

Sum-product phenomena in algebraic groups

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(Based on joint work with Harry Schmidt and Akshat Mudgal.)

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- (4) We will be interested in situations that combine the sumset operations in different groups.

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Theorem (Erdős–Szemerédi, 1983)

There exists some $\delta > 0$ such that for all finite, non-empty sets A of integers we have

$$|A + A| + |A \cdot A| \gg |A|^{1+\delta}.$$

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- (1) Many results replacing $\mathbb{Z}$ with $\mathbb{R}, \mathbb{C}, \mathbb{F}_p$ etc., or concerning the quality of δ .
- (2) Conjectured that we can take δ arbitrarily close to 1, but there have been recent counterexamples.

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Theorem (Breuillard–Bays, 2018)

*Let G and H be **non-isogenous**, connected algebraic groups over $\mathbb{C}$ of dimension 1, and let $f : G \rightarrow \mathbb{C}$ and $g : H \rightarrow \mathbb{C}$ be rational functions.*

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$$|f^{-1}(A) + f^{-1}(A)| + |g^{-1}(A) + g^{-1}(A)| \gg_{G,H,f,g} |A|^{1+\delta}.$$

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Theorem (H.–Mudgal–Schmidt, 2026)

*Suppose that C is **not** a translate of an algebraic subgroup of $G \times H$. There exists $D(d)$, depending only on d , such that for all $\delta > 0$ and finite, non-empty $A \subseteq G$ one has either*

$$|A + A| > |A|^{1+\delta} \text{ or } |C[A] + C[A]| > |A|^{2-D\delta}.$$

Ideas of the proof

- (1) Weak PFR over $\mathbb{Z}$ (Gowers, Green, Manners and Tao, 2023): if $|A + A| \leq K|A|$ then a subset of size $\gg K^{-O(1)}|A|$ is contained a finitely generated subgroup Γ of G of rank $r \ll \log K$.

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- (3) $r(x) = |\{(y, z) \in C[A] : y + z = x\}|$, construct $V \subseteq G^2 \times H$ a subvariety such that $r(x) = |V_x \cap A^2| \leq |V_x \cap \Gamma^2|$ where V_x is the fibre over the projection to H .

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- (4) Call x bad if V_x contains a translate of an algebraic subgroup of G^2 of positive dimension and good otherwise.
- (5) Uniform Mordell–Lang (Evertse–Schlickewei–Schmidt (2002), David–Philippon (2007)): there exists C_2 , depending on d , such that if x is good then $|V_x \cap \Gamma^2| \leq C_2^{1+r}$.

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- (2) Hence have

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- (3) Hence $|C[A] + C[A]| \gg_d C_1^{-1-r} |A|^2$, as required.

Thank you for listening!