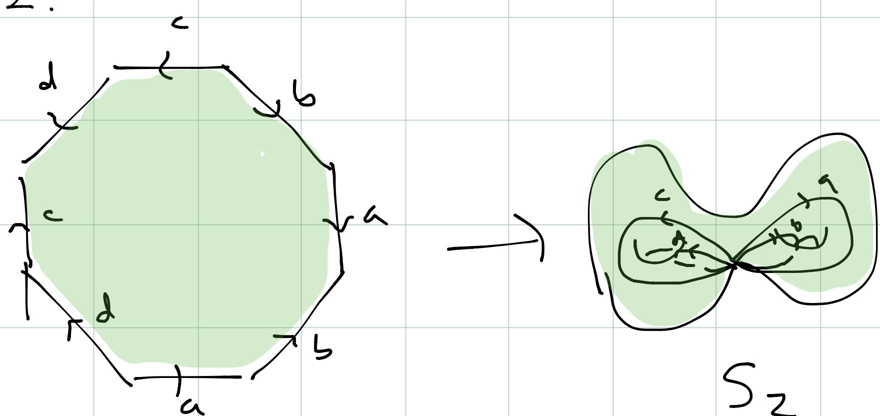
Thurs, Jan. 30

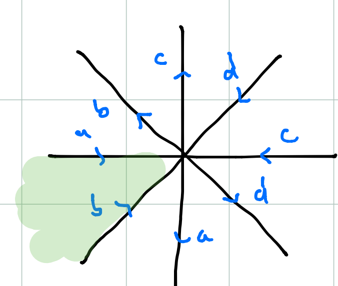
Last time: Defined hyperbolic metric space, hyperbolic group.

Started **Example:** surface group $\pi_1(S_g)$, $g \geq 2$.

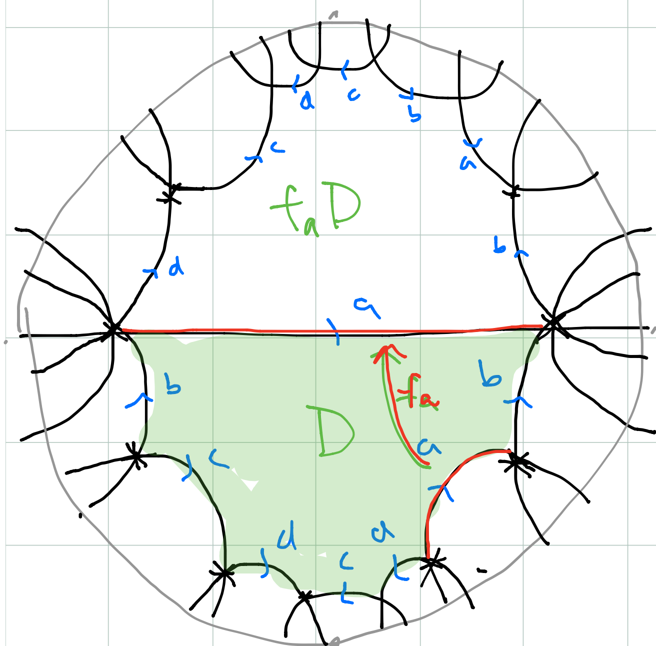


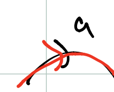
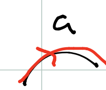
S_2

We showed you can tile $\mathbb{H}^2$ with regular hexagons, w/ internal angle $\frac{\pi}{4}$, so 8 fit around each vertex:

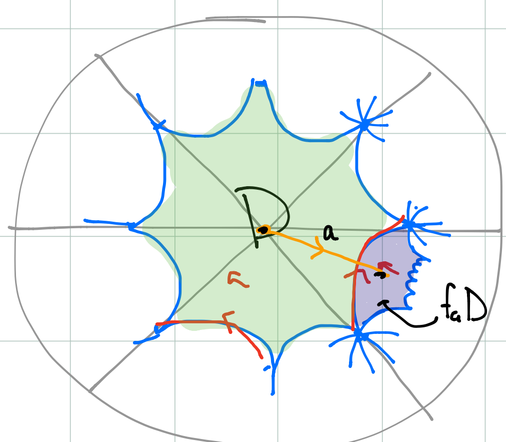


So the <u>combinatorial</u> $\tilde{S}_2$ is recreated in $\mathbb{H}^2$:



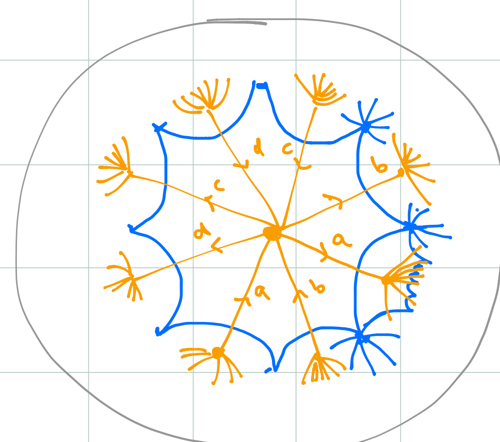
There is an orientation-preserving isometry f_a taking  to 

It moves D to an adjacent tile



So we can identify $\mathbb{H}^2$ with $\widetilde{S}_2$ and f_a with a deck transformation corresponding to $a \in \pi_1 S_g$

Now let Γ be the dual graph to the tiling:



$\pi_1 S_g$ acts freely on $\tilde{S}_g = \mathbb{H}_2$.

So the vertices of $\Gamma \longleftrightarrow$ elements of $\pi_1 S_g$

At each vertex There is one edge for each generator of $\pi_1 S_g$ (including inverses)

In other words, Γ is the Cayley graph $\mathcal{G}(\pi_1 S_g, \{a_1, b_1, \ldots, a_n, b_n\})$

It is embedded isometrically in $\mathbb{H}^2$.

<u>Heuristics</u>: distance in Γ is $\approx$ distance in $\mathbb{H}^2$

$\therefore$ trianges in Γ are δ-thin for some $\delta \approx \ln 3$.

so surface groups are hyperbolic – a good intuitive example to keep in mind throughout our discussion of hyperbolic groups.

Let X and X' be metric spaces. Recall that a map $f: X \to X'$ is an isometric embedding if it preserves distances, i.e.

$$d'(f(x), f(y)) = d(x,y) \text{ for all } x, y \in X,$$

and an isometry if in addition f is surjective.

We want an "approximate" version of isometry called quasi-isometry:

$f: X \to X'$ is a quasi-isometric map if there are constants $C \geq 0$ and $\lambda \geq 1$ s.t.

$$\frac{1}{\lambda} d(x,y) - C \leq d(f(x), f(y)) \leq \lambda\, d(x,y) + C$$

f is a quasi-isometry if in addition it is quasi-surjective: $\exists$ constant K s.t.

$$\forall\, y \in X',\ \exists\, x \in X \text{ s.t. } d(f(x), y) \leq K$$

example: The inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$ is a quasi-isometry, with $\lambda=1$, $c=0$ and $K=\frac{1}{2}$

Note the definition says nothing about continuity.

example: The map
$$\begin{aligned} \mathbb{R} &\longrightarrow \mathbb{Z} \\ x &\longmapsto [x] \end{aligned}$$

is a quasi-isometry with $K=0$, $\lambda=1$ and $C=1$

$$d(x,y)-1 \leq d([x],[y]) \leq d(x,y)+1$$

Def X, X' geodesic metric spaces are quasi-isometric if there is a quasi-isometry $f: X \to X'$.

Def: $f: X \to X'$ and $g: X' \to X$ are quasi-inverses if $\exists\, C, \lambda, K$ s.t.

$$d(f(x), f(y)) \leq \lambda\, d(x,y)$$

$$d(g(x'), g(y')) \leq \lambda\, d(x', y')$$

$$d(g(f(x)), x) \leq K$$

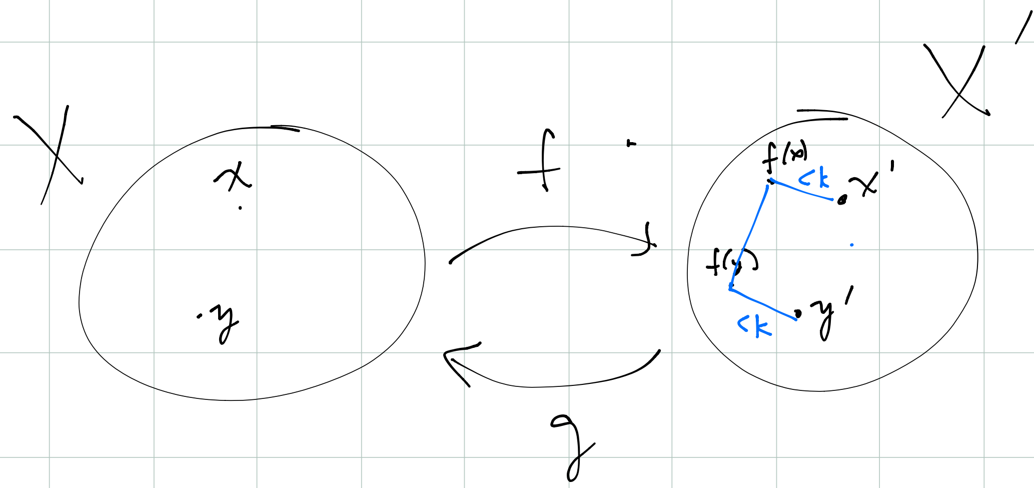
$$d(f(g(x))) \leq K$$

Lemma f is a quasi-isometry $\iff$ $\exists\, g$ s.t. f and g are quasi-inverses

pf

eg, Suppose $f: X \to X'$ is a quasi-isometry

Define a quasi-inverse g by: $g(x') =$ any $x \in X$ with $d(f(x), x') \leq K$



$$\frac{1}{\lambda} d(x,y) - c \le d(f(x), f(y)) \le \lambda d(x,y) + c$$

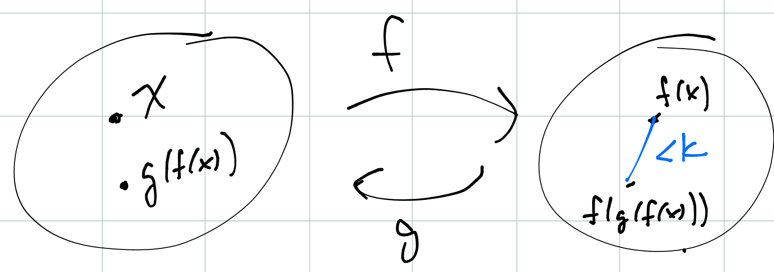
$$\Rightarrow \quad \frac{1}{\lambda} d(x,y) - c - 2k \le d(x', y')$$

$$\Rightarrow \quad d(g(x'), g(y')) = d(x,y) \le \lambda\, d(x', y') + (c - 2k)\cdot \lambda$$

(sim, $\quad d(x', y') \le \lambda d(x,y) + c + 2k$

$$\Rightarrow \quad \frac{1}{\lambda} d(x', y') - \left(\frac{c+2k}{\lambda}\right) \le d(x,y) = d(g(x'), g(y'))$$

$\Rightarrow$ g is a quasi-isometric map) and



$$d(x, g(f(x))) < \lambda_g \cdot K + C$$

sim $$d(x', f(g(x'))) < \lambda_f \cdot K + C$$

Converse is similar (exercize).

Example: S, S' finite generating sets for G

Then $\mathcal{G}(G,S) \sim \mathcal{G}(G,S')$

pf: Write each element of S' as a word in S,

$s = w'(s) = s_1' \cdots s_k'$ let $\lambda = \max(\text{length of } w'(s))$

Define $f : \mathcal{G} \longrightarrow \mathcal{G}'$ by

on vertices: $f(g) = g$

on edges: $f(\underset{g}{\bullet}\!\!\longrightarrow\!\!\underset{gs}{\bullet}) \longmapsto \underset{g}{\bullet}\overset{s_1'}{—}\overset{s_2'}{—}\cdots\overset{s_k'}{—}\underset{gs}{\bullet}$

Then $d(f(x), f(y)) \leq \lambda d(x,y) + 2$

Similarly define $g : \mathcal{G}' \longrightarrow \mathcal{G}$, get $d(g(x), g(y)) \leq \lambda\, d(x,y) + 2$

$f(g(x')) = x'$ if x' is a vertex, and otherwise

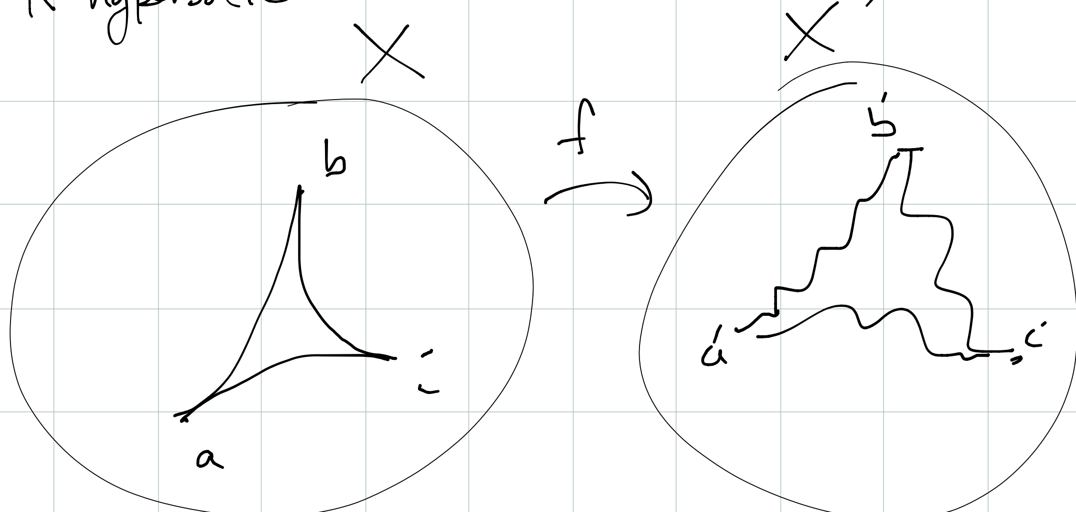
$$d(f(g(x')), x') \leq \lambda_f \cdot \lambda_g$$

(same for $g(f(x))$)

So f and g are quasi-inverses

so f is a quasi-isometry.

We want to show: $X \sim X'$, X' hyperbolic $\Longrightarrow$ X hyperbolic



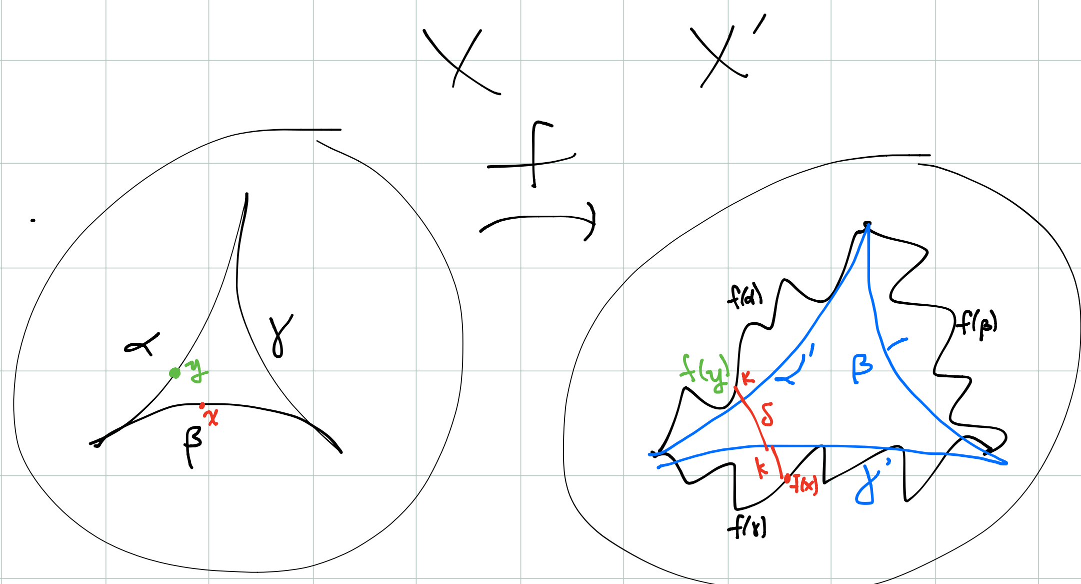
Take a geodesic Δ in X, map it to X'

f is a quasi-isometry, so the image of each geodesic is not too distorted

Key claims: Let γ be a geodesic joining the end pts of $f(\alpha)$.

Then $f(\alpha) \subset N_k(\gamma)$ and $\gamma \subset N_k(f(\alpha))$

That will do it:



$x \in \gamma$, $\Rightarrow \exists y$ $\quad d(f(x), f(y)) < 2k + \delta$

$$\Rightarrow \quad d(x,y) < \lambda(2k+\delta) + C$$