Monday, Feb. 24

Last time: Showed all 3 def'ns of ∂X for X a locally finite hyperbolic graph are $\sim$.

Assume X is loc. fin. hyp graph for all of today.

Extended Gromov product to ∂X using sequence definition: $x, y \in \partial X$, $w \in X$

$$\Rightarrow (x,y)_w = \sup_{\substack{x_n \to x \\ y_n \to y}} \liminf_{i,j} (x_i, y_j)_w$$

$x \in \partial X$, $y \in X$ $\quad (x,y)_w = \sup_{x_n \to x} \liminf_i (x_i, y)_w$

Then for $x \in \partial X$, $N_r(x) = \{ y \in X \cup \partial X : (x,y)_w \geq r \}$

(for $x \in X$, $B_r(x) = \{ y \in X \mid d(x,y) < r \}$)

The $N_r(x)$ and $B_r(x)$ form a neighborhood base for a topology on $\hat{X} = X \cup \partial X$. If we restrict to rational r, we have a countable neighborhood base.

Prop: $\hat{X}$ is metrizable

Pf Have to show $\hat{X}$ is regular Hausdorff. Then $\hat{X}$ is metrizable by Urysohn's lemma.

(Regular: closed sets can be separated from points not in them.

Hausdorff: Points can be separated.

Example of regular not Hausdorff: $\{\emptyset, X\}$ are all the open sets)

Exercise: Separate $x, y \in X$, $x \in X, \bar{y} \in \partial X$, $\bar{x}, \bar{y} \in \partial X$.

Prop $\hat{X} = X \cup \partial X$ is compact.

Pf $\hat{X}$ metrizable $\Rightarrow$

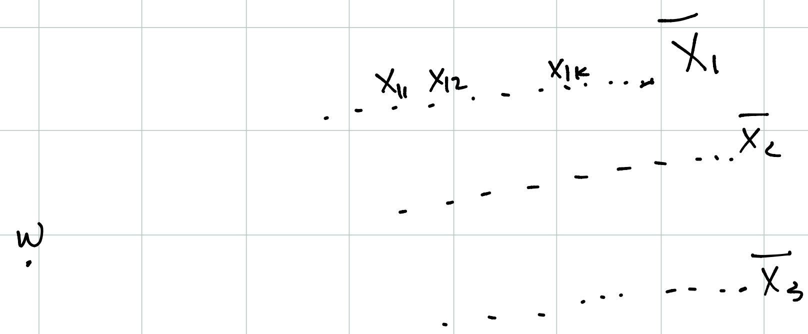
compact is equivalent to sequentially compact.

Let $\{x_i\}$ be a sequence. If all $x_i \in X$ and $\{x_i\}$ is bounded, then some edge of X contains ∞ many x_i, which has a convergent subsequence.

If $\{x_i\}$ is unbounded, we know how to extract a subsequence $x_{i_k} \longrightarrow \infty$, ie $\{x_{i_k}\}$ converges to a point in ∂X

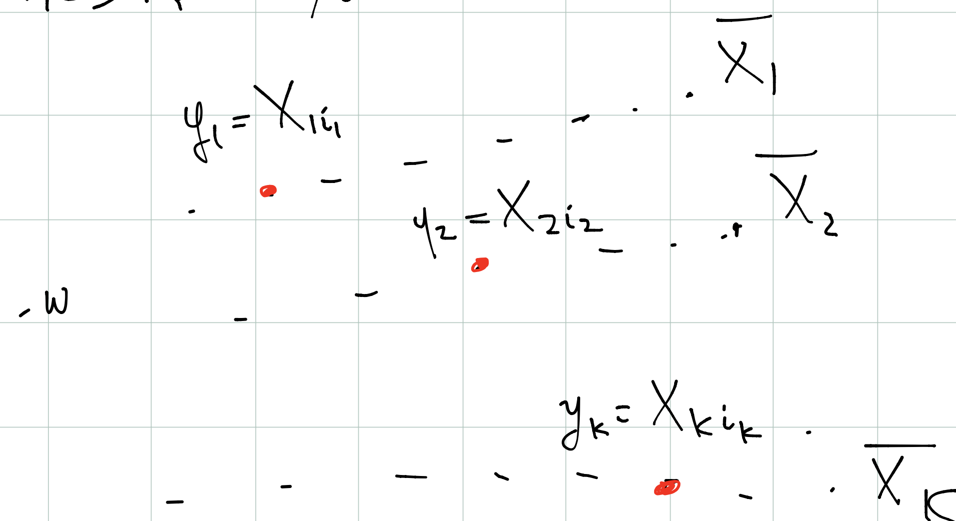
If $\{x_i\}$ contains only finitely many points in X, throw them out, to get a subsequence $\{\overline{x}_i\}$ of points at ∞, ie there are sequences

$x_{i_k} \xrightarrow[k]{} \overline{x}_i$



$(x_{k_i}, x_{kj})_w \xrightarrow[i,j]{} \infty$

For each i, choose k s.t. $(x_{i_r}, x_{i_s})_p > i$ for $r,s \geq k$, set $y_i = x_{i_k}$.



$\{y_i\}$ is unbounded, since $d(y_i, p) \geq i$ for all i.

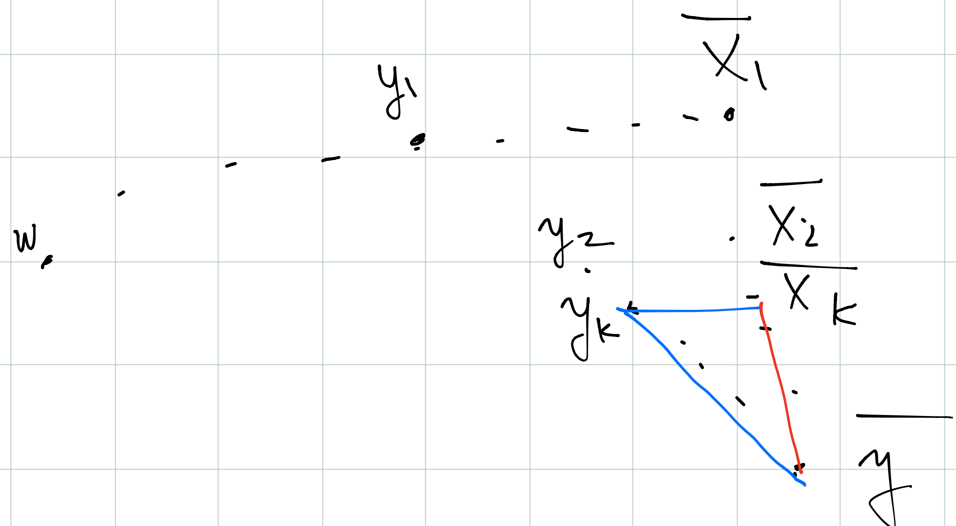
Passing to a subsequence, we may assume $y_i \to \infty$

say $\overline{y} = \{y_i\}$.

<u>Claim</u> $\{\overline{x_j}\} \to \overline{y}$

Ie $\forall n\ \exists R$ s.t. $j > R \Rightarrow \overline{x_j} \in N_n(\overline{y})$,

ie $(\overline{y}, \overline{x_j})_w \geq R$



<u>Recall</u> (Feb 18, remark 2)

$$(\overline{x_k}, \overline{y}) \geq \min\{(y_k, \overline{y}), (y_k, \overline{x_k})\} - 2\delta$$

for all k

so for any n, k sufficiently large $\Rightarrow \overline{x_k} \in N_n(\overline{y})$,

ie $\{\overline{x_k}\} \to \overline{y}$.