

GGT course - Warwick Sp 2014

Intro

GGT - primarily studies infinite discrete groups - algebraic objects - by studying their actions on geometric objects

Action: each $g \in G$ determines a symmetry of X

Multiplying elts \leftrightarrow composing symmetries

Uses tools from geometry, topology, algebra, analysis, ... to study these actions and draw conclusions about the groups.

Historically, group theory has always been linked with geometry -

Klein's Erlangen program: the geometry of a metric space is encoded in its group of isometries.

Study the group algebraically in order to deduce geometric properties of the space.

GGT is a "backwards Erlangen program": study the geometry of the space to deduce algebraic properties of the group.

Outline

Need background in

Metric spaces (assumed)

Free groups and presentations

Fundamental groups and covering spaces

Hyperbolic geometry

Then can do

Geometrical hyperbolic metric spaces

quasi-isometries

Cayley graphs

Hyperbolic groups

Examples

Properties

Genericity of hyperbolic groups

Hyperbolic spaces related to non-hyperbolic groups

Curve complex and mapping class group

Sphere complex and $\text{Out}(F_n)$

Applications of curve complex hyperbolicity

Asymptotic dimension (Bestvina-Bromberg-Fujiwara)

Geometry of Outer space and sphere complex

$\text{CAT}(0)$ spaces

$\text{CAT}(0)$ groups

Boundary of a hyperbolic space, hyperbolic group

RAAG example: no $\text{CAT}(0)$ -boundary

Applications of ∂

Free groups

It is often possible to write the identity as a product of non-trivial elements of a group G

eg $G = \mathbb{Z}$, $7 - 11 + 4 = 0$

Any such equality $a_1 \dots a_k = 1$, $a_i \neq 1$ is called a relation in G

Here's an obvious relation that holds in any group: $a a^{-1} = 1$

Informally, a free group is a group which has no other relations.

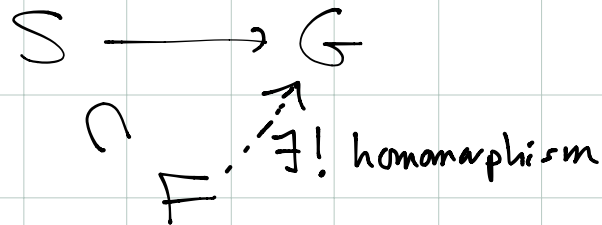
Here's the formal definition:

F is a free group if there is a subset $S \subset F$

such that every set map from S to a group G

extends to a unique homomorphism $F \rightarrow G$

Picture:



We say F is free on S , or free w/ basis S ,
and write $F = F\langle S \rangle$.

This is an example of a universal property,
which is a common notion in algebra

eg you may have seen tensor product defined

by a universal property: for vector spaces

V and W , the tensor product is a vector space $V \otimes W$

and bilinear map $V \times W \longrightarrow V \otimes W$

st. any bilinear map

$$V \times W \rightarrow U$$

extends uniquely to a linear map

$$\downarrow \\ V \otimes W \xrightarrow{\exists!}$$

Why does this say what we want?

(1) Every elt of F is a product of elts of SUS^{-1} :
if $x \in F$ were not such a product, then you could find a homomorphism $f' : F \rightarrow G$ sending x to $g' \neq f(x)$, contradicting uniqueness of f .

(2) Suppose there was a non-trivial relation $a_1 \dots a_k = 1$

Take any group G and elements g_1, \dots, g_k which do not satisfy $g_1 \dots g_k = 1$. Consider the set map $X_i \mapsto g_i$. This does not extend to a homomorphism of $F!$ *

Exercise: verify statements (1) and (2)

Next question: Do free groups exist?

To answer this, we construct a free group starting with a set S

$S = \text{set}$. Take two copies S, S' ← (for inverses)

$i: S \rightarrow S'$ a bijection

Define an involution $x \mapsto \bar{x}$ on $S \cup S'$

by $\bar{s} = i(s), \overline{i(s)} = s$

A word is a finite string $s_1 \dots s_k$ of elements of $S \cup S'$

A word is reduced if $s_{i+1} \neq \bar{s}_i$ for all i

$F\langle S \rangle$: (Group axioms: set, operation, rules)

set = all reduced words, including \emptyset

operation: $x_1 \dots x_k y_1 \dots y_l = x_1 \dots x_{k-r} y_{r+1} \dots y_l$

where $r = \max \{ i \mid y_i = \bar{x}_{k-i+1} \}$ ($0 \leq r \leq \min(k, l)$)
note \uparrow

identity: \emptyset

inverse: $(x_1 \dots x_k)^{-1} = \bar{x}_k \dots \bar{x}_1$

associative law: (this is the hard one)

Done next time

