coordinates

Consider the integral of $f(x, y) = \cos(x^2 + y^2)$ over the region

$$\Omega = \big\{(x, y) |\ -3 \le x \le 3, 0 \le y \le \sqrt{9 - x^2}\big\}.$$

This is a Type 1 region, so we can write this as

$$\iint_{\Omega} f\, dA = \int_{-3}^{3} \int_{0}^{\sqrt{9-x^2}} \cos(x^2 + y^2)\, dy\, dx.$$

Sketching the region, we see that Ω is actually half of a disk of radius 3. This should suggest that changing to polar coordinates may be a sensible move.

1. We choose polar coordinates, so $x = r \cos \theta$, $y = r \sin \theta$.

2. We substitute $x = r\cos\theta$, $y = r\sin\theta$ into the integrand $f(x, y)$.
$$f(x, y) = \cos(r^2\cos^2\theta + r^2\sin^2\theta) = \cos(r^2)$$

3. We change the limits of integration to those corresponding to Ω the polar coordinates
$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} \quad \to \quad \int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=3}$$

4. We recall that the correct area element for polar coordinates
$$dA = r\,dr\,d\theta$$

5. Finally, we compute
$$\begin{aligned}
&\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} \cos(x^2+y^2)\,dy\,dx \\
&= \int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=3} \cos(r^2) r\,dr\,d\theta \\
&= \tfrac{1}{2}\pi\cdot\sin(r^2)\Big|_{r=0}^{r=3} = \tfrac{1}{2}\pi\sin 9.
\end{aligned}$$

We have rewritten the integral in a more useful way by making a **change of variables**, or a **change of coordinates**.

A second example: an elliptical region

In this example, we consider integrating $f(x, y) = x^2y^2$ over the elliptical region
$$\Omega = \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; \frac{x^2}{2^2} + \frac{y^2}{3^2} \le 1\right\}.$$
We again follow the same steps:

1. A nice way to choose coordinates here is to 'stretch' our polar coordinates, defining
$$\mathbf{F}(r, \theta) = \big(2r\cos\theta, 3r\sin\theta\big).$$

2. This means that
$$\Omega = \left\{\mathbf{F}(r, \theta) \in \mathbb{R}^2 \;\middle|\; 0 \le r \le 1, 0 \le \theta \le 2\pi\right\}.$$

3. We write f in terms of the new coordinates:
$$\begin{aligned}
f(x, y) &= f(2r\cos\theta, 3r\sin\theta) \\
&= 36r^4\sin^2\theta\cos^2\theta \\
&= 9r^4\sin^2 2\theta \\
&= \tfrac{9}{2}r^4(1-\cos 4\theta).
\end{aligned}$$

4. In this case, the Jacobian determinant is
$$\det\mathbf{DF}(r, \theta) = \det\begin{pmatrix} 2\cos\theta & -2r\sin\theta \\ 3\sin\theta & 3r\cos\theta \end{pmatrix} = 6r.$$

5. Combining all of this, we have
$$\begin{aligned}
\iint_\Omega f\,dA &= \int_0^{2\pi}\int_0^1 \tfrac{9}{2}r^4(1-\cos 4\theta)\,6r\,dr d\theta \\
&= \frac{9}{2}\int_0^{2\pi}(1-\cos 4\theta)d\theta = 9\pi.
\end{aligned}$$

7.4 Explaining coordinate transformation for integrals

As we have discussed earlier this week, the Jacobian matrix is the derivative of a multivariable function as it provides a linear approximation to the function near to the point at which it is computed:

$$\mathbf{F}(\mathbf{x}+\mathbf{u}) \approx \mathbf{F}(\mathbf{x}) + \mathbf{DF}(\mathbf{x})\,\mathbf{u}.$$

We will now explore why the determinant of the Jacobian matrix appears in the change of variables formula for integrals.

For simplicity, we consider $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$. If $\mathbf{F}$ is a coordinate transformation, then it moves $\mathbf{x}$ to $\mathbf{F}(\mathbf{x})$, and areas near $\mathbf{x}$ will be distorted. To see how this distortion occurs, we can consider a square box with side lengths $\triangle x$ and $\triangle y$ and corners at

$$\mathbf{x},\ \mathbf{x}+\triangle x\mathbf{i},\ \mathbf{x}+\triangle y\mathbf{j}.$$

It has area $dA = \triangle x \triangle y$. After applying $\mathbf{F}$, these corners move to

$$\mathbf{F}(\mathbf{x}),\ \mathbf{F}(\mathbf{x}+\triangle x\,\mathbf{i}),\ \mathbf{F}(\mathbf{x}+\triangle y\,\mathbf{j}).$$

Using our linear approximation, these become approximately

$$\mathbf{F}(\mathbf{x}),\ \mathbf{F}(\mathbf{x}) + \triangle x\,\mathbf{DF}(\mathbf{x})\mathbf{i},\ \mathbf{F}(\mathbf{x}) + \triangle y\,\mathbf{DF}(\mathbf{x})\mathbf{j}.$$

While $\mathbf{i}$ and $\mathbf{j}$ were originally perpendicular, they may not be any more. To express the area of the parallelogram dA' after applying $\mathbf{F}$, we compute

$$\begin{aligned} dA' &\approx \big\|\mathbf{DF}(\mathbf{x})\mathbf{i} \times \mathbf{DF}(\mathbf{x})\mathbf{j}\big\| \triangle x \triangle y \\ &= \left\| \left(\frac{\partial F_1}{\partial x}, \frac{\partial F_2}{\partial x}\right) \times \left(\frac{\partial F_1}{\partial y}, \frac{\partial F_2}{\partial y}\right) \right\| dA \\ &= \left| \frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial y} - \frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial x} \right| dA \\ &= \big|\det \mathbf{DF}(x,y)\big|\, dA. \end{aligned}$$

We see that the ratio of the original area dA to the new area dA' is exactly given by the determinant of the Jacobian matrix, and so we have to take this transformation of the area into account when integrating. A similar calculation can be performed in the 3D case, but this time we need to consider the volume of the parallelipiped after transforming, which is given by the vector triple product

$$\Big|\mathbf{DF}(\mathbf{x})\mathbf{i} \cdot \Big(\mathbf{DF}(\mathbf{x})\mathbf{j} \times \mathbf{DF}(\mathbf{x})\mathbf{k}\Big)\Big| \triangle x \triangle y \triangle z.$$

It can be checked that this is indeed $\big|\det \mathbf{DF}(\mathbf{x})\big|\, dV$.

7.5 Vector fields

There are many situations in which a vector is associated to each point in some region of space. Familiar examples come from fluid motion such as the wind or the motion of water in a river. Wind has both a magnitude and a direction (blowing North-East at 18 miles/hour) and hence is a vector quantity whose value generally varies with location.

The assignment of a vector to each point in a region of space is described by a function of the type we defined above,

$$\mathbf{F}\colon U \subseteq \mathbb{R}^3 \to \mathbb{R}^3.$$

In this case (where $n = m$) we say that such a function is a **vector field**. We typically use boldface to denote vector fields, (usually $\mathbf{F}$, $\mathbf{V}$, or $\mathbf{v}$), to emphasise that to each point in U the function $\mathbf{F}$ assigns a vector.

While the majority of physical examples are vector fields on $\mathbb{R}^3$, the general case is for any dimension n

$$\mathbf{F}\colon U \subseteq \mathbb{R}^n \to \mathbb{R}^n$$

We will primarily be interested in vector fields on the plane ($n = 2$) or in space ($n = 3$).

Some examples

- The planar vector field

  $$\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$$

  corresponds to vectors pointing counterclockwise around the origin.

  The vector field,

  $$\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

  corresponds to vectors in space pointing away from the origin.

- An example we have already seen is the gradient of a function of several variables $f(\mathbf{x})$. The gradient is a vector field since $\nabla f(\mathbf{x})$ is a vector whose value depends on the point $\mathbf{x}$. On the plane for example

  $$\mathbf{F}(\mathbf{x}) = \nabla f(\mathbf{x}) = \frac{\partial f}{\partial x}(x, y)\mathbf{i} + \frac{\partial f}{\partial y}(x, y)\mathbf{j}$$

- The 'right-hand side' of a system of ordinary differential equations

  $$\begin{aligned} \dot{x}_1 &= f_1(x_1, \cdots, x_n) \\ &\vdots \\ \dot{x}_n &= f_n(x_1, \cdots, x_n) \end{aligned}$$

  can be viewed as a vector field on the corresponding phase space. To each point in phase space $\mathbf{x} = (x_1, \cdots, x_n)$ there is an associated vector

  $$\mathbf{F}(\mathbf{x}) = \Big(f_1(x_1, \cdots, x_n), \cdots, f_n(x_1, \cdots, x_n)\Big)$$

  so that the system of ODEs can be written concisely as

  $$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}).$$

- Vector fields are highly important for realistic applications. Fluid flows, gravitational fields, electric fields, and magnetic fields are all vector fields.

7.6 Divergence and curl

As we discussed above, the Jacobian matrix provides a notion of derivative for a general vector-valued function $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^m$, but there are other types of derivative which are important in the specific case of vector fields. These are mainly important because of their applications in mathematical modelling and geometry, and we will see more of them later on.

The **divergence** of a vector field $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is scalar-valued, and is defined to be

$$\operatorname{div} \mathbf{F}(\mathbf{x}) = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i}(x_1, \ldots, x_n).$$

This is defined for any dimension $n \geq 2$. In the cases where $n = 2$ or 3, we have

$$\operatorname{div} \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}$$
$$\operatorname{div} \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Note that this is just the trace of the Jacobian matrix (i.e. the sum of the diagonal entries).

The **curl** of a vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is vector-valued, and is defined to be

$$\begin{aligned}\operatorname{curl} \mathbf{F} &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{pmatrix} \\ &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right).\end{aligned}$$

Sometimes we also refer to the curl of a vector field $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$; in this case we define it to be

$$\begin{aligned}\operatorname{curl} \mathbf{F} &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & 0 \end{pmatrix} \\ &= \left(0, 0, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right).\end{aligned}$$

Some authors use a slightly different notation for the divergence and curl, writing

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} \quad \text{and} \quad \operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F}.$$

The view behind this notation is that ∇ is treated as a vector of operations of partial differentiation, i.e.

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right),$$

and then the divergence and curl then indeed correspond to the dot and cross product of ∇ with $\mathbf{F}$ in these cases.

Example: The vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$\mathbf{F}(x, y, z) = -\tfrac{1}{2} y \mathbf{i} + \tfrac{1}{2} x \mathbf{j} + z \mathbf{k}$$

has divergence and curl

$$\operatorname{div} \mathbf{F}(x, y, z) = 1$$
$$\operatorname{curl} \mathbf{F}(x, y, z) = (0, 0, 1).$$

Additional Material

The Jacobian matrix and the general chain rule

Now that we have considered general functions $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$, we can consider composition of these functions. In general, we may wish to compose two functions $\mathbf{h} : \mathbb{R}^k \to \mathbb{R}^m$ and $\mathbf{f} : \mathbb{R}^m \to \mathbb{R}^n$ to form a function $\mathbf{g} : \mathbb{R}^k \to \mathbb{R}^n$, $\mathbf{g} = \mathbf{f} \circ \mathbf{h}$ and then take its derivative. The question is, what is the derivative of $\mathbf{g}$ in terms of the derivatives of $\mathbf{f}$ and $\mathbf{h}$? The answer is given by the general Chain Rule:

The **General Chain Rule**.

$$\mathbf{Dg}(\mathbf{x}) = \mathbf{Df}(\mathbf{h}(\mathbf{x}))\,\mathbf{Dh}(\mathbf{x})$$

where $\mathbf{Dg}(\mathbf{x})$ is an $m \times k$ derivative matrix, $\mathbf{Df}(\mathbf{h}(\mathbf{x}))$ is an $m \times n$ derivative matrix, and $\mathbf{Dh}(\mathbf{x})$ is an $n \times k$ derivative matrix.

The Chain Rule is expressed very simply as the product of derivative matrices. It is frequently expressed in component form using summation

$$\frac{\partial g_i}{\partial x_j} = \sum_{k=1}^{p} \frac{\partial f_i}{\partial h_k}\frac{\partial h_k}{\partial x_j}$$

where $\frac{\partial f_i}{\partial h_k}$ is understood to mean the derivative of f_i with respect to its k^{th} argument evaluated at the appropriate point $\mathbf{h}(\mathbf{x})$.

In the case $k = n = 1$ the general Chain Rule reduces to the special case considered in Week 4. Using the notation from Week 4 where $g = f \circ \mathbf{r}$, this special case is the following matrix product

$$\underbrace{\begin{bmatrix}\frac{dg}{dt}\end{bmatrix}}_{1\times 1} = \underbrace{\begin{bmatrix}\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m}\end{bmatrix}}_{1\times m} \underbrace{\begin{bmatrix}\frac{dx_1}{dt}\\ \vdots \\ \frac{dx_m}{dt}\end{bmatrix}}_{m\times 1}$$

Previously n was used where m appears here, and we have suppressed notation for where the various derivatives are evaluated. This product could be written as we did before using summation notation, or using the dot product

$$\frac{dg}{dt} = \sum_{i=1}^{p} \frac{\partial f}{\partial x_i}\frac{dx_i}{dt}, \qquad \frac{dg}{dt} = \nabla f \cdot \frac{d\mathbf{r}}{dt}.$$

Week 8: Green's Theorem and Stokes' Theorem

Introduction

This week we return to parametrised curves and consider integration along such curves. In Week 2 we saw one example of this when we integrated along a curve to find its length. Here we generalise this idea with a particular emphasis on integrating over vector fields.

8.1 Line Integrals

The basic line integral can be motivated as follows. Given an interval $[a, b]$ and a function $f(x)$ which is positive over the interval, $\int_a^b f(x)dx$ is the area under the graph $y = f(x)$. Intuitively one understands that $f(x)\,dx$ is the area of a tall skinny rectangle of height $f(x)$ and width dx and $\int_a^b$ means "add these up" for x's in the interval $[a, b]$.

We need not restrict ourselves to just integrating along straight lines however. As we now know how to work with curves in $\mathbb{R}^n$, let us generalise and consider a curve in the plane and a function $f(x, y)$ that is positive in some region containing the curve. A surface is formed by f over the curve. Think of a curtain hanging down from f to the curve. We want to compute the area of this curtain by integration.

The formula for the integral is easy once one recalls the formulas from Week 2. Recall the length of a curve $\mathcal{C}$ is given by

$$\ell(\mathcal{C}) = \int_{\mathcal{C}} ds$$

where ds is the infinitesimal arc length, or distance, along the curve. To find the area of the curtain formed above $\mathcal{C}$ beneath the surface $z = f(x, y)$, we simply multiply the height f times the infinitesimal arc length ds and integrate over the curve, computing the integral

$$\int_{\mathcal{C}} f\, ds$$

I ntegrals of this type are evaluated by parametrising the curve. Given a parametrisation of the curve $\mathbf{r}(t)$, $t \in [a, b]$, recall that the infinitesimal arc length ds can be expressed in terms of the infinitesimal change dt via

$$ds = \|\mathbf{r}'(t)\|dt, \quad \text{so} \quad \int_{\mathcal{C}} f\, ds = \int_a^b f(\mathbf{r}(t)) \left\|\mathbf{r}'(t)\right\| \, dt.$$

The relationship $ds = \|\mathbf{r}'(t)\|dt$ holds in any dimension, so works for line integrals in 2D, 3D, or higher dimensions:

Given $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{r} : [a, b] \to \mathbb{R}^n$ a parametrisation of a curve $\mathcal{C}$ lying in U, the **line integral** of f along a curve $\mathcal{C}$ is given by

$$\int_{\mathcal{C}} f\, ds = \int_a^b f(\mathbf{r}(t)) \left\|\mathbf{r}'(t)\right\| \, dt$$

Similar comments to those made in Week 2 apply here: the curve may be piecewise regular and one must parametrise the curve in a sensible way.

Note that in our definition of the line integral f is a function defined on a region of $\mathbb{R}^n$. It also happens that one may have f defined only on the curve. For example, f might represent the linear density (mass per unit length) of a wire (the curve), and in this case f would only have meaning on the curve.

8.2 Integrating vector fields along curves

Given a vector field $\mathbf{F}$, it frequently occurs that one wants to compute a line integral where the function f is

$$f = \mathbf{F} \cdot \mathbf{T}$$

where $\mathbf{T}$ is the *unit tangent vector* to the curve $\mathcal{C}$. Examples of this type of integration are the work done on a particle and the circulation, and are discussed in more detail below.

In this case, we need to evaluate

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T}\, ds.$$

Recall that $\mathbf{r}'(t)$ is a tangent vector to the curve $\mathcal{C}$ at a point $\mathbf{r}(t)$, but it may not be a unit vector. To find $\mathbf{T}$, we therefore *normalise*, i.e. divide by it's length, so

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}.$$

As a result, we can write

$$\begin{aligned}\mathbf{F} \cdot \mathbf{T}\, ds &= \mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \|\mathbf{r}'(t)\|\, dt \\ &= \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt.\end{aligned}$$

The last expression is what we will use in practice to evaluate this type of line integral. However, to stress the independence of the line integral of the parametrisation corresponding to a chosen orientation, we often write $\mathbf{r}'\, dt$ as $d\mathbf{r}$.

Let $\mathbf{F}$ be a vector field defined in some region of $\mathbb{R}^n$, and let $\mathbf{r} : [a, b] \to \mathbb{R}^n$ be a parametrisation of a curve $\mathcal{C}$ in this region. Then we have the

Line integral of $\mathbf{F}$ along $\mathcal{C}$

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T}\, ds = \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

One important feature of line integrals of vector fields is that they are dependent on the orientation of the curve. The reason is that is that if one reverses the orientation of a curve, then the tangent vector changes sign. Denoting $-\mathcal{C}$ as the curve $\mathcal{C}$ with the opposite orientation, then

$$\int_{-\mathcal{C}} \mathbf{F} \cdot\, d\mathbf{r} = -\int_{\mathcal{C}} \mathbf{F} \cdot\, d\mathbf{r}.$$

8.3 Fundamental Theorem of Line Integrals

When we introduced vector fields it was noted that an important class of vector fields was that obtained as the gradient of a function of several variables: $\mathbf{F} = \nabla f$. Such vector fields are called **conservative** vector fields. They are important because they often arise in practice and because the following variant of the Fundamental Theorem of Calculus holds:

Fundamental Theorem of Line Integrals

Let $\mathcal{C}$ be a regular curve which is parametrised by $\mathbf{r} : [a, b] \to \mathbb{R}^n$. Let f be a differentiable function whose gradient vector is continuous on $\mathcal{C}$. Then

$$\int_{\mathcal{C}} \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

Proving the FTLI is not difficult as it primarily relies on the Chain Rule for parametrisations (see Week 4) and the FTC. The manipulations are

$$\begin{aligned}\int_{\mathcal{C}} \nabla f \cdot d\mathbf{r} &= \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt \\ &= \int_a^b \frac{d}{dt} f(\mathbf{r}(t)) dt = f(\mathbf{r}(b)) - f(\mathbf{r}(a))\end{aligned}$$

The FTLI tells use that if we know our vector field $\mathbf{F}$ is a conservative vector field, and hence given by gradient of some function f, then we can evaluate any line integral of $\mathbf{F}$ over $\mathcal{C}$ simply by evaluating f at the end points of $\mathcal{C}$. Call these points $\mathbf{r}_a$ and $\mathbf{r}_b$. The importance is not just that it simplifies our calculations, but the fact that since the integral depends only on the end points, it in fact must be the same for any curve that starts $\mathbf{r}_a$ and ends at $\mathbf{r}_b$. That is, if $\mathbf{F} = \nabla f$, then

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r},$$

for any two $\mathcal{C}_1$ and $\mathcal{C}_2$ that start at $\mathbf{r}_a$ and end at $\mathbf{r}_b$. The line integral is said to by **path independent**. Note in particular that if $\mathbf{F} = \nabla f$ then

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = 0$$

for any closed curve $\mathcal{C}$ because $\mathbf{r}_a = \mathbf{r}_b$ for a closed curve.

With some mild conditions, it can be shown that if all line integrals of a vector field $\mathbf{F}$ are path independent, or equivalently if the line integral around all closed curves is zero, then $\mathbf{F}$ is a conservative vector field and there is a function f such that a $\mathbf{F} = \nabla f$.

The converse is generally easier, although perhaps less important. If the line integral of $\mathbf{F}$ around a close path is not zero, then $\mathbf{F}$ is definitely not a conservative vector field and it cannot be expressed as the gradient of a function.

An example

We will consider the line integral of $\mathbf{F}(x, y) = (x, y)$ along a section of the Logarithmic spiral, parametrised by

$$\mathbf{r}(t) = (e^t \cos t, e^t \sin t) \quad t \in [0, 2\pi].$$

We can see directly that $\mathbf{F}(x, y) = \nabla V(x, y)$, where $V(x, y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$, so by the FTLI,

$$\begin{aligned}\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} &= V\big(\mathbf{r}(2\pi)\big) - V\big(\mathbf{r}(0)\big) \\ &= \tfrac{1}{2}(e^{2\pi})^2 - \tfrac{1}{2} \cdot 1^2 = \tfrac{1}{2}(e^{4\pi} - 1).\end{aligned}$$

Now the longer way. In this case, we have

$$\begin{aligned}\mathbf{r}'(t) &= \big(e^t(\cos t - \sin t), e^t(\sin t + \cos t)\big) \\ \mathbf{F}\big(\mathbf{r}(t)\big) &= (e^t \cos t, e^t \sin t),\end{aligned}$$

so that, doing the algebra, we have

$$\mathbf{F}\big(\mathbf{r}(t)\big) \cdot \mathbf{r}'(t) = e^{2t}.$$

Finally, we integrate over $[0, 2\pi]$, and find

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} e^{2t}\, dt = \tfrac{1}{2}(e^{4\pi} - 1),$$

in agreement with the result above.

8.4 Work and Potential Energy

Work is an important physical concept that you can learn all about in a mechanics module. It is a classic example of a case where one needs to do line integrals of a vector field.

If a force $\mathbf{F}(\mathbf{r})$ acts on a point particle and the particle moves from position $\mathbf{r}_a$ to $\mathbf{r}_b$ along a curve $\mathcal{C}$, then the work W_{ab} done by the force on the particle is

$$W_{ab} = \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

This definition is independent of whether or not the force $\mathbf{F}(\mathbf{r})$ is a conservative vector field.

In many situations, but not all, the force $\mathbf{F}(\mathbf{r})$ is a conservative vector field. It can be thus be written as a gradient of a function. One typically defines the function so that $\mathbf{F} = -\nabla V$ where V is a **potential**, and more specifically in this case, V is **potential energy**.

The work done in moving from $\mathbf{r}_a$ to $\mathbf{r}_b$ is given in terms of the potential at the end points

$$W_{ab} = (-V(\mathbf{r}_b)) - (-V(\mathbf{r}_a)) = V(\mathbf{r}_a) - V(\mathbf{r}_b)$$

independently of how the particle moved from $\mathbf{r}_a$ to $\mathbf{r}_b$.

You should visualise the potential V as the height of a hill, or more general landscape. Assume that $V(\mathbf{r}_a) > V(\mathbf{r}_b)$. This means the particle starts out a some high point and moves to some lower point. The work done on the particle is $W_{ab} = V(\mathbf{r}_a) - V(\mathbf{r}_b) > 0$, independently of the path followed from $\mathbf{r}_a$ to $\mathbf{r}_b$. W_{ab} is the energy that can be extracted from the particle as it moves downhill from $\mathbf{r}_a$ to $\mathbf{r}_b$.

Contrarily, if a particle starts at $\mathbf{r}_b$ then one must expend energy to push it uphill to $\mathbf{r}_a$. We must input energy equal to W_{ab}, All work (or energy) differences are encoded in the potential V and are independent of the path taken by the particle. Informally, the force conserves mechanical energy by converting work to potential energy, and back. Gravitational and Coulomb forces are two examples of fundamental forces that are conservative, so are described in terms of potentials.

8.5 Circulation

For many vector fields, line integrals around closed curves have physical significance. In fluid dynamics, for example, such integrals give what is known as the **circulation** of the fluid around the curve. In electricity and magnetism, such integrals appear in the integral statement of Maxwell's equations and correspond to a circulation of electric or magnetic fields.

We will focus on the fluids case. Let $\mathbf{v}$ be a vector field corresponding to the velocity of fluid in some region of space (or could be confined to a plane). Then the circulation Γ of $\mathbf{v}$ over a closed curve $\mathcal{C}$ is

$$\Gamma = \int_{\mathcal{C}} \mathbf{v} \cdot d\mathbf{r}$$

Intuitively this integral corresponds to the total amount the fluid is circulating around the curve. Sometimes, integrals corresponding to circulations are expressed using the notation

$$\oint_{\mathcal{C}} \mathbf{v} \cdot d\mathbf{r},$$

where the circle is used to indicate the fact that the curve is closed.

Knowing the circulation around a body such as a wing or a spinning ball, one can calculate the **lift force** on the body. In the case of a wing, the lift force is what holds the aeroplane up. In the case of a spinning ball, the lift force gives rise to a deflection, or bending, of its path through the air.

8.6 Green's Theorem

We now introduce one of the major *integral theorems* we will see in this module, known as Green's Theorem. Integral theorems in multivariable calculus usually relate integrals over objects of different dimensions. In the case of Green's theorem, an area integral is related to a line integral:

Green's Theorem

Suppose that $\mathbf{u} : \mathbb{R}^2 \to \mathbb{R}^2$ is a smooth vector field with components $\mathbf{u} = (P, Q)$. Consider an area $S \subset \mathbb{R}^2$, and suppose that $\mathcal{C} \subset \mathbb{R}^2$ is a closed curve tracing the boundary of S in an anti-clockwise direction (often called a 'right-handed orientation'). Then

$$\iint_S \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \int_{\mathcal{C}} \mathbf{u} \cdot d\mathbf{r}.$$

Traditionally, the equality is often denoted

$$\iint_S \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \int_{\mathcal{C}} P\, dx + Q\, dy,$$

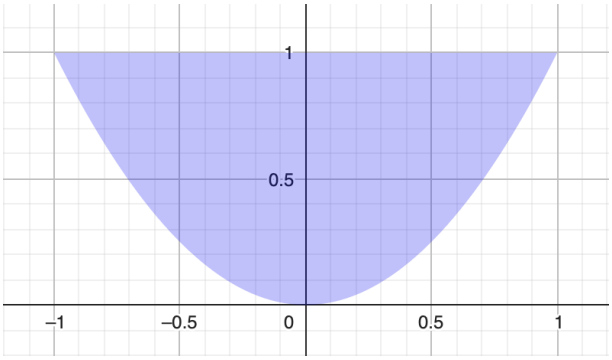
and this is the version you may find in textbooks.

An example

Let's verify that Green's theorem holds in a simple case. Consider the region

$$S = \{(x, y) \in \mathbb{R}^2 : x^2 < y < 1\}$$

which is illustrated below, and the vector field $\mathbf{u}(x, y) = (2y, x)$.



This is a region of Type I, so it makes sense to use Cartesian coordinates. In this case we have

$$P = 2y,\ Q = x \quad \text{so} \quad \frac{\partial Q}{\partial x} = 1 \quad \text{and} \quad \frac{\partial P}{\partial y} = 2.$$

The left-hand side in the statement of Green's Theorem becomes

$$\iint_S \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA = \int_{-1}^{1} \int_{x^2}^{1} -1\, dy\, dx = \int_{-1}^{1} x^2 - 1\, dx = \left[\tfrac{1}{3}x^3 - x\right]_{-1}^{1} = -\tfrac{4}{3}.$$

For the boundary, we must parametrise in two parts. For the top part of the boundary, we parametrise from right to left:

$$\mathbf{r}_1(t) = (-t, 1) \quad \text{so} \quad \mathbf{r}_1'(t) = (-1, 0) \quad \text{for} \quad t \in [-1, 1],$$

and for the the bottom part, from left to right:

$$\mathbf{r}_2(t) = (t, t^2) \quad \text{so} \quad \mathbf{r}_2'(t) = (1, 2t) \quad \text{for} \quad t \in [-1, 1].$$

These choices mean that the boundary curve $\mathcal{C}$ goes around S anti-clockwise. On the right-hand side of the equality in the theorem, we have:

$$\begin{aligned}\int_{\mathcal{C}} \mathbf{u} \cdot d\mathbf{r} &= \int_{-1}^{1} \mathbf{u}(\mathbf{r}_1(t)) \cdot \mathbf{r}_1'(t)\, dt + \int_{-1}^{1} \mathbf{u}(\mathbf{r}_2(t)) \cdot \mathbf{r}_2'(t)\, dt \\ &= \int_{-1}^{1} (2, -t) \cdot (-1, 0)\, dt + \int_{-1}^{1} (2t^2, t) \cdot (1, 2t)\, dt \\ &= \int_{-1}^{1} 4t^2 - 2\, dt = \left[\tfrac{4}{3}t^3 - 2t\right]_{-1}^{1} = -\tfrac{4}{3}.\end{aligned}$$

Unsurprisingly, the results match!

8.7 Stokes' Theorem

Looking back at Green's theorem, lets define a 3D extension of the vector field $\mathbf{u}$. If we define a vector field $\mathbf{v} : \mathbb{R}^3 \to \mathbb{R}^3$ to be $\mathbf{v}(x, y, z) = (P, Q, 0)$, then

$$\nabla \times \mathbf{v} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k},$$

and treating S as an area and $\mathcal{C}$ as a curve in the (x, y)–plane in 3D space, we can actually write

$$\iint_S (\nabla \times \mathbf{v}) \cdot \mathbf{k}\, dA = \int_{\mathcal{C}} \mathbf{v} \cdot d\mathbf{r}$$

This result holds in more generality in 3D, so we don't need to consider only 2D vector fields and flat surfaces. The more general result is known as Stokes' theorem, and it relates the circulation of a vector field $\mathbf{v} : \mathbb{R}^3 \to \mathbb{R}^3$ around $\mathcal{C}$ to the flux integral of $\operatorname{curl} \mathbf{v}$ over S:

Stokes' Theorem

Suppose that $\mathbf{v} : \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field. Consider an oriented surface $S \subset \mathbb{R}^3$ with unit normal $\mathbf{n}$, and suppose that $\mathcal{C} \subset \mathbb{R}^3$ is a closed curve tracing the boundary of S with a right-handed orientation relative to $\mathbf{n}$. Then

$$\iint_S (\nabla \times \mathbf{v}) \cdot \mathbf{n}\, dS = \int_{\mathcal{C}} \mathbf{v} \cdot d\mathbf{r}.$$

- We have yet to define surface integrals, but we explore how to compute them next. However, with the help of Stokes' theorem, you can already directly compute some surface integrals without actually knowing how to compute them in general!
- Notice that the integrals on both sides of this equation yield scalar results, as $(\nabla \times \mathbf{v}) \cdot \mathbf{n}$ is a scalar, and $\mathbf{v} \cdot \mathbf{r}'$ is also scalar.
- The remark in the statement about right-handed orientation means that we should orient the curve so that it travels around the surface boundary anti-clockwise relative to the direction of $\mathbf{n}$, just as the fingers of a right hand do relative to the thumb.

- A vector field is said to be **curl-free** if $\nabla \times \mathbf{v} = 0$. This does not mean that $\mathbf{v}$ is constant. However, it does mean that the integral around any closed curve is zero. If $\mathbf{r} : [a, b] \to \mathbb{R}^3$ with $\mathbf{r}(a) = \mathbf{r}(b) = \mathbf{r}_0$, the FTLI tells us that this is also true whenever $\mathbf{v} = \nabla V$, since

$$\int_{\mathcal{C}} \mathbf{v} \cdot d\mathbf{r} = \int_{\mathcal{C}} \nabla V \cdot d\mathbf{r} = V(\mathbf{r}_0) - V(\mathbf{r}_0) = 0.$$

  It turns out that the converse is also true: if $\nabla \times \mathbf{v} = \mathbf{0}$, then we can define $V : \mathbb{R}^3 \to \mathbb{R}$ such that $\mathbf{v} = \nabla V$. To do this in practice, we define curves $\mathcal{C}(\mathbf{x})$ connecting $\mathbf{0}$ to $\mathbf{x}$, fix a value for $V(\mathbf{0})$ of our choice, and then defining

$$V(\mathbf{x}) := \int_{\mathcal{C}} \mathbf{v} \cdot d\mathbf{r} + V(\mathbf{0}).$$

  With a bit more work, thanks to Stokes' Theorem, we can then show that the value of $V(\mathbf{x})$ doesn't depend on the particular path we chose, and indeed that $\mathbf{v} = \nabla V$.

Week 9: Surface Integrals and the Divergence Theorem

Introduction

We have seen surfaces a few times already, as graphs of functions of two variables and as level sets of functions of three variables. We now approach surfaces through parametrisations - extending the ideas from parametrised curves. You will see many similarities both to parametrised curves from Part I and to coordinate transformation from last week.

9.1 Parametric Surfaces

Recall that a curve in $\mathbb{R}^3$ is parameterised by a continuous map from an interval in $\mathbb{R}$ into $\mathbb{R}^3$.

$$\mathbf{r}\colon [a,b] \to \mathbb{R}^3$$

By extension, a surface S in $\mathbb{R}^3$ is parameterised by a continuous map from a region in Ω in $\mathbb{R}^2$ into $\mathbb{R}^3$, that is

$$\mathbf{r}\colon \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}^3$$

We think of the region Ω getting mapped to the surface in $\mathbb{R}^3$.

In terms of components, we write

$$\begin{gathered}\mathbf{r}(u,v) = \big(x(u,v), y(u,v), z(u,v)\big) \\ \text{with} \quad (u,v) \in \Omega\end{gathered}$$

We say that

$$S = \big\{\mathbf{r}(u,v) \in \mathbb{R}^3 \,\big|\, (u,v) \in \Omega\big\}$$

is a **parametric surface** parametrised (u,v) in the region (or domain) Ω.

Comments

- One must put conditions on $\mathbf{r}$, or its component functions $x(u,v)$, $y(u,v)$, and $z(u,v)$, to guarantee niceness of the surface. We will not focus on this, but implicitly assume that the component functions are differentiable, except possibly at the boundaries of Ω. Self intersections are ruled out by requiring $\mathbf{r}$ to be injective, except at the boundaries of Ω.
- One often draws and analyses surfaces in terms of individual parameterised curves corresponding to the parameters u and v separately. For example, one fixes v to some value v_0 and varies u. $\mathbf{r}(\,\cdot\,, v_0)$ is a parameterised curve in S. We call this a **u-curve**. There is a family of these since the value v_0 can be varied. Similarly there is a family of **v-curves**. These curves are sometimes collectively called **grid curves**.
- It is not sufficient to specify the mapping alone, one must specify the parameter domain Ω.

  Similarly to what we encountered with multiple integration, in the simplest case Ω will be rectangle $[a,b] \times [c,d] \in \mathbb{R}^2$. More complex domains, such as those of the form

  $$\Omega = \big\{(u,v) \,\big|\, a \le u \le b,\ g_1(u) \le v \le g_2(u)\big\}$$

  can easily be dealt with.

- Just as with curves in multiple segments, some surfaces of interest (see examples) will be most naturally specified as several distinct pieces, each with their own domain and parametrisation.
- The parametric representation of a surface S is not unique. Just as with curves, we can choose many different ways to represent a surface. We will see some examples of this below.
- Finally, be aware that many of the derivations and formulas that follow are similar to those encountered last week with coordinate transformations. There are important differences, however. Coordinate transformations are mappings between spaces of the same dimensions, $\mathbb{R}^2 \to \mathbb{R}^2$ or $\mathbb{R}^3 \to \mathbb{R}^3$. Parametrised surfaces are mappings between spaces of different dimension, $\mathbb{R}^2 \to \mathbb{R}^3$.

Examples

As with parametrised curves, one best learns how to parametrise surfaces by working through examples.

Unit sphere. We obtain the parametrisation for the unit sphere by putting $r = 1$ in the transformation into spherical coordinates:

$$\mathbf{r}(u, v) = (\cos u \sin v, \sin u \sin v, \cos v)\,.$$

Here $\Omega = [0, 2\pi] \times [0, \pi]$. We could of course have labelled the parameters as (θ, ϕ) rather than (u, v).

$$\mathbf{r}(\theta, \phi) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)\,.$$

This is more natural and generally preferred when a parametrisation is closely connected with particular coordinate system. If we restrict ϕ to take values in $[0, \pi/2]$, we obtain a parametrisation of the upper unit hemisphere.

Another parametrisation of this hemisphere, not based on spherical coordinates, is given by:

$$\mathbf{r}(u, v) = \big(u, v, \sqrt{1 - u^2 - v^2}\big),\ u^2 + v^2 \leqslant 1\,.$$

Cylinder of radius R. This is obtained by using cylindrical coordinates. We put $r = R$ and let z vary over any range, $[0, h]$ for example, obtaining

$$\mathbf{r}(\theta, t) = (R\cos\theta, R\sin\theta, t) \quad (\theta, t) \in [0, 2\pi] \times [0, h]$$

Cylinder of radius R with end caps. Frequently one wants to consider surfaces that are closed (fully enclose some volume). For example, a cylinder with end caps. In this case we would specify the surface in 3 pieces, S_1, S_2 S_3, where S_1 would be the above cylinder and S_2 and S_3 would be disks:

$$\mathbf{r}_2(r, \theta) = (r\cos\theta, r\sin\theta, 0) \quad (r, \theta) \in [0, R] \times [0, 2\pi]$$

$$\mathbf{r}_3(r, \theta) = (r\cos\theta, r\sin\theta, h) \quad (r, \theta) \in [0, R] \times [0, 2\pi]$$

Surfaces of revolution. Consider a function $f : [a, b] \to \mathbb{R}^+$. As you know, the graph of f is a curve

$$\mathcal{C} = \{(x, y) \in \mathbb{R}^2 \,|\, x \in [a, b], y = f(x) > 0\}.$$

We can generate a surface by rotating this curve around an axis in 3D, which is called a **surface of revolution.** This surface is given parametrically by

$$\mathbf{r}(x, \theta) = \big(x\,,\ f(x)\cos\theta\,,\ f(x)\sin\theta\big),$$
$$\text{with} \quad (x, \theta) \in [a, b] \times [0, 2\pi].$$

Torus. We obtain our parametrisation by taking a circle $C = \{(0, a + b\cos u, b\sin u) \mid 0 \leqslant u \leqslant 2\pi\}$ of radius b and centred at $(0, a, 0)$ in the (y, z) plane, $0 < b < a$, and rotating it about the z-axis:

$$\mathbf{r}(u, v) = \big((a + b\cos u)\cos v, (a + b\cos u)\sin v, b\sin u\big)$$

where $v \in [0, 2\pi]$ is the angle in the x-y plane measured from the x-axis.

Graphs. As we saw in Part II, the graph of a function $f\colon \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ is a surface. This fits into our definition by parametrising in the following way:

$$\mathbf{r}(u, v) = \big(u, v, f(u, v)\big).$$

The hemisphere $f(u, v) = \sqrt{1 - u^2 - v^2}$ above was such an example. You should compare this approach with our discussion of graphs of functions of one variable and curves.

9.2 Tangent Plane and Normal to a Surface

We have already touched on this when discussing partial derivatives. In the case of a parametrised surface we have grid curves from which we can easily obtain the tangent plane and normal vector at a given point on a surface.

Tangent plane

Consider a point $\mathbf{u}_0 = (u_0, v_0)$ and a particular u-curve $\mathbf{r}(\cdot, v_0)$ passing through $\mathbf{r}(\mathbf{u}_0)$ on a surface S. We know from our study of parametrised curves that differentiating this with respect to u will give a vector tangent to the curve. Here the derivative will be a partial derivative. Thus $\dfrac{\partial \mathbf{r}}{\partial u}(\mathbf{u}_0)$ is tangent to the curve generated by varying u with v fixed. Since the curve is in S, this vector will be tangent to S at point $\mathbf{r}(u_0, v_0)$. Likewise, $\dfrac{\partial \mathbf{r}}{\partial v}(\mathbf{u}_0)$ will be another vector tangent to S at $\mathbf{r}(\mathbf{u}_0)$.

These two vectors form a basis for the **tangent plane** to S at $\mathbf{r}(\mathbf{u}_0)$. That is, any point in the plane tangent to S at $\mathbf{r}(\mathbf{u}_0)$ can be represented by a linear combination of these two vectors, plus the point $\mathbf{r}(\mathbf{u}_0)$ itself. We can express the position vector of every point of the tangent plane at $\mathbf{r}(\mathbf{u}_0)$ as

$$\mathbf{p}(h, k) = \mathbf{r}(\mathbf{u}_0) + h\frac{\partial \mathbf{r}}{\partial u}(\mathbf{u}_0) + k\frac{\partial \mathbf{r}}{\partial v}(\mathbf{u}_0)\,,$$

where $(h, k) \in \mathbb{R}^2$. Note, the plane is itself a parametrised surface: $\mathbf{p}$ is a mapping $\mathbb{R}^2 \to \mathbb{R}^3$ here written with parameters (h, k).

Using a matrix-vector product, we can write the parametrisation of the plane directly in terms of the Jacobian matrix of $\mathbf{r}$:

$$\begin{aligned}\mathbf{p}(h, k) &= \mathbf{r}(\mathbf{u}_0) + \begin{pmatrix} \dfrac{\partial \mathbf{r}}{\partial u}(\mathbf{u}_0) & \dfrac{\partial \mathbf{r}}{\partial v}(\mathbf{u}_0) \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} \\ &= \mathbf{r}(\mathbf{u}_0) + \mathbf{D}\mathbf{r}(\mathbf{u}_0) \begin{pmatrix} h \\ k \end{pmatrix}.\end{aligned}$$

This demonstrates a geometric fact about the Jacobian matrix of a function $\mathbf{r} : \mathbb{R}^2 \to \mathbb{R}^3$. Just as $\mathbf{r}'(t)$ provided a description of the direction for the tangent line to a curve at the point $\mathbf{r}(t)$, so the columns of $\mathbf{D}\mathbf{r}(\mathbf{u}_0)$ provide vectors which describe the tangent plane to the parametrised surface at a point $\mathbf{r}(\mathbf{u}_0)$.

Normal vector

It is often convenient to represent a plane in terms of its **normal vector**. This is a vector which is perpendicular to the plane, and play an important role in integration, as we will see shortly.

From our previous calculations for the tangent plane and the properties of the cross product, we know that the vector

$$\mathbf{n} = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$$

is perpendicular to both $\dfrac{\partial \mathbf{r}}{\partial u}$ and $\dfrac{\partial \mathbf{r}}{\partial v}$. This vector is thus **normal to** S at $\mathbf{r}(u, v)$. It is perpendicular to any vector in the tangent plane. Of course there are infinitely many normals because we can multiply the above vector by any constant and obtain another normal. Normalising the normal in the above calculation gives a unit normal vector. In fact there are two possible unit normal vectors to a surface at a point

The two **unit normal vectors** to a surface at a point are

$$\mathbf{n} = \pm \frac{\dfrac{\partial \mathbf{r}}{\partial u} \times \dfrac{\partial \mathbf{r}}{\partial v}}{\left\| \dfrac{\partial \mathbf{r}}{\partial u} \times \dfrac{\partial \mathbf{r}}{\partial v} \right\|}$$

In practice, either it will not be important which of these two vectors to use, or else it will be clear which one to choose. For a closed surface it is convention to choose the outward-pointing normal.

From the normal vector at a point $\mathbf{r}_0$ on the surface, one can write an expression for the tangent plane as

$$\mathcal{P} = \{\mathbf{r} \in \mathbb{R}^3 \,|\, \mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0\}$$

9.3 Surface Area

Given a parameterised surface S, we would like to find the area of the surface. That is, we want to compute something like

$$\iint_S dS$$

where dS is an infinitesimal element of surface area.

The approach follows closely our previous treatment double integrals in generalised coordinates. Let $\mathbf{r}(u, v), (u, v) \in \Omega$ be a parametrisation of S. To integrate over the surface, we perform a double integral over Ω.

$$\iint_S \to \iint_\Omega$$

What we need to work out what is infinitesimal element of surface area dS, in terms of an infinitesimal area $du\, dv$ in Ω.

We can proceed as we did for double integrals in generalised coordinates. Consider a small rectangle in Ω of width Δu and height Δv. This gets mapped by $\mathbf{r}$ to a "curved parallelogram", which is well approximated by the parallelogram based at $\mathbf{r}(u_0, v_0)$ and with sides

$$\mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0) \approx \frac{\partial \mathbf{r}}{\partial u}(u_0, v_0)\Delta u$$
$$\mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0) \approx \frac{\partial \mathbf{r}}{\partial v}(u_0, v_0)\Delta v.$$

The area ΔS of the surface is given approximately by the area of the small parallelogram. Then, using the fact that the magnitude of the cross product of two vectors is the area of the parallelogram formed by them, we have

$$\Delta S \approx \left\| \frac{\partial \mathbf{r}}{\partial u}\Delta u \times \frac{\partial \mathbf{r}}{\partial v}\Delta v \right\| = \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| \Delta u \Delta v$$

or since $\Delta u\, \Delta v$ is the area ΔA in Ω,

$$\Delta S = \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| \Delta A$$

Thus, we have derived the

Surface area element:

$$dS = \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| dA$$

We have dropped the explicit dependence on the point in question, (u_0, v_0), but you should understand that in general the partial derivatives on the right-hand-side are functions of (u, v), and hence so is dS.

Do not confuse this with similar expressions from generalised coordinates. Here $\dfrac{\partial \mathbf{r}}{\partial u}$ and $\dfrac{\partial \mathbf{r}}{\partial v}$ is each a three-component vector. In our treatment of two-dimensional integration in generalised coordinates, we had two two-component vectors.

Using this expression for dS we have

$$\begin{aligned} \text{Area of S} = A(S) &= \iint_S dS \\ &= \iint_\Omega \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| dA \end{aligned}$$

where in practice the double integral over Ω is carried out by usual iterated integration over u and v is some suitable order.

One can show that the area of S is independent of the parametrisation of S. The proof is simply an exercise in the use of the change of variables formula.

Example: The surface area of a circular cone of height h and base of radius R. We can parametrise a sloped surface of this cone S by considering

$$\mathbf{r}(\theta, z) = \left(\frac{R}{h}(h-z)\cos\theta, \frac{R}{h}(h-z)\sin\theta, z \right),$$
$$(\theta, z) \in [0, 2\pi] \times [0, h].$$

In this case, we have

$$\frac{\partial \mathbf{r}}{\partial \theta} = \left(-\frac{R}{h}(h-z)\sin\theta, \frac{R}{h}(h-z)\cos\theta, 0 \right)$$
$$\frac{\partial \mathbf{r}}{\partial z} = \left(-\frac{R}{h}\cos\theta, -\frac{R}{h}\sin\theta, 1 \right),$$

and the cross product is

$$\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial z} = \left(\frac{R}{h}(h-z)\cos\theta, \frac{R}{h}(h-z)\sin\theta, \frac{R^2}{h^2}(h-z) \right),$$

with norm

$$\left\|\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial z}\right\| = \frac{R}{h}(h-z)\sqrt{1+\frac{R^2}{h^2}}.$$

Integrating, we have

$$\begin{aligned}\iint_S dS &= \int_0^h \int_0^{2\pi} \frac{R}{h}(h-z)\sqrt{1+\frac{R^2}{h^2}}\, d\theta dz \\ &= \pi R\sqrt{h^2+R^2}.\end{aligned}$$

Finally, adding the area of the base, πR^2, we have that the surface area is

$$A = \pi\Big(R^2 + R\sqrt{h^2+R^2}\Big).$$

9.4 Surface Integrals

Given a continuous scalar function $f\colon \mathbb{R}^3 \to \mathbb{R}$ and a surface S in $\mathbb{R}^3$, we can evaluate f at a each point on S. Thus we have a real value associated to each point on S and we can integrate this over the surface. Rather than again going through all the arguments of summing up pieces and taking the limit as the number of pieces goes to infinity, we simply state the unsurprising result

Integrating scalar functions over a surface:

Given a scalar function $f : \mathbb{R}^3 \to \mathbb{R}$, and a parametrisation $\mathbf{r}\colon \Omega \to \mathbb{R}^3$ of a surface S, the surface integral of f over S is given by

$$\iint_S f\, dS = \iint_\Omega f\big(\mathbf{r}(u,v)\big) \left\|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right\| dA$$

Remarks:

- When f takes the value 1 everywhere, (or equivalently is just left out), we have $\iint_S f\, dS = \iint_S dS$ and so reduces to the previous formula for the area of S.
- We have given the definition assuming that f was defined in a region of $\mathbb{R}^3$ containing the surface. In practice, all that is required is that f is defined on the points of the surface alone.

9.5 Flux Integrals

In applications, we often care about the flow of a vector-valued quantity across a surface. To make mathematical sense of this, we define what are known as **flux integrals**.

Given a vector field $\mathbf{v}\colon \mathbb{R}^3 \to \mathbb{R}^3$ and a surface S in $\mathbb{R}^3$, we define the

Flux integral:

$$\text{Flux of } \mathbf{v} \text{ through S} = \iint_S (\mathbf{v}\cdot\mathbf{n})\, dS$$

where $\mathbf{n}$ is a unit normal vector to S whose direction will be set by the context.

Flux integrals are surface integrals in which the scalar function we integrate is $f = \mathbf{v}\cdot\mathbf{n}$. Notice that this f is an example of a function that is only defined on the surface; while $\mathbf{v}$ may be defined away from the surface, $\mathbf{n}$ has no meaning except at points on the surface.

As always, we will evaluate flux integrals by resorting to some parametrisation of S.

$$\iint_S (\mathbf{v}\cdot\mathbf{n})\,dS = \iint_\Omega (\mathbf{v}\cdot\mathbf{n}) \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| dA$$

Recalling our expression of the unit normal vector $\mathbf{n}$ this can be written

$$\begin{aligned}\iint_\Omega \left(\mathbf{v} \cdot \pm \frac{\dfrac{\partial \mathbf{r}}{\partial u} \times \dfrac{\partial \mathbf{r}}{\partial v}}{\left\| \dfrac{\partial \mathbf{r}}{\partial u} \times \dfrac{\partial \mathbf{r}}{\partial v} \right\|} \right) \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| dA \\ = \pm \iint_\Omega \mathbf{v} \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) dA\end{aligned}$$

Hence to find the flux of $\mathbf{v}$ through a surface S we evaluate:

The flux of $\mathbf{v}$ through a surface S is

$$\iint_S (\mathbf{v}\cdot\mathbf{n})\,dS = \pm \iint_\Omega \mathbf{v} \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) dA$$

The direction of $\mathbf{n}$, and hence the choice of $\pm$ in the expression above will be specified for any given problem. Usually for closed surfaces, we demand that the flux is measured in the direction of the outward-pointing normal.

9.6 The Divergence Theorem

The divergence theorem (sometimes called Green's theorem or Gauss's theorem) is one of the most important and useful theorems for modelling physical systems with PDEs. It relates the flux of a vector field $\mathbf{v}$ out of a closed surface to the integral of the divergence of $\mathbf{v}$ over the volume enclosed. We will not prove this theorem here (see MA259 Multivariable Calculus for further discussion), but you should know it.

The Divergence Theorem

Suppose that $\mathbf{v} : \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field. Consider a set $\Omega \subset \mathbb{R}^n$, and let $S \subset \mathbb{R}^3$ be the surface enclosing Ω. Suppose that $\mathbf{n}$ is the outward-point unit normal to the surface S; then

$$\iiint_\Omega \operatorname{div} \mathbf{v}\, dV = \iint_S \mathbf{v}\cdot\mathbf{n}\, dS.$$

- Notice that both side of this equation yield scalar results, as $\operatorname{div}\mathbf{v}$ is a scalar, and the flux is also a scalar.
- This result can be viewed as being analogous to the Fundamental Theorem of Calculus. Recall that this states that if $f : [a, b] \to \mathbb{R}$ is differentiable, then

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

Looking back to the definition of the divergence, we see that $f'(x)$ is exactly the divergence for $n = 1$, so the left-hand sides are analogous. Likewise, the right-hand side depends at

the ends of the interval $[a, b]$, just as the right-hand side in the expression of the divergence theorem depends only on the values of the function $\mathbf{v}$ on the boundary of Ω. Finally, the signs on the right-hand side act just like the outward pointing normal: at b, the normal would point to larger real numbers, and at a it would point to smaller real numbers.

- The relationship with the Fundamental Theorem of Calculus actually points to something even bigger. While we stated the Divergence Theorem for 3D vector fields, it holds in any dimension. To give a more general statement we would need to make sense of the notion of surface integral in more generality.
- A vector field is said to be **divergence-free** if $\operatorname{div} \mathbf{v} = 0$. This does not mean that $\mathbf{v}$ is constant (consider for example $\mathbf{v}(x, y, z) = y^2\mathbf{i}$), but as a consequence of the Divergence Theorem, it does mean that the total flux of $\mathbf{v}$ over any closed surface is zero. When modelling, we use this fact to derive a certain class of PDE (partial differential equation) called a **conservation law**. For more details, see MA250 Introduction to PDEs.

Orientation

In our definition of the flux integral we implicitly assumed that we had a well-defined normal vector at each point of S. This sometimes fails. One way it can fail is that S has corners. For example, along the edges of a cube. The tangent plane and normal vector are not defined at corners. In most practical cases it is clear how to divide the surface into finite number of pieces where each piece has a well defined normal and the normals in the different pieces have a sensible relationship with one another, e.g. the normals on all faces of a cube can be taken in the outward direction.

A more serious problem arise for certain surfaces that cannot be oriented. You are probably familiar with a Möbius strip. Such a surface has only one side. This means that if one picks one of the two possible unit normal vectors at a point and then tries to vary that vector continuously over the surface, the choice will fail even though the surface is perfectly smooth. We can only consider orientable surfaces in the computation of flux integrals. Precise definitions of these concepts appear in later modules.