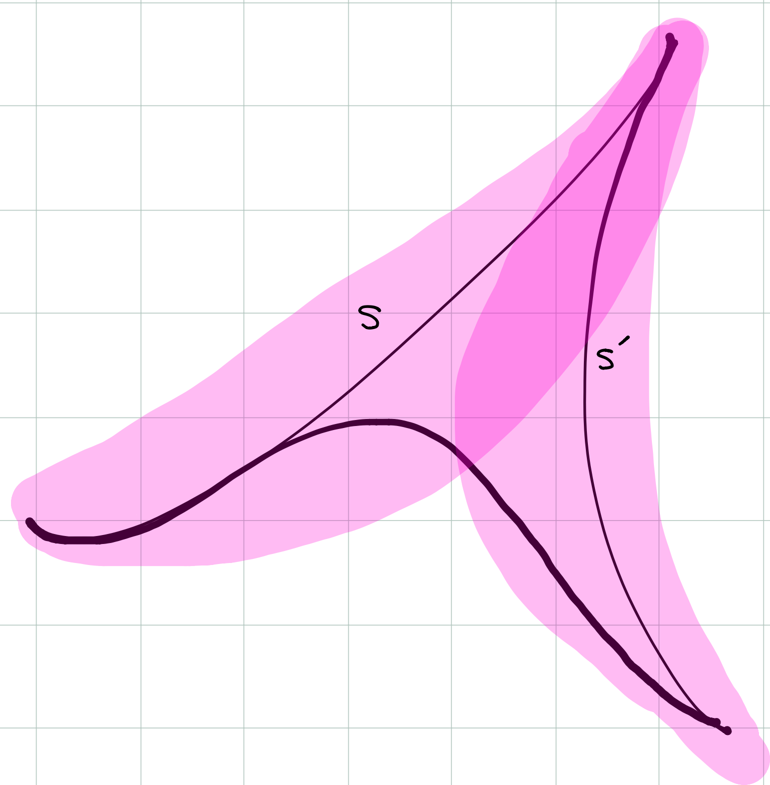
Thurs Jan 23

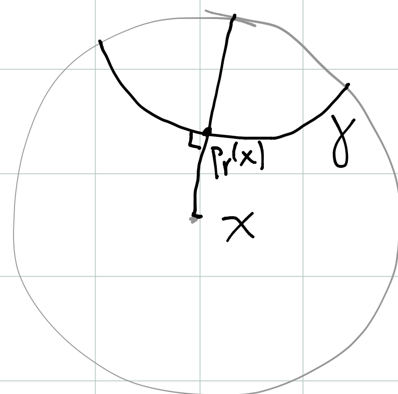
Last thing we did Tuesday was to show geodesic triangles in $\mathbb{H}^2$ are $\log 3$-thin

Another way to say C-thin: Let s and s' be any two sides of Δ. Then $\Delta \subset N_C(s \cup s')$:

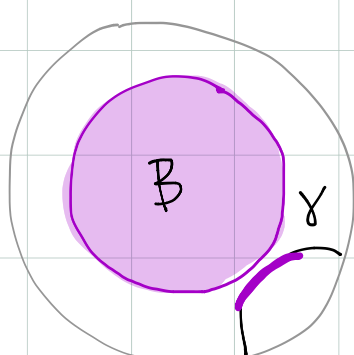


Recall we defined <u>Projection onto a geodesic</u>:

$x \in \mathbb{H}^2$ and $\gamma \subset \mathbb{H}^2$ a geodesic $\rightsquigarrow P_\gamma(x)$

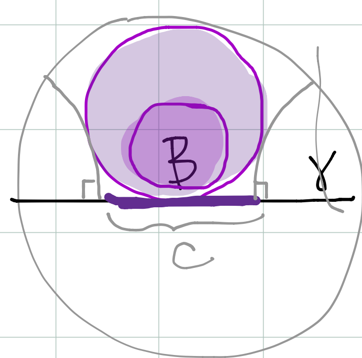


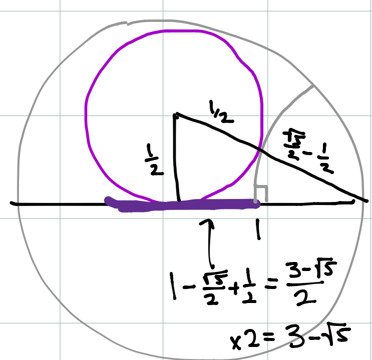
<u>Claim</u> $B \subset \mathbb{H}^2$ a ball <u>disjoint</u> from γ. Then there is a constant C s.t. $P_\gamma(B)$ has diameter $< C$.



<u>Terminology</u>: γ is <u>contracting</u>.

Proof: Move γ to the x-axis by an isometry of $\mathbb{H}^2$. Then move the center of B to the y-axis by an isometry preserving γ. B is contained in the ball tangent to S^1, so its projection is bounded by the projection of that arc:





$(\approx .76)$

$r = 3-\sqrt{5} \Rightarrow$

$$C = \ln\left(\frac{1+r}{1-r}\right) = \ln\left(\frac{4-\sqrt{5}}{\sqrt{5}-2}\right)$$

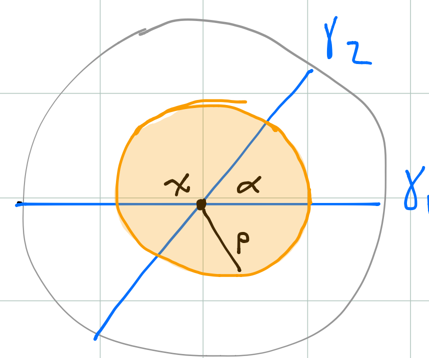
$$\approx 2.01$$

(that's <u>hyperbolic</u> distance)

One more observation about the geometry of $\mathbb{H}^2$:

geodesics starting at x diverge very fast:

Put x at origin with an isometry.



Measure divergence by shortest path from γ_1 to γ_2 outside the ball B_ρ

We've calculated circumference $= 2\pi \sinh(\rho)$

so length of arc subtended by α is $\alpha \sinh \rho$

$$= \alpha\left(\frac{e^{\rho}-e^{-\rho}}{2}\right)$$

: Grows exponentially fast as a function of ρ.

Summary: In $\mathbb{H}^2$

- triangles are thin
- geodesics are contracting
- geodesics diverge exponentially fast.

$\boxed{\text{Negative curvature in } \underline{\text{groups}}?}$

How do you think of a group as a metric space?

We will confine ourselves to finitely generated groups

ie there is a finite set $S \subseteq G$ such that the homomorphism $F(S) \longrightarrow G$ is surjective.

Def Let S be a finite generating set for G.

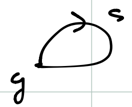
The Cayley graph $\mathcal{G}(G,S)$ is defined by:

Vertices = elements of G

edges : There is an edge from g to gs for each $g \in G$, $s \in S \cup S^{-1}$

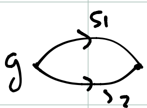
involution : $\overline{(g, gs)} = (gs, g) = ((gs), (gs)s^{-1})$

Notation: $\iota(g, gs) = g \quad \tau(g, gs) = gs$

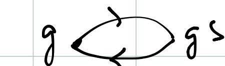
Note wma there are no loops 

$\Rightarrow gs = g \Rightarrow s = id$

similarly, wma there are no multiple edges



$gs_1 = gs_2$
$\Rightarrow s_1 = s_2$

Usually, the two oriented edges 

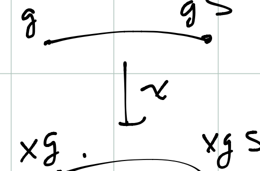
are considered a single <u>un</u>oriented edge

$\mathcal{C}(G,S)$ is made into a metric space by:

- each edge is isometric to $[0,1]$
- The distance between two points is the length of the shortest path joining them.

G acts on $\mathcal{C}(G,S)$ on the left:

$x \in G$ Then x acts by <u>left</u> multiplication:



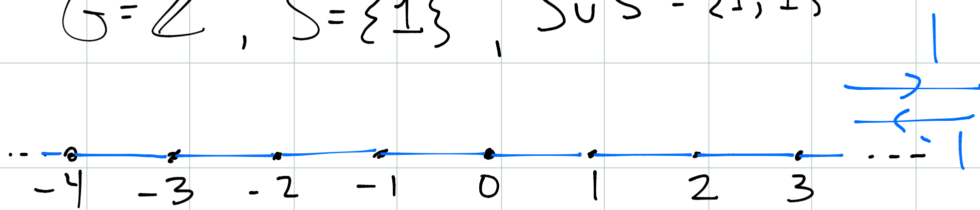
$(xy)g = x(yg)$, so this <u>is</u> an action,

ie a homomorphism $G \longrightarrow \mathrm{Isom}(\mathcal{C}(G,S))$

Remark: This is a <u>free</u> action: $\text{stab}(x) = 1 \ \forall x$, ie $gx = x \Rightarrow g = 1$

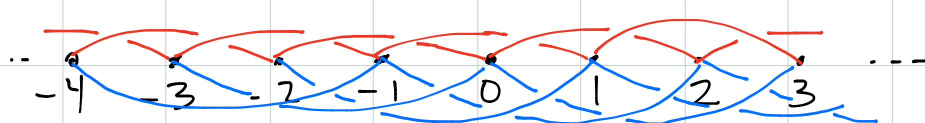
Examples

$G = \mathbb{Z}$, $S = \{1\}$, $S \cup S^{-1} = \{1, -1\}$



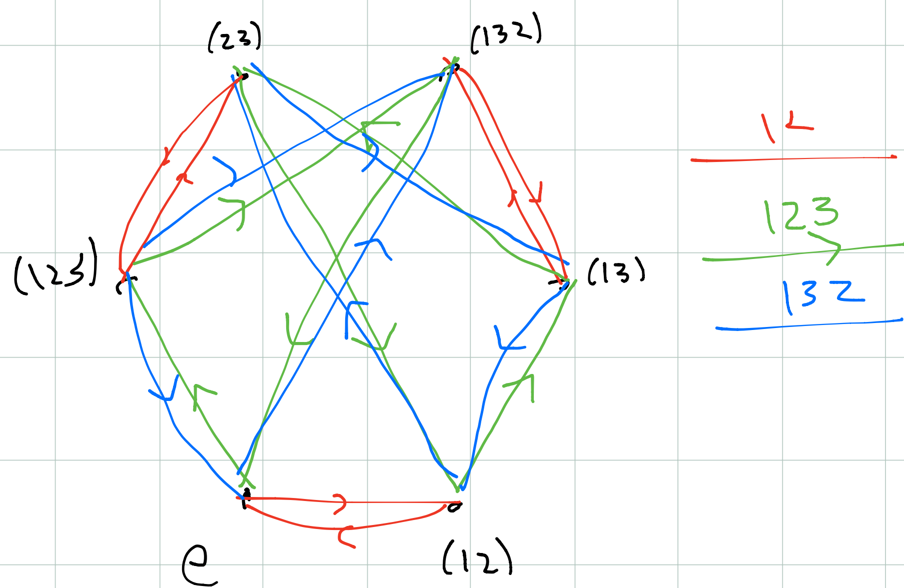
Action of G: n translates line to the right by n

$G = \mathbb{Z}$, $S = \{2, 3\}$



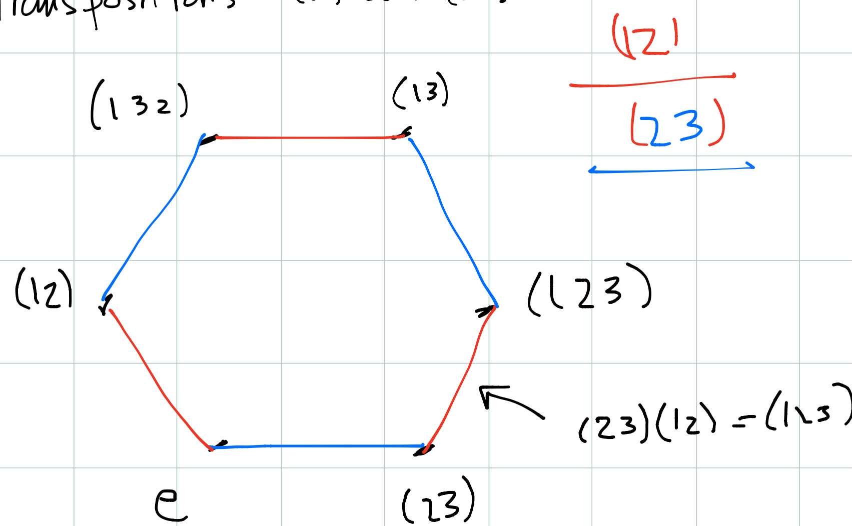
Example: S_3 = symmetric group on 3 letters

generators $S = (12), (123)$. $S \cup S^{-1} = \{(12), (123), (132)\}$

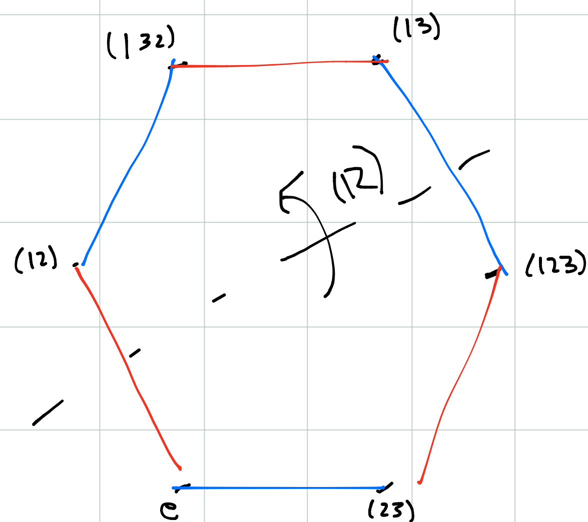


<u>Example</u>: S_3 = symmetric group on 3 letters

Generated by transpositions (12) and (23) ($s = s^{-1}$)!



(12) acts by



(23) acts by vertical flip.

(123) acts by rotation.