Mon, Feb 10

We're showing how to use geometry of Cayley graphs to prove algebraic properties of groups, in particular

Thm If G is hyperbolic then G is finitely presented.

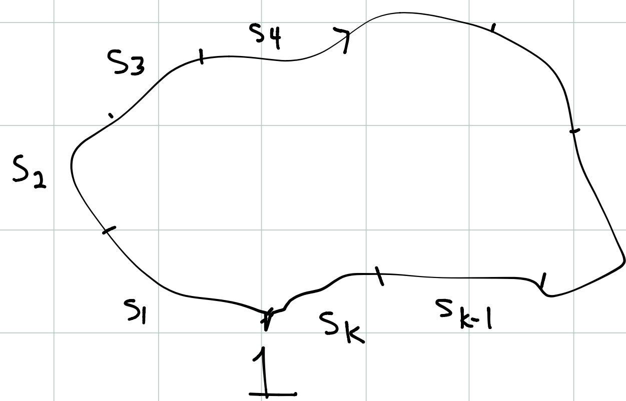
Proof $S =$ (finite) generating set for G,

$$F(S) \twoheadrightarrow G.$$
$$w \longmapsto \overline{w}$$

$\mathcal{C} = \mathcal{C}(G,S) =$ Cayley graph

We are assuming $\mathcal{C}$ is δ-hyperbolic

Recall a word $w \in F(S)$ gives a <u>loop</u> in $\mathcal{C}$ if and only if $\overline{w} = 1$ in G.



$w = s_1 \dots s_k$, $\overline{w} = 1$

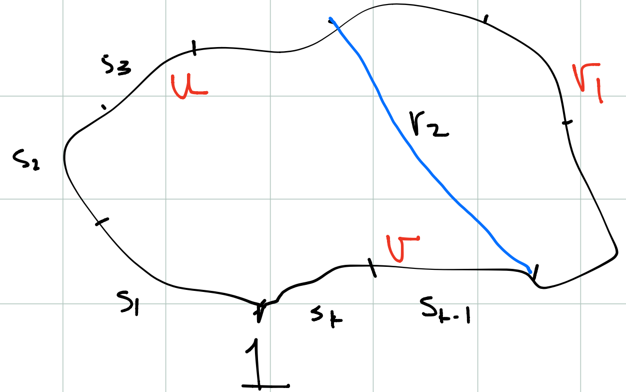
Let $R = \{ w \in F(S) \mid \overline{w} = 1$ and $\text{length}(w) \leq 16\delta$

<u>Claim</u> $\langle S \mid R \rangle$ is a presentation of G, ie R normally generates $\ker(F(S) \to G)$, ie every w with $\overline{w} = 1$ is a product of conjugates of elements of length $\leq 16\delta$ //

<u>pf</u>

Induct on $\text{length}(w) = |w|$. $|w| \leq 16\delta \Rightarrow$ true!

w long $\Rightarrow$ want to find a <u>shortcut</u> of a segment of length $\leq 8\delta$.



$w = u r_1 v$

$|r_2| < |r_1| \le 8\delta$

ie want $w = u r_1 v$ with $|r_1| \le 8\delta$ and r_1 not a geodesic. Then $\exists\, r_2$ with same endpoints and $|r_2| < |r_1|$

$$\text{So } w = u r_1 v = u r_2 v (v^{-1} r_2^{-1} r_1 v)$$

with $r_2^{-1} r_1 \in R$.

Now $\overline{u r_2 v} = \overline{w} = 1$ and $|u r_2 v| < |w| \Rightarrow$

$$u r_2 v \in \langle\langle R \rangle\rangle \Rightarrow w \in R \checkmark$$

What if you <u>can't</u> find such a segment r_1?

ie what if every segment of length $\leq 8\delta$ is a geodesic?

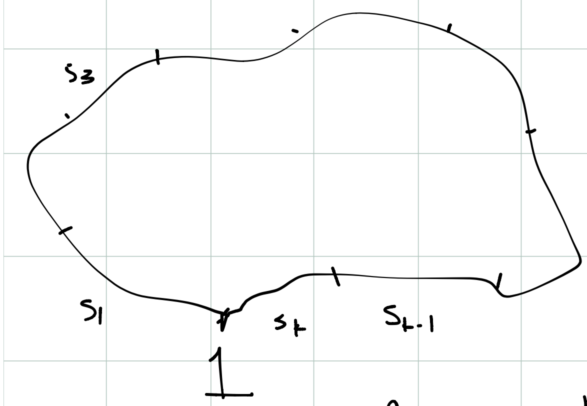
<u>Def</u> X a geodesic metric space, k a constant.

A <u>k-local geodesic</u> is a path σ s.t. every sub-path of length $\leq k$ is a geodesic.

eg. A great circle arc on S^2.

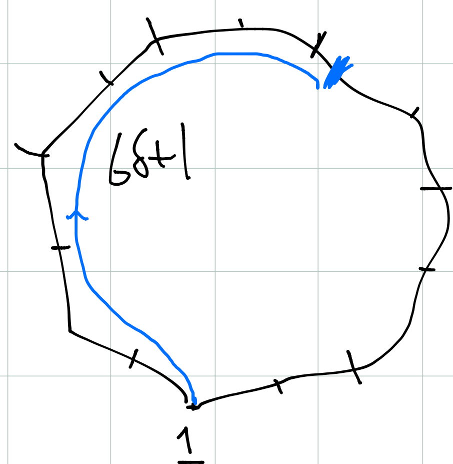
<u>Lemma</u> In a hyperbolic metric space, an 8δ-local geodesic from x to y is within 6δ of any geodesic from x to y.

(proof later)



If w is an 8δ-local geodesic, by the lemma it's in a 6δ-nbd of the (unique) geodesic from 1 to 1, ie the constant path.

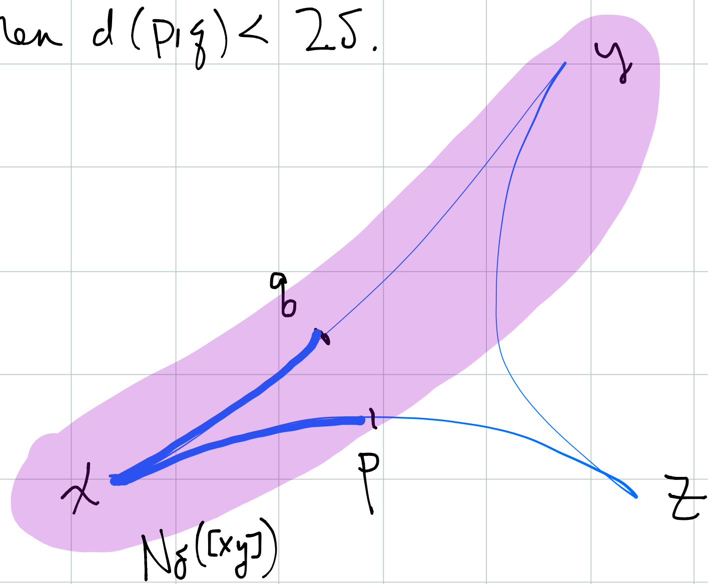
So it has length $\leq 12\delta$. If longer, it would contain a $6\delta+\varepsilon$-length geodesic from 1, contradicting $w \subseteq N_{6\delta}(1)$.



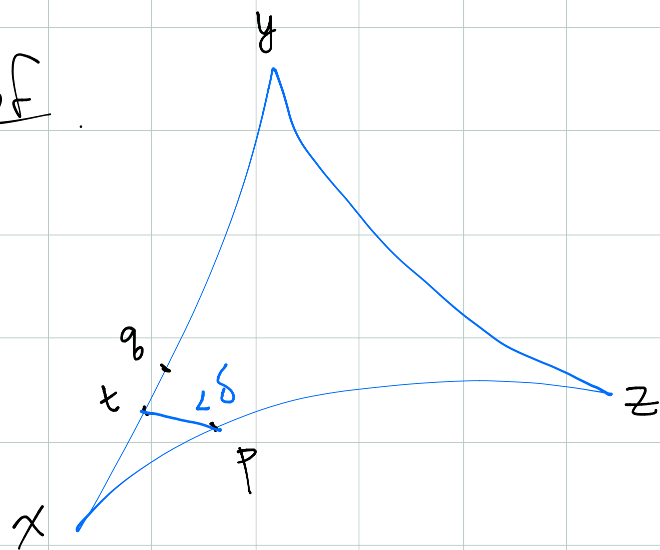
So $w \in R$ ✓

Now we need to prove the Lemma. We will use the following observation about hyperbolic spaces X

Lemma X δ-hyperbolic, Δ a geodesic triangle with vertices x, y and z. If $p \in [x,z]$ is within δ of $[x,y]$, and $q \in [x,y]$ satisfies $d(x,q) = d(x,p)$, then $d(p,q) < 2\delta$.



Pf.



By hypothesis $\exists t \in [x,y]$ with $d(p,t) \leq \delta$

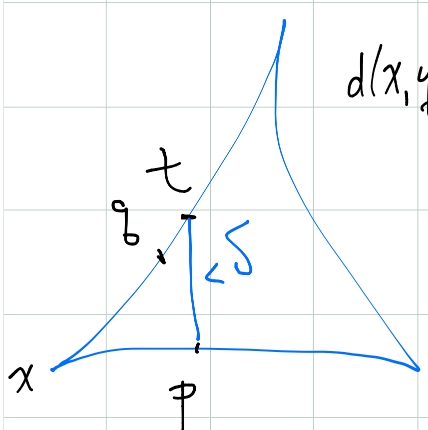
If t is between x and q, we have

$$d(x,t) + d(t,q) = d(x,q) = d(x,p) \leq d(x,t) + \delta$$

$$\Rightarrow d(t,q) \leq \delta$$

$$\Rightarrow d(p,q) \leq d(p,t) + d(t,q) = 2\delta$$

If t is on the other side of q get



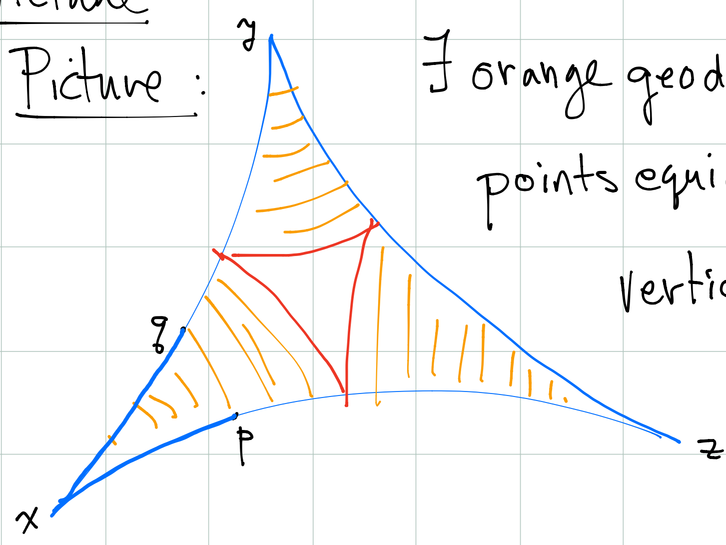
$$d(x,q) + d(q,t) = d(x,t)$$

$$\leq d(x,p) + \delta = d(x,q) + \delta$$

$$\Rightarrow d(q,t) \leq \delta$$

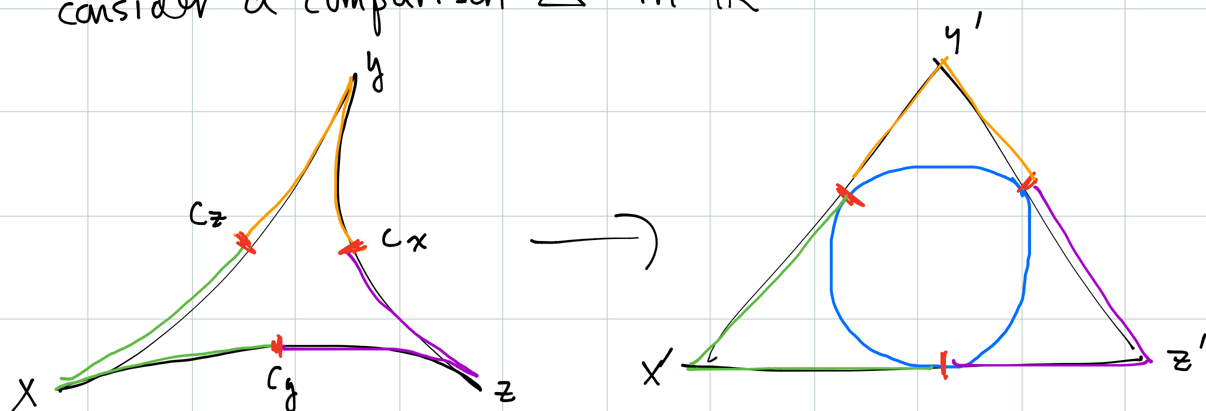
$$\Rightarrow d(p,q) \leq d(p,t) + d(t,q) \leq 2\delta$$

Actual Picture:



$\exists$ orange geodesics joining points equidistant from vertices, all of length $\leq 6\delta$

Pf To find the vertices of the red $\triangle$, consider a comparison $\triangle$ in $\mathbb{R}^2$



Inscribed circle determines 3 points.

$$d(x, c_y) = d(x, c_z), \quad d(y, c_x) = d(y, c_z)$$

$$d(z, c_x) = d(z, c_y).$$

Argument in lemma shows: <u>either</u> $d(c_y, c_x) < 2\delta$ or $d(c_y, c_z) < 2\delta$. Same for c_x and c_z. So

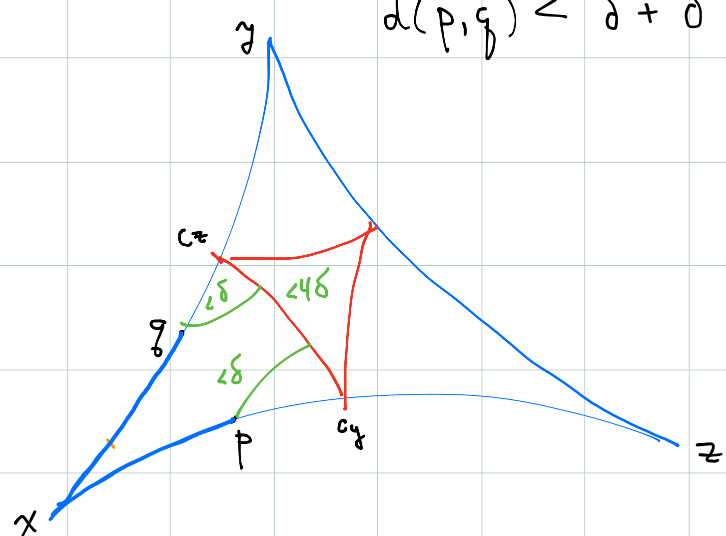
So $d(c_y, c_z) < 4\delta$ at worst.

If $d(p, [x, c_z]) < \delta$, then $d(p, q) < 2\delta$ by the Lemma

Otherwise $d(p, [c_z, c_y]) < \delta$.

Similarly, $d(q, [c_z, c_y]) < \delta$. So

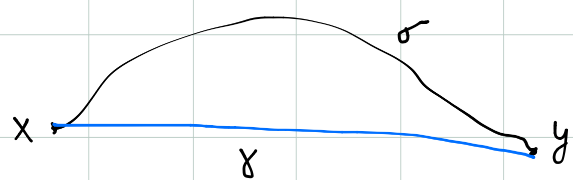
$$d(p,q) < \delta + \delta + 4\delta = 6\delta \checkmark$$



Now we have to prove the Lemma:

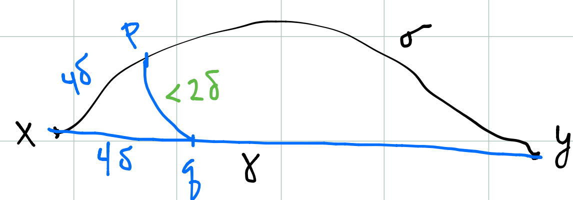
Lemma X δ-hyperbolic, $k = 8\delta \Rightarrow$ a k-local geodesic from x to y is within 5δ of any geodesic from x to y.

Pf



Induct on $\text{length}(\sigma) = |\sigma|$. $|\sigma| \le k \Rightarrow \sigma$ is a geodesic $\Rightarrow$ it is within δ of any geodesic!

Claim $p \in \sigma$, $q \in \gamma$ at distance 4δ from $x \Rightarrow$ $d(p,q) < 2\delta$

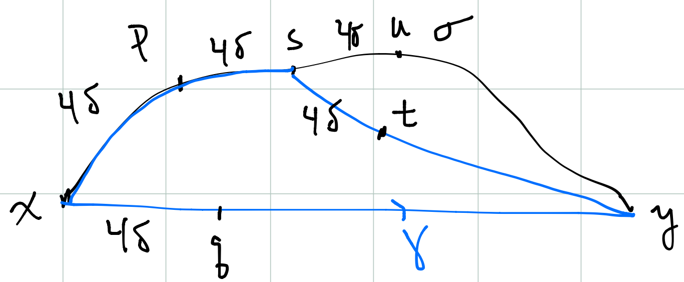


pf "Induct" on length of $\sigma = |\sigma|$

$|\sigma| < 8\delta \Rightarrow \sigma$ is a geodesic. $\Rightarrow$ true (exercise)

Now suppose true if $|\sigma| < N$ and suppose

$N \leq |\sigma| \leq N + 8\delta$



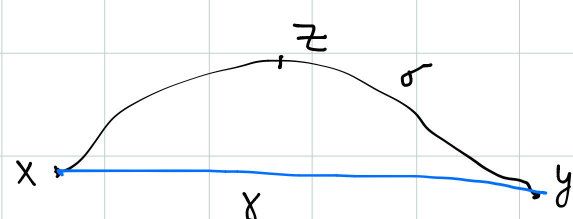
Blue is a geodesic Δ. $\left| \sigma|_{[s,y]} \right| \leq N$

So by induction, $d(u,t) < 2\delta$

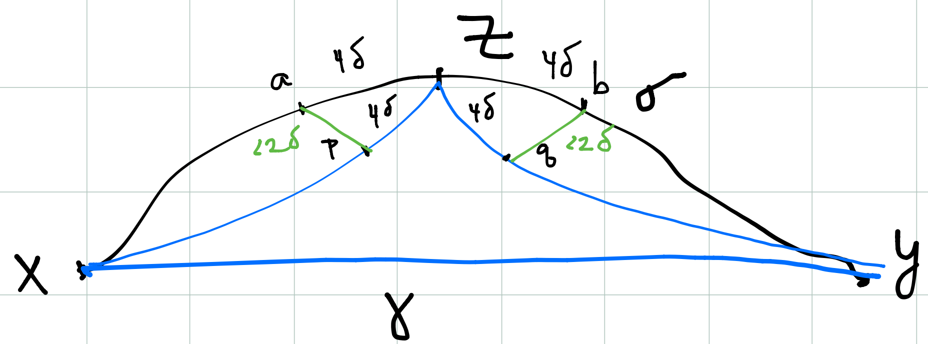
So $d(u,p) = 8\delta \Rightarrow d(t,p) \geq 6\delta$

So by our lemma, $d(p,q) < 2\delta$ ✓

Now:



want to show $d(z,\gamma) < 5\delta$.



$\Rightarrow d(p,q) \geq 4\delta$

$\Rightarrow d(p,\gamma) \leq \delta$ (again by Lemma)

$\Rightarrow d(z,\gamma) \leq 5\delta$