Monday, Feb 17

Last time used Švarc–Milnor to show various groups are quasi-isometric.

To show two metric spaces (or groups) are not quasi-isometric is harder - need to find some invariants of quasi-isometry. Hyperbolicity is one. But how do you show two hyperbolic groups are not quasi-isometric? Is a free group qi to a surface group?

One obvious invariant: Boundedness:

If X is bounded and Y is not, then $X \not\sim Y$

Next goal: given X hyperbolic we will define a "boundary" ∂X and prove $X \sim X'$ $\Rightarrow$ ∂X is homeomorphic to $\partial X'$.

Giving a very powerful quasi-isometry invariant.

For $X = \mathbb{H}^2$, this will be S_∞. We already saw this can be useful, eg the fact that an isometry of $\mathbb{H}^2$ which fixes S_∞ is the identity $\Rightarrow$ Stab of a point in $\mathbb{H}^2 \cong O(2) = \mathrm{Isom}(S^1)$.

In $\mathbb{H}^2$, can identify S_∞ with the set of geodesic rays from any point.

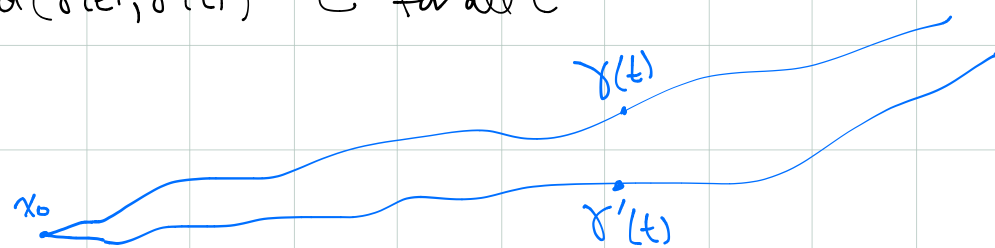
In a general hyperbolic space geodesics are not unique, so have to modify this a little.

Let X be a metric space.

<u>Def</u> A <u>geodesic ray</u> at $x_0 \in X$ is a continuous map $\gamma : [0,\infty) \longrightarrow X$ with $\gamma(0) = x_0$ satisfying $d(\gamma(t), \gamma(t')) = |t - t'|$ for all t, t'

Two geodesic rays $\gamma(t), \gamma'(t)$ at x_0 are <u>equivalent</u> if there is a constant C s.t.

$d(\gamma(t), \gamma'(t)) < C$ for all t



Then <u>want</u> to say

<u>Def</u> 1 Fix $x_0 \in X$. Then ∂X is the set of equivalence classes of geodesic rays from x_0.

But it is not clear this is independent of x_0.

Given $x_1 \neq x_0$, the concatenation of a geodesic from x_1 to x_0 with a geodesic ray at x_0 is not a geodesic ray, but is a <u>quasi-geodesic</u> ray.

<u>Def</u> σ, σ' quasi-geodesic rays are <u>equivalent</u> if there is a constant C s.t. $\sigma \subseteq N_C(\sigma')$ and $\sigma' \subseteq N_C(\sigma)$, ie the Hausdorff distance between σ and σ' is $\leq C$.

<u>Def</u> ② ∂X is the set of equivalence classes of quasi-geodesic rays in X

Advantages of this definition: no longer depends on choice of x_0, and a quasi-isometry $X \to X'$ takes ∂X to $\partial X'$.

But it's hard to define a topology or metric on ∂X or say how to glue ∂X to X – so we'll try a <u>third</u> definition, in terms of <u>sequences</u>.

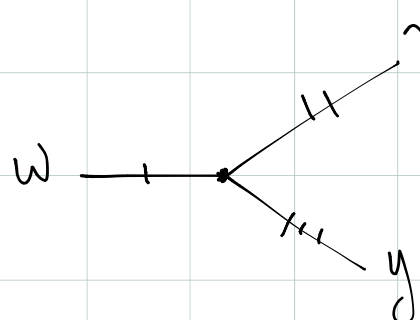
We want to consider sequences that stay close to geodesic rays and leave every bounded set.

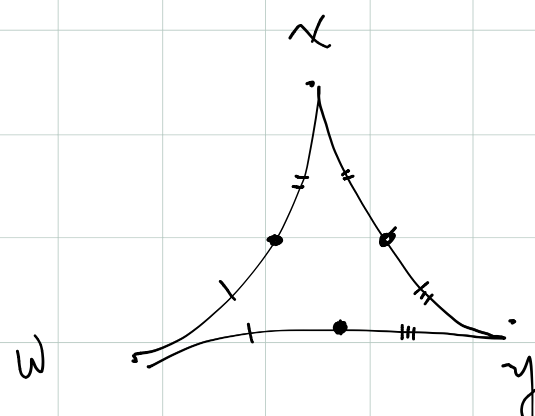
This idea is captured by the <u>Gromov product</u>.

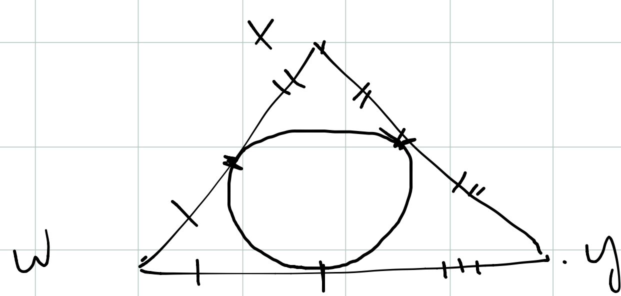
<u>Def</u> Fix a base point $w \in X$. Then

$$(x.y)_w := \frac{1}{2}\left[d(x,w) + d(y,w) - d(x,y)\right]$$

<u>Idea</u>: If you consider only 3 points in any metric space X, you can't tell X from a tree:

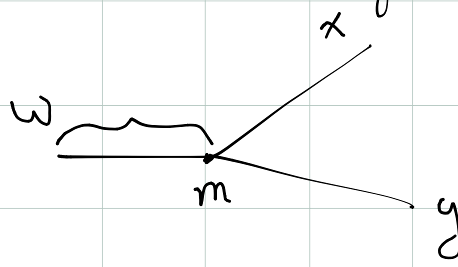




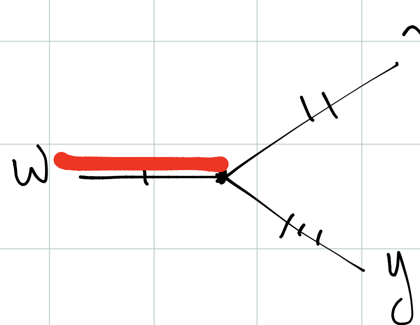


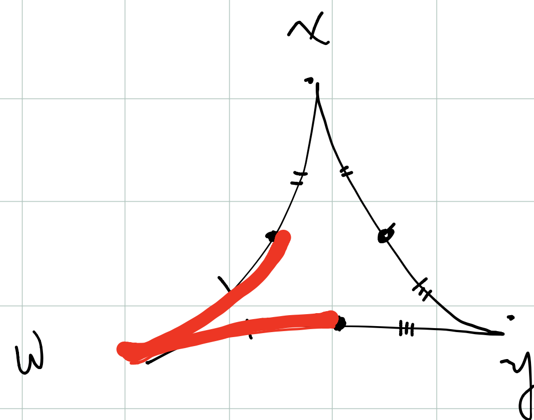
In the tree, $d(w,x)+d(w,y)-d(x,y) = 2\cdot$ length of

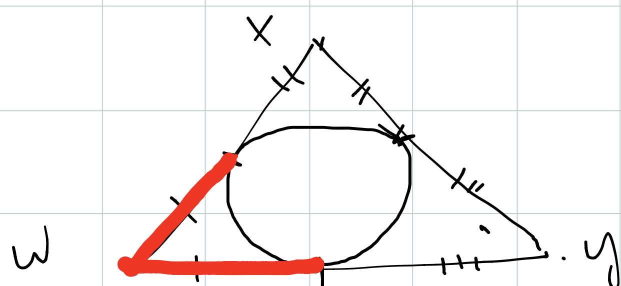
overlap:



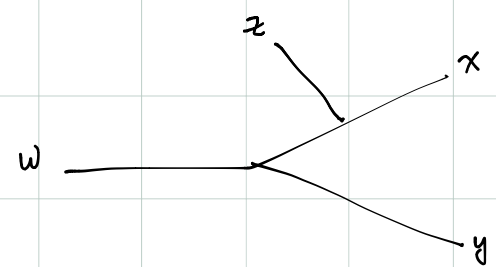
In all cases, $(x,y)_w$ is visible:







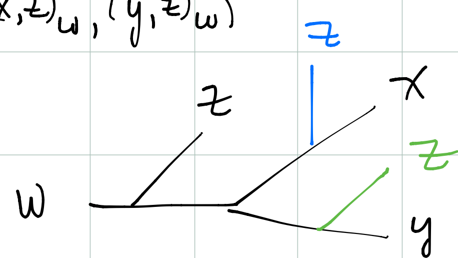
For 4 points, however, you can tell the difference



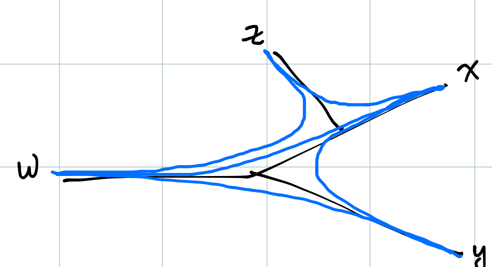
$$(x,y)_w = \min\left((x,z)_w, (y,z)_w \right)$$

In fact, no matter what the configuration, always have

$(x,y)_w \geq \min\left((x,z)_w, (y,z)_w\right)$



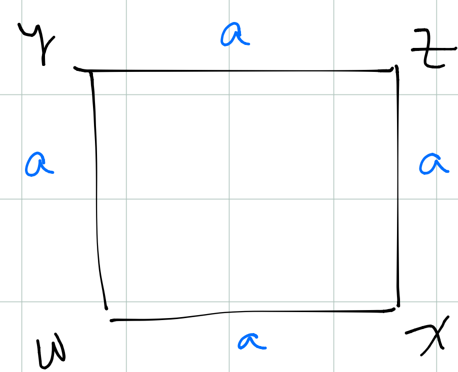
In a hyperbolic space the distances are off slightly:



Def X is $\underline{\delta\text{-hyperbolic}}$ if $\forall\, x, y, z, w$,

$$(x,y)_w \geq \min\{(x,z)_w, (y,z)_w\} - \delta$$

So $\mathbb{R}^2$ is <u>not</u> hyperbolic, for any δ:



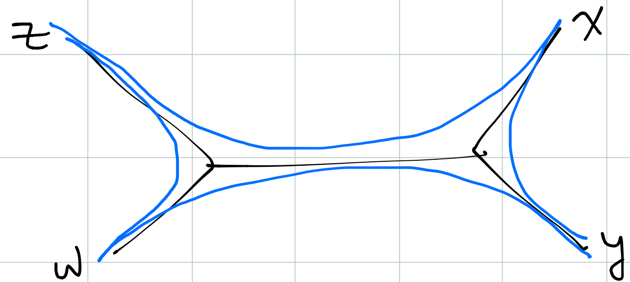
$$\begin{array}{ccccc} (x,y)_w & < & \min\{ & (x,z)_w, & (y,z)_w \} \\ \| & & & \| & \| \\ \frac{(2-\sqrt{2})}{2}a & & & \frac{\sqrt{2}}{2}a & \frac{\sqrt{2}}{2}a \end{array}$$

$$\left(1-\tfrac{\sqrt{2}}{2}\right)a \geq \tfrac{\sqrt{2}}{2}a - \delta \quad \underline{\text{for all } a}$$

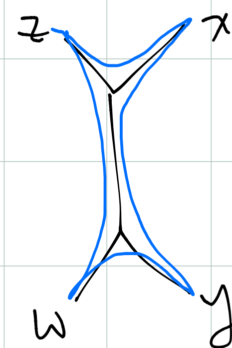
$$\delta \geq (\sqrt{2}-1)a \quad \text{for all } a \quad \times$$

Note: To make this look more symmetric, could rewrite it as: $\forall x, y, z, w \in X$

$$d(x,w) + d(y,z) \le \max\{d(x,y)+d(w,z),\ d(x,z)+d(y,w)\} + 2\delta$$



or



Gory details: write xy for $d(x,y)$:

$(x.y)_w \geq \min\{(x.z)_w, (y.z)_w\} - 2\delta$ means

$*$ $wx + wy - xy \geq \min(wx + wz - xz,\ wy + wz - yz) - 2\delta$

Case 1 $\quad wx + \cancel{wz} - xz \geq wy + \cancel{wz} - yz$

Then $wx + yz \geq wy + xz$

$* \iff wx + \cancel{wy} - xy \geq \cancel{wy} + wz - yz - 2\delta$

ie $wx + yz \geq wz + xy - 2\delta$

ie $xy + wz \leq wx + yz + 2\delta$ ✓

Case 2 $wx + wz - xz \leq wy + wz - yz$

$$wx + yz \leq wy + xz$$

$* \iff wx + wy - xy \geq wx + wz - xz - 2\delta$

$$wy + xz \geq wz + xy - 2\delta$$

$$xy + wz \leq wy + xz + 2\delta \checkmark$$

So $* \iff xy + wz \leq \max\{yw + xz, xw + yz\} + 2\delta$

Prop The Gromov product definition of hyperbolic metric space is equivalent to the thin triangles definition.

Pf. Deferred

Back to the boundary: Fix a basept $w \in X$

Set $(a.b) = (a.b)_w$

Consider sequences $\{a_i\}$ of points in X

Say $\{a_i\} \to \infty$ if $\lim_{i,j\to\infty} (a_i.a_j) = \infty$

ie $\forall n, \exists N$ st. $i,j > N \Rightarrow (a_i . a_j) > n$

Note that "going to infinity is independent of the choice of w:

$$\left| (x,y)_w - (x,y)_{w'} \right| \leq d(w,w')$$

$$\Big(pf: \left| (wx + wy - xy) - (w'x + w'y - xy \right|$$

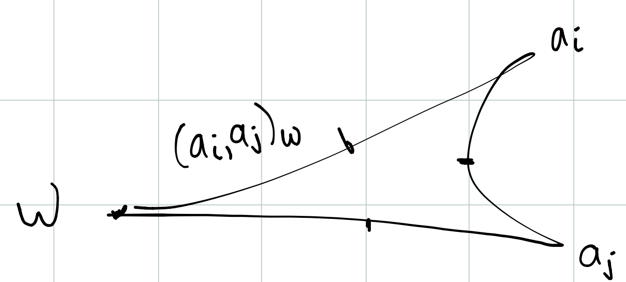
$$\leq |wx - w'x| + |wy - w'y|$$

$$\leq |ww'| + |ww'| = 2d(w,w') \Big)$$

Note also $\{a_i\} \to \infty \Rightarrow \lim_{i \to \infty} d(a_i, w) = \infty$

since $x.y \leq \min\big(d(x,w), d(y,w)\big)$

so $(a_i . a_j) \leq \min\big(d(w,a_i), d(w,a_j)\big)$



Let $S_\infty(X)$ = set of sequences $\to \infty$

Define $\{a_i\} R \{b_i\}$ iff $\lim_{i\to\infty} (a_i . b_i) = \infty$

R is symmetric and reflexive, but not (for general metric spaces X) transitive.

<u>Lemma</u> X hyperbolic $\Rightarrow$ R is transitive

<u>pf</u> $a_i R b_i$, $b_i R c_i$

$$(a_i . c_i) \geq \min\{(a_i, b_i), (b_i, c_i)\} - \delta$$

$$\searrow \infty \quad \searrow \infty .$$

$\therefore (a_i, c_i) \to \infty$ ✓

So R is an equivalence relation if X is hyperbolic, and we can give our third definition of ∂X

<u>Def</u> 3 $\partial X = R$-equivalence classes of sequences in $S_\infty(X)$