Geometric group theory

Introduction

What *is* Geometric group theory?

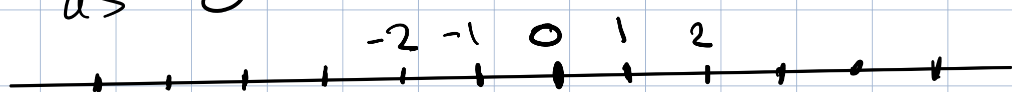
We study finitely generated infinite groups by finding nice spaces on which they act as symmetries - then use the geometry and topology of these spaces to deduce algebraic properties of the groups.

eg The integers $\mathbb{Z}$ is infinite

If you're a

- <u>number theorist</u> you think of $\mathbb{Z}$ as $\{0, \pm 1, \pm 2, \pm 3, \dots\}$ with <u>addition</u> and <u>multiplication</u> (but it's only a group under addition.)

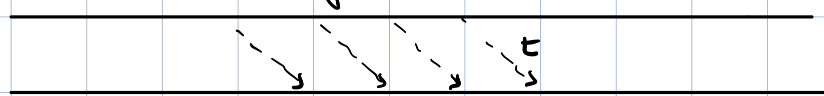
- <u>geometer</u> you think of $\mathbb{Z}$ as evenly spaced points on the real line, with one point arbitrarily chosen as 0



- <u>group theorist</u> - you write the group operation as multiplication, call $\mathbb{Z}$ the infinite cyclic group

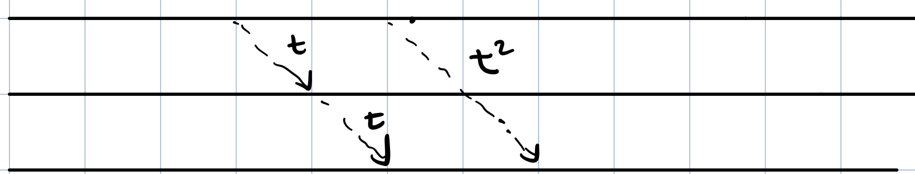
$$\{1, t, t^{-1}, t^2, t^{-2}, t^3, \ldots\}$$

- $\boxed{\text{geometric group theorist}}$ - think of $\mathbb{Z}$ as a group of <u>translations</u> of $\mathbb{R}$



(no need to choose an origin)

The group operation is composition: $t^2 = t \circ t$



We will study some examples in detail, including free groups, free abelian groups, surface groups and some matrix groups

Then find general geometric and topological criteria (like contractibility or some form of negative or non-positive curvature) that determine properties of a group action.

I expect that you are equipped with a basic knowledge of

- Linear algebra, Euclidean space
- Groups
- Fundamental groups and covering spaces
- The geometry of the hyperbolic plane

I also posted lecture notes from the last time I taught the course — but this course is organized differently, and it is these notes you should use to study for the exam.

Free groups and presentations

Let G be a group. A subset $S \subset G$ <u>generates</u> G if every $g \in G$ can be written as a product of elements of S and their inverses.

G is <u>finitely generated</u> if some finite $S \subset G$ generates G

We are interested in finitely generated groups that have an infinite number of elements.

<u>Example</u> $G = \langle t \rangle = \{ \ldots t^{-2}, t^{-1}, t^0, t, t^2, t^3, \ldots \}$.
$\langle t \rangle$ is generated by one element (namely t)

This is the first example of a <u>free group</u>

In intro to topology you probably learned that the fundamental group of a graph is a free group.— you may have even defined "free group" in that way.

We're going to define it in a more abstract— but very useful—way, via a <u>universal property</u>.

Definition A group F is free on a subset $S \subset F$ if any set map from S to a group G extends uniquely to a homomorphism $F \longrightarrow G$.

Picture:

$$\begin{tikzcd} F \arrow[drr, dashed, "\exists!\ \text{homomorphism } \hat{f}"] & & \\ \cup & & \\ S \arrow[rr, "\text{set map}", "f"'] & & G \end{tikzcd}$$

Example $t \in \langle t \rangle$: Claim $\langle t \rangle$ is free on t

If G is any group, any map $t \xrightarrow{s} G$

extends to a homomorphism $f: \langle t \rangle \longrightarrow G$
(if $s(t) = g$, define $f(t^k) = g^k$.)

$$\begin{tikzcd} \langle t \rangle \arrow[dr, dashed] & t^k \arrow[r, mapsto] & g^k \\ \cup & & \\ \{t\} \arrow[r] & G & \\ t \arrow[r, mapsto] & g & \end{tikzcd}$$

Claim this is a good definition because it makes it easy to prove basic facts about free groups.

Questions:

1. If we can find such an $S \subset F$, does it generate F?

2. Can F be free on a different set $S' \subset S$? If so, what is the relation between S and S'?

3. Do free groups exist?

4. Can I realize a free group as a group of isometries of some metric space?

Answers:

Suppose F is free on $S \subset F$

(0) The identity $F \xrightarrow{id} F$ extends the inclusion $i : S \hookrightarrow F$, so is the unique homomorphism extending i, by the universal property.

$$
\begin{tikzcd}
F \arrow[dr, "\mathrm{id} = \hat{\imath}"] & \\
S \arrow[u, phantom, "\cup" description] \arrow[r, hook, "i"'] & F
\end{tikzcd}
$$

(1) S generates F.

Products of elements of S and their inverses form a subgroup $\langle S \rangle$ of F (closed under $\circ$, inverse)

The inclusion $S \hookrightarrow \langle S \rangle$ extends to a unique homomorphism $F \to \langle S \rangle$

$$
\begin{tikzcd}
F \arrow[dr, "\hat{\imath}"] & \\
S \arrow[u, phantom, "\cup" description] \arrow[r, hook, "i"'] & \langle S \rangle
\end{tikzcd}
$$

The inclusion $\langle S \rangle \overset{I}{\subseteq} F$ is a homomorphism.

If $\langle S \rangle \subsetneq F$ the composition $I \circ \hat{\imath}$ extends the inclusion $i : S \hookrightarrow F$ but is not surjective, so is not the identity, contradicting (0) above.

$$
\begin{tikzcd}
F \arrow[dr, "\hat{\imath}"'] \arrow[drr, bend left, "I \circ \hat{\imath}"] & & \\
S \arrow[u, phantom, "\cup" description] \arrow[r, hook, "i"'] & \langle S \rangle \arrow[r, hook, "I"'] & F
\end{tikzcd}
$$

(2) Suppose F is free on S and on S' (both finite, or at most countably infinite)

<u>Claim</u> $|S| = |S'|$

<u>Proof</u>:

Any homomorphism $F \to \mathbb{Z}/2 = \{0,1\}$ restricts to a unique set map $S \to \mathbb{Z}/2$ ($S \subset F$)

Conversely, any set map $S \to \mathbb{Z}/2$ extends to a unique homomorphism $F(S) \to \mathbb{Z}/2$

$$\boxed{\text{ie } \left(\mathrm{Hom}(F, \mathbb{Z}/2)\right) \leftrightarrow \left(\text{maps } S \to \{0,1\}\right)}$$

Now count: there are $2^{|S|}$ set maps $S \to \{0,1\}$

so $|\mathrm{Hom}(F, \mathbb{Z}/2)| = 2^{|S|}$

Similarly, $|\mathrm{Hom}(F, \mathbb{Z}/2)| = 2^{|S'|}$

$\therefore 2^{|S|} = 2^{|S'|}$ so $|S| = |S'|$ ✓

<u>Definition</u>: If F is free on S, then $|S|$ is the <u>rank</u> of F

Theorem Two finitely-generated free groups are isomorphic if and only if they have the same rank.

proof ($\Rightarrow$) Let F, F' be finitely-generated free groups, F free on S, and let $f : F \to F'$ be an isomorphism

Claim F' is free on $S' = f(S) = \{f(s) \mid s \in S\}$

pf We verify the universal property.

Given
$$\begin{tikzcd} f(S) \arrow[r, "g"] \arrow[d, phantom, "\cup" rotate=90] & G \\ F' \arrow[ur, dashed, "\hat{g}"'] & \end{tikzcd}$$
a set map, we need a unique homomorphism $\hat{g} : F' \to G$ extending g.

existence

Define g by: $\hat{g}(y) = \widehat{gf}(f^{-1}(y))$

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, phantom, "\cap" rotate=-90] & f(S) \arrow[r, "g"] \arrow[d, phantom, "\cap" rotate=-90] & G \\ F \arrow[r, "f"] \arrow[urr, "\widehat{gf}"] & F' \arrow[ur, dashed, "\hat{g}"'] & \\ f^{-1}(y) & y \arrow[l, mapsto] & \end{tikzcd}$$

This extends g: if $y = f(s)$, then

$$\widehat{gf}(f^{-1}(s)) = g \circ f \circ f^{-1}(s) = g(s) \checkmark$$

<u>uniqueness</u> follows from the uniqueness of f^{-1} and $\widehat{gf}$. (Exercise)

$\Leftarrow$ Suppose $|S| = |S'|$. Choose a bijection $b : S \leftrightarrow S'$

$$\begin{tikzcd}
F(S) \arrow[r] & F(S') \\
S \arrow[u, hook, "i"] \arrow[r, "b"] \arrow[ur, "i' \circ b"] & S' \arrow[u, hook, "i'"']
\end{tikzcd}$$

extends $i' \circ b$.

$$\begin{tikzcd}
F(S') \arrow[rr] & & F(S) \\
S' \arrow[u, hook, "i'"] \arrow[rr, "b^{-1}"] & & S \arrow[u, hook, "i"']
\end{tikzcd}$$

extends $i \circ b^{-1}$

The composition

$$\begin{tikzcd}
F(S) \arrow[r, "\widehat{i' \circ b}"] & F(S') \arrow[r, "\widehat{i \circ b^{-1}}"] & F(S) \\
S \arrow[u, hook] \arrow[r, "b"] & S' \arrow[u, hook] \arrow[r, "b^{-1}"] & S \arrow[u, hook]
\end{tikzcd}$$

extends id_S, so $= id_{F(S)}$ by (0) $\checkmark$

③ Next question: Do free groups exist?

- We saw $\mathbb{Z} = \langle t \rangle$ is free.
  Any others?

To construct a group, need.

- A set G
- An operation $\cdot : G \times G \longrightarrow G$
  satisfying
  1. there is an identity element
  2. every element has an inverse
  3. The associative law holds
     $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

Construction of $F(S)$

Let S be a set, and $\overline{S}$ another copy

ie $$S \longleftrightarrow \overline{S} \quad \text{a bijection.}$$
$$a \longleftrightarrow \overline{a}$$

Let $A = S \cup \overline{S}$ (A for "alphabet")

we have an involution
$$A \to A$$
$$a \to \overline{a}$$
$$\overline{a} \to a$$

We will construct a group F containing S, then prove it is free on S

A <u>word</u> in A is a finite string of elements of A $a_1 \cdots a_k$

A word is <u>reduced</u> if $a_{i+1} \neq \overline{a}_i$ for all i.

The <u>elements</u> of F are the reduced words in A plus the empty word $\emptyset$

The <u>operation</u> $F \times F \longrightarrow F$

is "juxtaposition followed by reduction"

ie

if $X = x_1 \cdots x_k$ $\quad Y = y_1 \cdots y_\ell$ $\quad$ reduced words

Let $r(X, Y) = \begin{cases} 0 & \text{if } y_1 \neq \bar{x}_k \\ \max\{j : y_i = \bar{x}_{k-i+1} \ \forall i \leq j\} & \text{otherwise} \end{cases}$

(eg $\quad x_1 \cdots \cancel{x_k}\, \cancel{y_1} \cdots y_\ell \quad r=1$)

$$x_1 \cdots x_k y_1 \cdots y_\ell = \overbrace{x_1 \cdots x_{k-r}}^{a}\, \overbrace{x_{k-r+1} \cdots x_k}^{v}\, \overbrace{y_1 \cdots y_r}^{v^{-1}}\, \overbrace{y_{r+1} \cdots y_\ell}^{b}$$

Define

not inverses

$$XY = x_1 \cdots x_{k-r}\, y_{r+1} \cdots y_\ell = ab$$

this is reduced so is in F.

<u>Claim</u> F with this product is a group

① identity $= \phi$

② inverses: $(x_1 \cdots x_k)^{-1} = \bar{x}_k \cdots \bar{x}_1$

③ associative law

$$\left.\begin{array}{l} X = x_1 \cdots x_k \\ Y = y_1 \cdots y_\ell \\ Z = z_1 \cdots z_m \end{array}\right\} \quad (XY)Z \overset{?}{=} X(YZ)$$

Check: let $r = r(x, y)$

$s = r(y, z)$

Look at $$x_1 \cdots \underbrace{x_k}_{r} \underbrace{y_1}_{r} \cdots \underbrace{y_\ell}_{s} \underbrace{z_1}_{s} \cdots z_m$$

Suppose first $r + s < \ell$:

$$\overbrace{x_1 \cdots}^{a} \overbrace{\underbrace{\cancel{x_k}}_{r}}^{u} \overbrace{\underbrace{\cancel{y_1}}_{r}}^{\bar{u}} \overbrace{\cdots}^{b} \overbrace{\underbrace{y_\ell}_{s}}^{v} \overbrace{\underbrace{z_1}_{s}}^{\bar{v}} \overbrace{\cdots z_m}^{c}$$

$XY = (au)(\bar{u}bv) = abv$ $\quad (XY)Z = (abv)(\bar{v}c) = abc$

$YZ = (\bar{u}bv)(\bar{v}c) = \bar{u}bc$ $\quad X(YZ) = (au)(\bar{u}bc) = abc$

abc is reduced, so these are equal in F ✓

If $r + s = \ell$ $\quad$ ($b = \emptyset$ in the above picture)

$$\overbrace{x_1 \cdots}^{a} \overbrace{\underbrace{\cancel{x_k}}_{r}}^{u} \overbrace{\underbrace{\cancel{y_1}}_{r}}^{\bar{u}} \overbrace{\underbrace{y_\ell}_{s}}^{v} \overbrace{\underbrace{z_1}_{s}}^{\bar{v}} \overbrace{\cdots z_m}^{c}$$

$XY = (au)(\bar{u}v) = av \Rightarrow (XY)Z = (av)\bar{v}c = ac$

and $YZ = (\bar{u}v)(\bar{v}c) = \bar{u}c \Rightarrow X(YZ) = (au)(\bar{u}c) = ac$

ac may not be reduced, but a and c are so $ac \in F$ is well-defined

If $r+s > l$

$$x_1 \cdots x_t \, y_1 \cdots y_l \, z_1 \cdots z_m$$

$$a\bar{v}\bar{u}\,uvw\,\bar{w}\bar{v}c$$

$$XY = (a\bar{v}\bar{u})(uvw) = aw$$

$$(XY)Z = (aw)(\bar{w}\bar{v}c) = a\bar{v}c \quad \text{reduced}$$

$$YZ = (uvw)(\bar{w}\bar{v}c) = uc$$

$$X(YZ) = (a\bar{v}\bar{u})(uc) = a\bar{v}c \checkmark$$

so associativity holds, and we have a group. Call it $F(S)$

<u>Claim</u> $F(S)$ is free on S

<u>Proof</u> We have to show: for any group G, any <u>set map</u> $f: S \longrightarrow G$ extends to a unique <u>homomorphism</u> $\hat{f}: F(S) \to G$.

How do you define $\hat{f}$?

Let $a_1 \cdots a_k \in F(S)$ with $a_i \in S \cup \bar{S}$

define $\hat{f}: F(S) \to G$ by

- $\hat{f}(\bar{s}) = f(s)^{-1}$ and
- $\hat{f}(a_1 \cdots a_k) = \hat{f}(a_1) \cdots \hat{f}(a_k)$

Then $\hat{f}(\bar{w}) = \hat{f}(w)^{-1}$.

check $\hat{f}(vw) = \hat{f}(v)\hat{f}(w)$:

If $v = v_0 u$ and $w = \bar{u} w_0$ with

with $v_0 w_0$ reduced, then $v_0 u$ is the reduced word $v_0 w_0$

So $\hat{f}(vw) = \hat{f}(v_0 w_0) = \hat{f}(v_0)\hat{f}(w_0)$

$$
\begin{aligned}
&= \hat{f}(v_0)\hat{f}(u)\hat{f}(u)^{-1}\hat{f}(w_0) \\
&= \hat{f}(v_0 u)\hat{f}(\bar{u} w_0) \\
&= \hat{f}(v)\hat{f}(w) \checkmark
\end{aligned}
$$

So $\hat{f}$ is a homomorphism

It is unique because <u>any</u> homomorphism h must

send $\bar{s}_i \mapsto h(s_i)^{-1}$ and

$a_1 \cdots a_k \mapsto h(a_1) \cdots h(a_k)$ ✓

<u>Prop</u> Free groups have no <u>torsion</u>,

ie $w \in F(S) \Rightarrow w^n \neq \phi$, for all n.

<u>Pf</u> Since $id : F(S) \longrightarrow F(S)$ is

the <u>unique</u> homomorphism extending

$id : S \xrightarrow{1} S$, the only <u>reduced</u> word

in $A = S \cup \bar{S}$ that maps to 1 is 1.

If w is a reduced word, then $w = u v \bar{u}$

with $v \neq \phi$ <u>cyclically</u> <u>reduced</u>

(ie first letter of v is not the inverse

of the last letter)

so $w^2 = u v \bar{u} u v \bar{u} = u v^2 \bar{u}$ is reduced

similarly $w^n = u v^n \bar{u}$ is reduced,

so $w^n \neq 1$ in $F(S)$ ✓

Here is another way to say G is generated by S:

There is a surjective homomorphism $F(S) \to G$ for some finite set S.

<u>Definition</u> A subgroup $H < G$ is <u>normally generated</u> by a subset $R \subset G$ if H is generated by elements of R and conjugates of elements of R

We write $H = \langle\langle R \rangle\rangle$

<u>Note</u>: If $H = \langle\langle R \rangle\rangle$, $h \in H$ and $g \in G$
Then ghg^{-1} is a product of conjugates of elements of R, so $hgh^{-1} \in H$, ie H is a normal subgroup of G.

<u>Definition</u> Let G be a group
A <u>presentation</u> of G consists of a generating set S for G and a normal generating set R for $\ker(F(S) \to G)$.

we write $G = \langle S \mid R \rangle \cong F(S)/\langle\langle R \rangle\rangle$
If both S and R are finite, we say G is <u>finitely presented</u>

If G has <u>some</u> finite presentation, we say G is <u>finitely presentable</u>.

A presention determines G – but G can have many different presentatoons

eg $\langle t, s \mid ts^{-1} \rangle = \mathbb{Z} = \langle t \rangle$

and it's not always clear whether two presentations determine the same group

$$1 = \langle a \mid a \rangle = \langle a, b \mid a^{-1}b^2ab^{-3}, ba^{-1}b \rangle$$

This is the <u>isomorphism problem</u> in (combinatorial) group theary

The main result in this area is that there is no general algorithm that can take any two presentations and tell you whether or not they determine isomorphic groups (in finite time).

Group actions

So now we have a free group.
We want to think of it as a group of symmetries of a geometric object.

Here is some terminology

If X is a vector space, the set of invertible linear maps $X \to X$ forms a group $GL(X)$, with composition as its operation. If X has a finite basis, then $GL(X)$ = matrices, the operation = matrix multiplication

If X is a topological space, the set of homeomorphisms $X \to X$ forms a group $Homeo(X)$, with composition as its operation.

If X is a metric space, the set of isometries forms a group $Isom(X)$.

If X is a smooth manifold, the set of diffeomorphisms forms a group $Diff(X)$.

Let $Sym(X)$ denote one of these groups

Def A (left) action of G on X is a homomorphism
$G \xrightarrow{\rho} \mathrm{Symm}(X)$, ie $\rho(gh) = \rho(g) \circ \rho(h)$
"first apply h, then g"

ie: For each $g \in G$ there is a symmetry $X \longrightarrow X$, written $x \mapsto g \cdot x$, such that
$$gh \cdot x = g(h \cdot x) \quad \text{and} \quad 1_G \cdot x = x.$$

Notation a left action of G on X will often be denoted $G \curvearrowright X$.

Example $\langle t \rangle$ acts on $\mathbb{R}$ by isometries
$$\rho : \langle t \rangle \longrightarrow \mathrm{Isom}(\mathbb{R})$$
$$t^n \longmapsto (x \mapsto x + n)$$
This is a left action

$$\rho(t^n t^m) = \rho(t^{n+m}) : x \longmapsto x + (n+m)$$
$$\rho(t^n) \circ \rho(t^m) : x \longmapsto x + m \longmapsto x + m + n$$

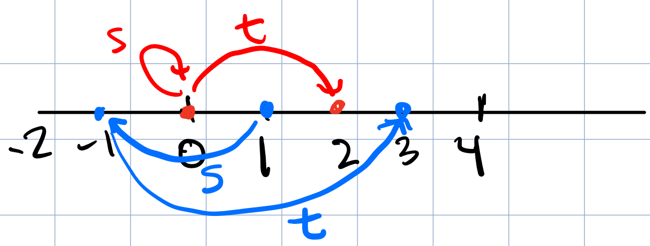
Example $D_\infty = \langle t, s \mid t^2, s^2 \rangle$ acts on $\mathbb{R}$

by isometries:

$t \longmapsto$ reflection in 0

$s \longmapsto$ reflection in 1

$ts \longmapsto$ translation $x \mapsto x+2$



$st \longmapsto$ translation $x \mapsto x-2$

D_∞ is the <u>infinite dihedral group</u>

<u>Example</u> $SL(2,\mathbb{Z})$

$$=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a,b,c,d \in \mathbb{Z},\ ad-bc=1\right\}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

acts on $\mathbb{R}^2$ by linear transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$$

This is a <u>left</u> action : ie $\rho : SL_2\mathbb{Z} \longrightarrow GL(\mathbb{R}^2)$ is a <u>homomorphism</u>

$$\rho(AB)(v) = \rho(A) \circ \rho(B)(v)$$

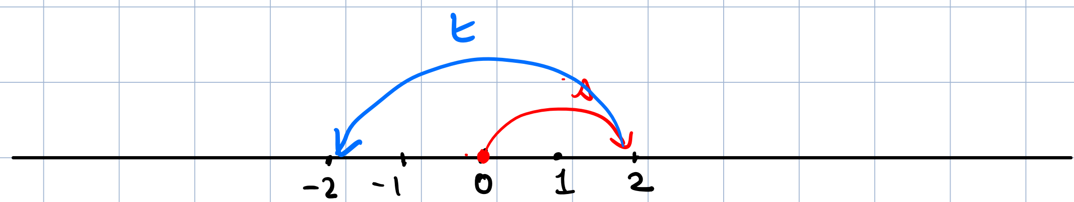
<u>First</u> apply B, then apply A.

(A **right action** of G on X is a map
$\rho: G \longrightarrow \mathrm{Sym}(X)$ such that
$\rho(gh) = \rho(h) \circ \rho(g)$ and $\rho(1_G) = \mathrm{id}_X$

ie ρ is an "anti-homomorphism"

ie for every $g \in G$, there is a symmetry $X \to X$
written $x \longmapsto x \cdot g$
satisfying $x \cdot 1_G = x$ and $x \cdot (gh) = (x \cdot g) \cdot h$
("first apply g, then h")

eg **right** action of $D_\infty = \langle s, t \mid s^2, t^2 \rangle$ on $\mathbb{R}$ given by
t = refl in 0 s = refl in 1 $st \longmapsto$ translation
$x \mapsto x - 2$



right action of $SL_2\mathbb{Z}$ on $\mathbb{R}^2$
$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{Z}$ sends

$$(x, y) \longmapsto (x, y)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ax + cy, bx + dy)$$

Suppose $G \curvearrowright X$ is a left action.

Let $x \in X$. The stabilizer $G_x = \{g \in G \mid gx = x\}$ (This is a subgroup).

eg $\langle t \rangle \curvearrowright \mathbb{R}$ $\quad G_x = 1$ for all x

$D_\infty \curvearrowright \mathbb{R}$ $\quad G_0 = \mathbb{Z}/2\mathbb{Z} = \{1, t\} \subseteq D_\infty$

$G_1 = \mathbb{Z}/2\mathbb{Z} = \{1, s\} \subseteq D_\infty$

$SL_2\mathbb{Z} \curvearrowright \mathbb{R}^2$ $\quad G_{(1,0)} = \left\{\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \,\middle|\, n \in \mathbb{Z}\right\}$

$G_{(0,0)} = SL_2\mathbb{Z}$!

The action is free if $G_x = \{1_G\}$ for all x

ie every point of X is "moved" by every group element.

(only $\langle t \rangle \curvearrowright \mathbb{R}$ above is free)

The action is faithful if $\rho : G \to \mathrm{Sym}(X)$ is injective

ie $g \cdot x = x$ for <u>all</u> $x \in X \Rightarrow g = 1_G$.

"the only group element that fixes every $x \in X$ is the identity"

$\mathbb{Z} \curvearrowright \mathbb{R}$, $D_\infty \curvearrowright \mathbb{R}$ and $SL_2\mathbb{Z} \curvearrowright \mathbb{R}^2$

are all faithful.

An action $G \curvearrowright X$ is proper if for every compact set $K \subset X$, $\{g \mid gK \cap K \neq \emptyset\}$ is finite.

$\mathbb{Z} \curvearrowright \mathbb{R}$, $D_\infty \curvearrowright \mathbb{R}$ are proper,

$SL_2\mathbb{Z} \curvearrowright \mathbb{R}^2$ is not.

An action $G \curvearrowright X$ is cocompact if there is a compact set $K \subset X$ whose translates cover X:

$$X = \bigcup_{g \in G} gK$$

eg $\langle t \rangle \curvearrowright \mathbb{R}^2$ is not cocompact

$t \mapsto [(x,y) \longmapsto (x+1, y)]$

$SL_2\mathbb{Z} \curvearrowright \mathbb{R}^2$ defined above is not cocompact

(homework)

If $G \curvearrowright X$ and $x \in X$, the orbit of x is $Gx = \{gx \mid g \in G\}$

eg $\underline{\mathbb{Z} \curvearrowright \mathbb{R}}$ The orbit of 0 is all integers

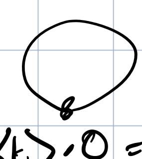
eg $\underline{D_\infty \curvearrowright \mathbb{R}}$ The orbit of 0 is all <u>even</u> integers

If $G \curvearrowright X$, define an equivalence relation on X by $x \sim x'$ if $x' \in Gx$.

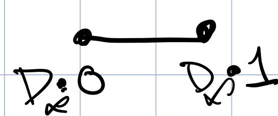
The orbit space (or quotient space) $G \backslash X$ is the set of equivalence classes, with the quotient topology

($U \subset G\backslash X$ is open $\iff$ $p^{-1}(U)$ is open in X where
$$p : X \longrightarrow G\backslash X$$
$$x \longmapsto G \cdot x$$
.)

eg quotient $\langle t \rangle \backslash \mathbb{R} \approx S^1$



quotient $D_\infty \backslash \mathbb{R} =$



<u>HW</u>: If an action is cocompact, the quotient space is compact.

<u>Next</u>: How do you find a space for G to act on?

FYI (information only)

Actions by isometries on metric spaces

(reference: M. Kapovich, A note on properly discontinuous actions, arXiv: 2301.05325)

Saying $G \backslash X$ is compact is a common way to define "cocompact action."
If X is a metric space and G acts by isometries, then this is equivalent to our definition.

But it is not always equivalent!

If X is a metric space and G acts by isometries, "proper" is equivalent to:

every $x \in X$ has a neighborhood U_x such that $\{g \mid gU_x \cap U_x \neq \emptyset\}$ is finite.

But it is not always equivalent!

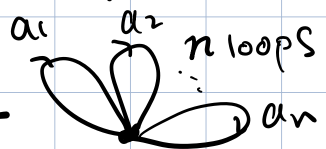
Examples can be found in the reference above.

Covering spaces (review)

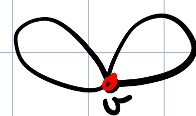
Here is one way to find a space for G to act on:

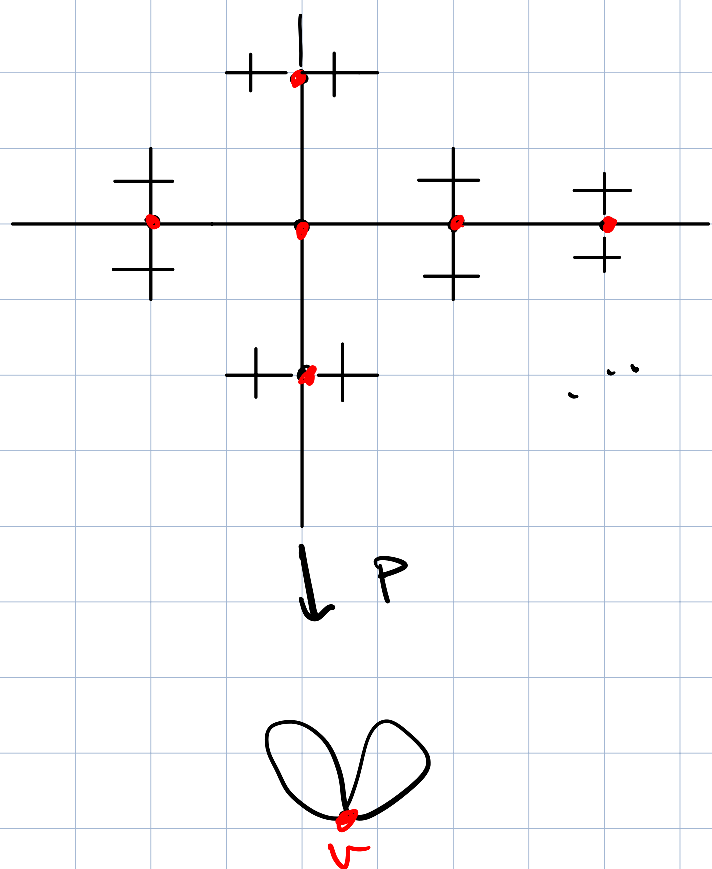
If $G=\pi_1(Y,b)$, then $G \curvearrowright \tilde{Y}$ by deck transformations

<u>eg</u> $G = F_n = F(a_1,\dots,a_n) \cong \pi_1(R_n)$, where

$R_n =$ 

so F_n acts on $T_n = \tilde{R}_n$.

Picture for $n=2$ $\qquad R_2 =$ 

$T =$ 

To define an action of $F\langle a,b\rangle$ on T:

1. Choose an isomorphism $F\langle a,b\rangle \xrightarrow{f} \pi_1(R_2, v)$
2. Choose a lift $\tilde{v} \in p^{-1}(v)$.
3. For $w \in F\langle a,b\rangle$, let γ_w be a loop at v representing $f(w)$

   Lift γ_w to a path $\tilde{\gamma}_w$ in T starting at $\tilde{v}$.
   The endpoint of $\tilde{\gamma}_w$ is $w \cdot \tilde{v}$.
4. For any other $x \in T$, choose a path σ from $\tilde{v}$ to x, and a lift $\tilde{\sigma}$ of $p(\sigma)$ starting at $w \cdot \tilde{v}$.

   The endpoint of $\tilde{\sigma}$ is $w \cdot x$.

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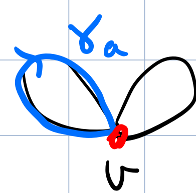
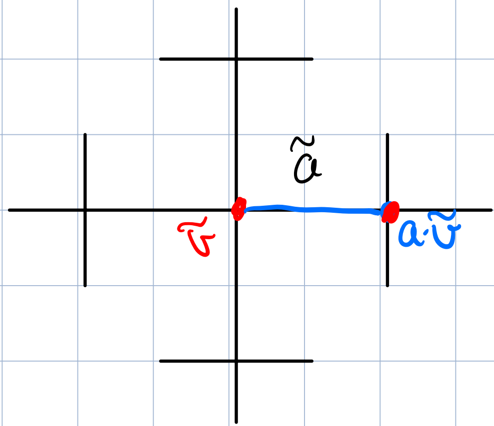
eg If $F(a,b) \longrightarrow \pi_1(R_2, v)$

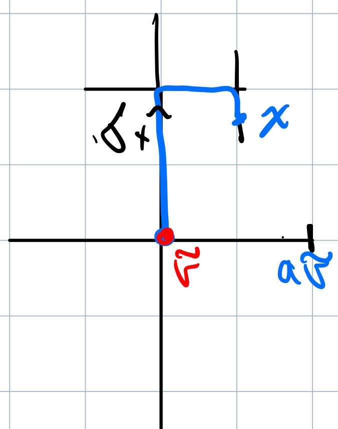
sends $a \longmapsto$ 

lift γ_a to a path in T starting at $\tilde{v}$.

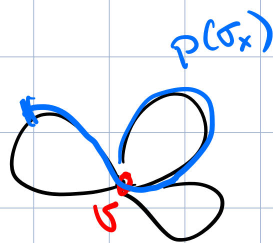


• The endpoint of the path is $a\cdot\tilde{v}$.

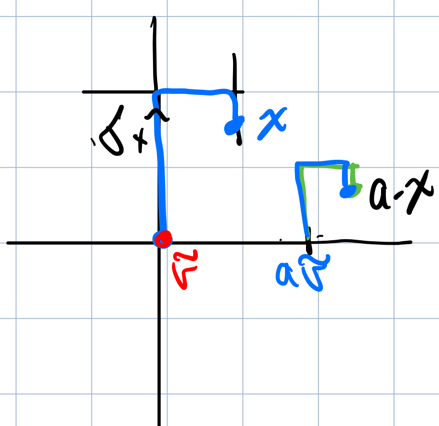
• For arbitrary $x \in T$:
choose a path σ_x from $\tilde{v}$ to x



Then $p(\sigma_x)$ is a path in R_n starting at v



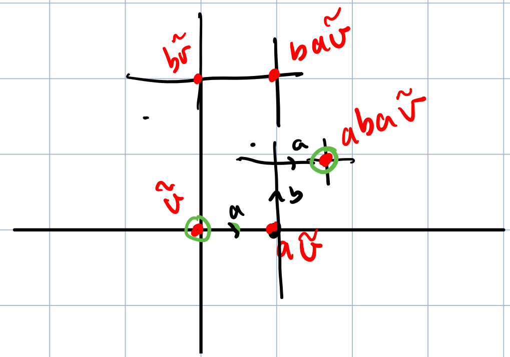
Lift $p(\sigma_x)$ to a path starting at $a\cdot\tilde{v}$ instead of $\tilde{v}$. The endpoint is $a\cdot x$



ie a shifts the whole tree to the right

Similarly, b shifts the whole tree <u>up</u>.

$aba \cdot \tilde{v}$



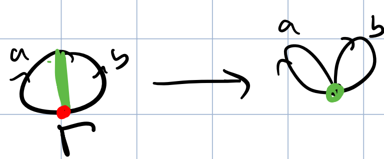
This is a free, faithful, proper and cocompact left action.

<u>Exercise</u>: ① What happens if you pick a different isomorphism $F(a,b) \to \pi_1(R_2, v)$?

② What if you pick a different lift of $\tilde{v}$. Is the action on T the same?

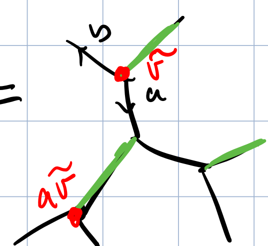
F_2 acts on many different trees.

<u>eg</u>



so $\pi_1(\Theta) \cong F_2 = F(a,b)$

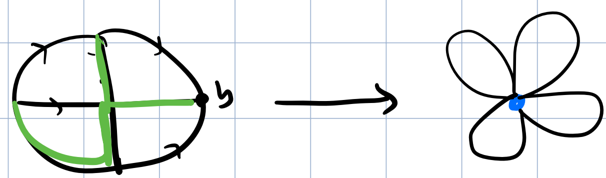
so $F\langle a,b \rangle \circlearrowright \tilde{\Gamma} =$



Every graph Γ has a maximal tree τ. (possibly several)
Collapsing τ to a point is a homotopy equivalence

$$\Gamma \longrightarrow \Gamma/\tau \simeq R_S$$

where R_S is a rose with a petal for each edge of Γ that is not in τ



so $\pi_1(\Gamma, b) \cong F(S)$

The universal cover $\tilde{\Gamma}$ is a tree, and $F(S)$ acts on it by deck transformations.

Galois correspondence for $G = \pi_1(Y, b)$

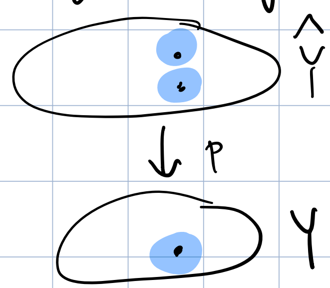
Theorem (Galois correspondence) There is a one-to-one correspondence

$$\text{Subgroups of } \pi_1(Y, b) \longleftrightarrow \text{Covering spaces of } Y$$

<u>idea of proof</u>.

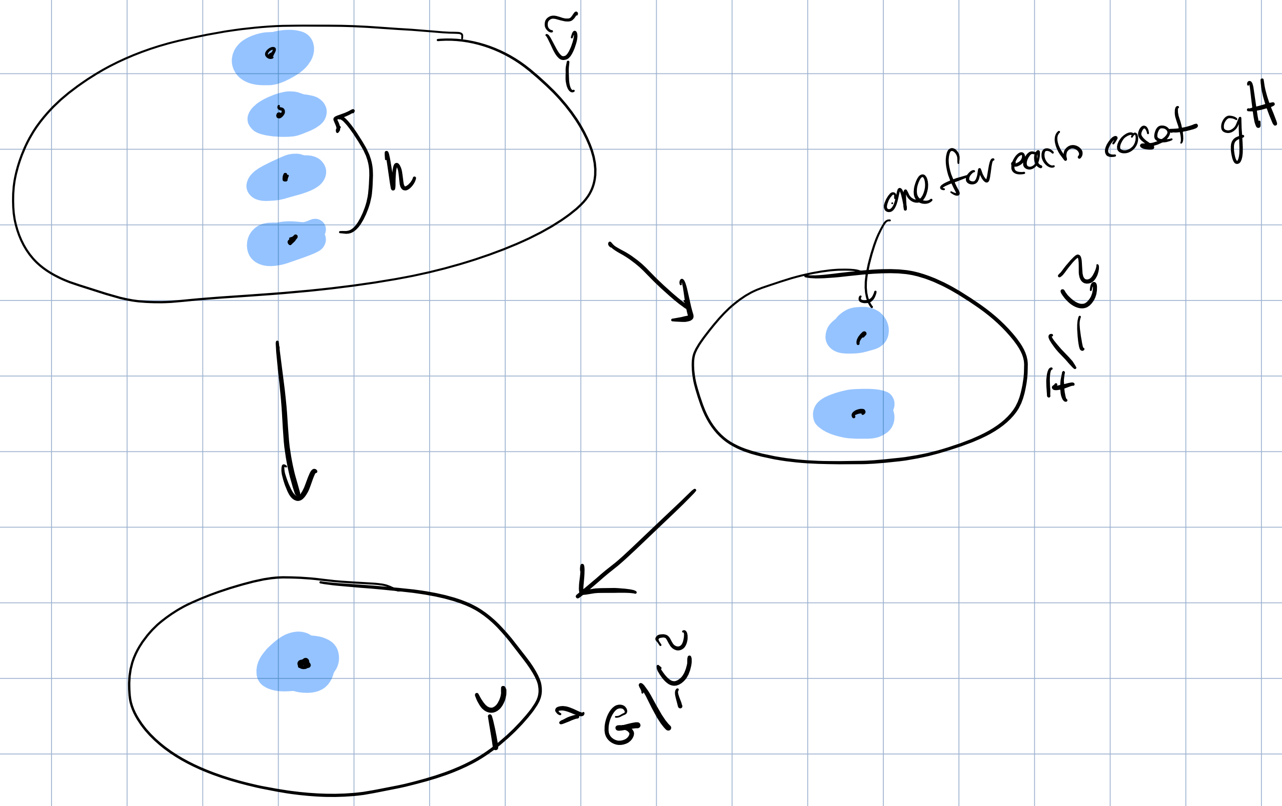
① Given a covering space $\hat{Y} \xrightarrow{p} Y$, $p_*: \pi_1(\hat{Y}, \hat{v}) \to \pi_1(Y, p(\hat{v}))$ is injective, so the image

$p_* \pi_1(\hat{Y}, \tilde{v})$ is a subgroup of $\pi_1(Y, v)$.



② If $H < G$ then the action of G on the universal cover $\tilde{Y}$ of Y restricts to an action of H. Since the action of G on $\tilde{Y}$ is free and proper, so is the action of H.

This implies that $\tilde{Y} \to H \backslash \tilde{Y}$ is a covering space, and so is $H \backslash \tilde{Y} \to G \backslash \tilde{Y}$ (exercise!)



$\tilde{Y}$ is 1-connected $\Rightarrow$ $\tilde{Y}$ = universal cover of $H\backslash\tilde{Y}$

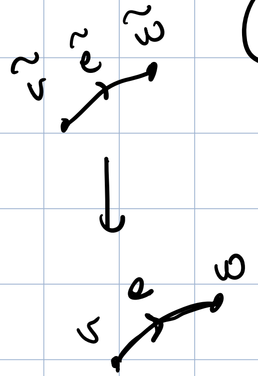
$\Rightarrow \pi_1(H\backslash\tilde{Y}) = H$. ✓

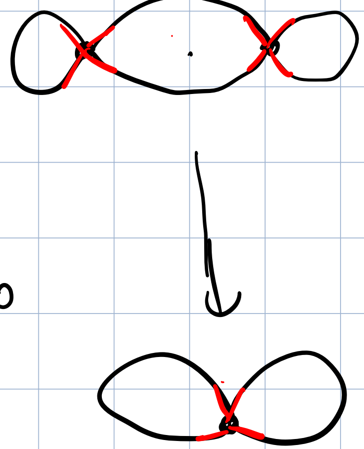
Back to free groups:

<u>Corollary</u> Every subgroup of a free group is a free group.

<u>Proof:</u> $F(S) = \pi_1(R_S, v)$

If $H < F(S)$, then $H = \pi_1(Z, \tilde{v})$ for some covering space Z of R_S.





A covering space of a graph is a graph

Vertices $= p^{-1}(\text{vertices})$

edges $= p^{-1}(\text{edges})$

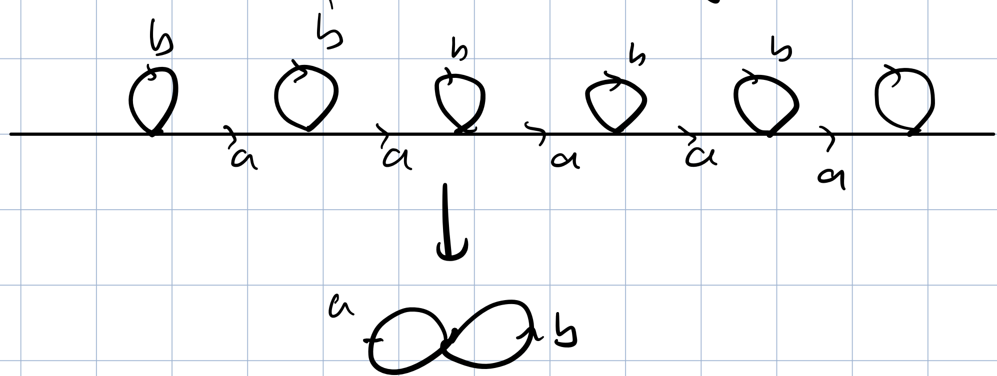
$e =$ edge v to w. Lift v to $\tilde v$.

lift e to $\tilde e$ starting at $\tilde v$. (ending at $\tilde w$)

$\therefore$ $\pi_1 Z$ is free ✓

Note: even if S is finite, a subgroup $H < F(S)$ might not be finitely generated.

eg $Z =$



is a covering space

$\pi_1(Z)$ is freely generated by $\{a^n b a^{-n}\}$

Exercise Prove that the commutator subgroup $[F_2, F_2]$ is not finitely generated.

In our example, $F(a,b) \cong \pi_1(R_2, v)$
the universal cover T is a tree, so is contractible.

Not all universal covers are contractible (eg the universal cover of the projective plane is S^2)

<u>But</u>: If $\tilde{Y}$ is contractible then a theorem of Hurewicz implies that
$G = \pi_1 Y$ determines Y up to homotopy
In particular, homotopy invariants of Y, such as homology and cohomology groups $H_i(Y)$, $H^i(Y)$ are invariants of G

These are also important in geometric group theory, but beyond the scope of this course.

Free abelian groups:

Let $F(S)$ be the free group on S
an element of the form $uvu^{-1}v^{-1}$ is called a commutator, denoted $[u,v]$

The commutator subgroup $[F(S),F(S)]$ is the subgroup generated by all commutators

We have $w[u,v]w^{-1} = [wuw^{-1}, wvw^{-1}]$, so $[F(S),F(S)]$ is normal.

The quotient $F(S)/[F(S),F(S)]$ is the free abelian subgroup on S. It has presentation

$$\langle S \mid \{[u,v] \mid u,v \in F(S)\} \rangle$$

It has the following universal property

* Any set map $S \longrightarrow A$ to an <u>abelian</u> group A extends to a unique homomorphism $F(S)/[F,F] \longrightarrow A$

(This follows from one of your homework problems)

<u>Claim</u>: $F(S)/[F,F] \cong \mathbb{Z}^{|S|}$

<u>proof</u> (for $S = \{a, b\}$)

Map $S \to \mathbb{Z}^2$

$$a \longmapsto (1,0)$$
$$b \longmapsto (0,1)$$

By the universal property, this extends to a unique homomorphism

$$F(S) \longrightarrow \mathbb{Z}^2$$
$$a^{n_1} b^{m_1} \cdots a^{n_k} b^{m_k} \longmapsto (\Sigma n_i, \Sigma m_i)$$

Surjective $\quad a^n b^m \longmapsto (n, m)$ ✓

$[u,v] \subset \ker.$ ✓

<u>Claim</u> $[u,v] = \ker$, ie any $w \in \ker$ can be written as a product of commutators.

proof

$w \in \ker \Leftrightarrow w = a^{n_1} b^{m_1} \dots a^{n_k} b^{m_k}$ $\sum n_i = 0 = \sum m_i$

Induct on word length of w

$l=4 \Rightarrow aba^{-1}b^{-1}$, $bab^{-1}a^{-1}$ or some conjugate.

$l>4$ wlogwma w starts with a

There must be a^{-1} somewhere in w,

so $w = aua^{-1}v$

$= (aua^{-1}u^{-1})(uaa^{-1}v)$

$= [a,u]\, uv$

Now uv is a shorter word, the sum of the exponents of a and b are still 0, so we can write uv as a product of commutators.

With more work (of a similar nature) can show

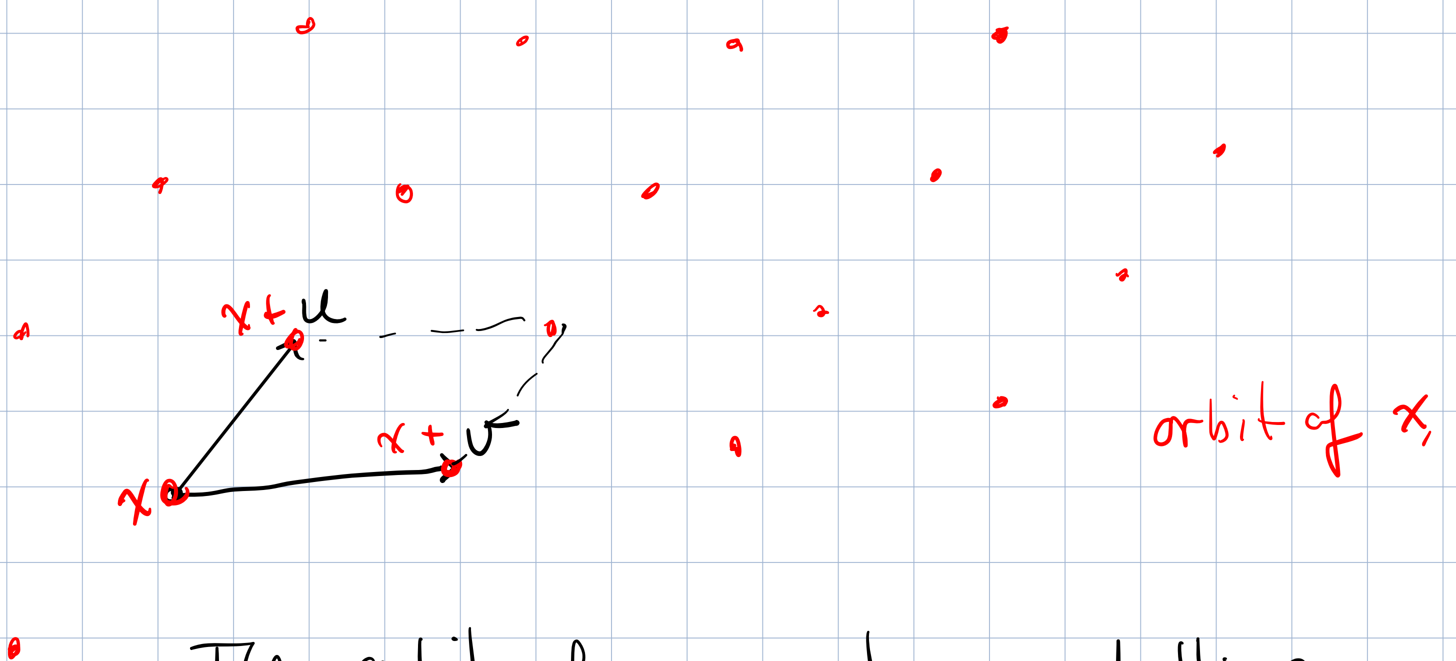
$$\mathbb{Z}^2 = \langle a, b \mid [a,b] = 1 \rangle$$

Now $\pi_1(T^2, v) \cong \mathbb{Z}^2$ and $\widetilde{T}^2 \cong \mathbb{R}^2$

$\mathbb{Z}^2$ acts on $\mathbb{R}^2$ by (independent) translations (in many different ways):

Pick two independent vectors (u, v)

a sends $x \mapsto x+u$
b sends $x \mapsto x+v$



The orbit of a point is a <u>lattice</u>

Copies of the parallelogram spanned by u and v tile the plane

The quotient is a torus

Ping-Pong

We've seen how a free proper action of G on a nice enough space Y can be used to give information about G.

But there are many actions on nice spaces that are not free. Can we still get information about G?

Here's one way:

Given an action of G on a space X (of any sort), and two elements $a, b \in G$, there is a criterion called the Ping-Pong lemma which will prove that the subgroup $\langle a,b \rangle$ of G generated by a and b is free.

This is often used to prove that G contains a (non-abelian) free group.

It suffices to prove that reduced words in a and b are never the identity in G, since then the homomorphism $F(a,b) \longrightarrow \langle a,b \rangle$ extending $\{a,b\} \longrightarrow \langle a,b \rangle$ is both surjective <u>and</u> injective.

$\boxed{\text{Ping-Pong Lemma}}$ Let $G \curvearrowright X$ and $a, b \in G$. Suppose X contains subsets A, B, $B \not\subset A$ such that

$$\left.\begin{aligned} a^n B &\subset A \\ \text{and } b^n A &\subset B \end{aligned}\right\} \text{ for all } n \in \mathbb{Z} \setminus \{0\}$$

Then the subgroup generated by a and b is a free group.

__Proof:__ Suppose $w = a^{n_1} b^{n_2} a^{n_3} \cdots b^{n_{k-1}} a^{n_k} = id_G$, with all $n_i \in \mathbb{Z} \setminus \{0\}$ (ie w is a reduced word starting and ending with a power of a.)

Let $x \in B \setminus A$. Then $w \cdot x \in A$, so $wx \neq x$, so $w \neq id_G$.

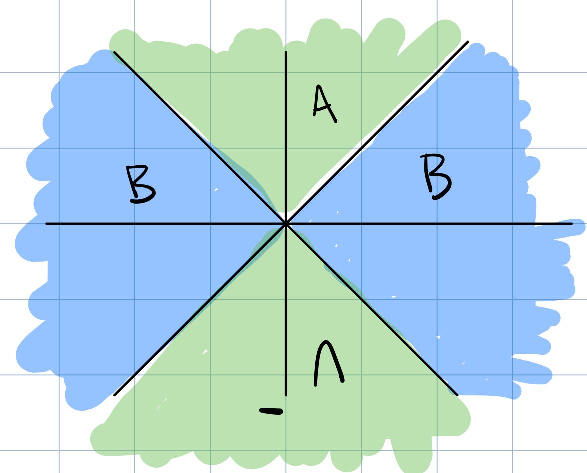
If w starts or ends with b, then for large enough N, $a^N w a^{-N}$ starts __and__ ends with a, so $a^N w a^{-N} \neq id_G$, so $w \neq id_G$.

$\blacksquare$

__Example__ $G = SL(2, \mathbb{Z})$, $a = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$

Use standard linear action of $G \curvearrowright \mathbb{R}^2$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

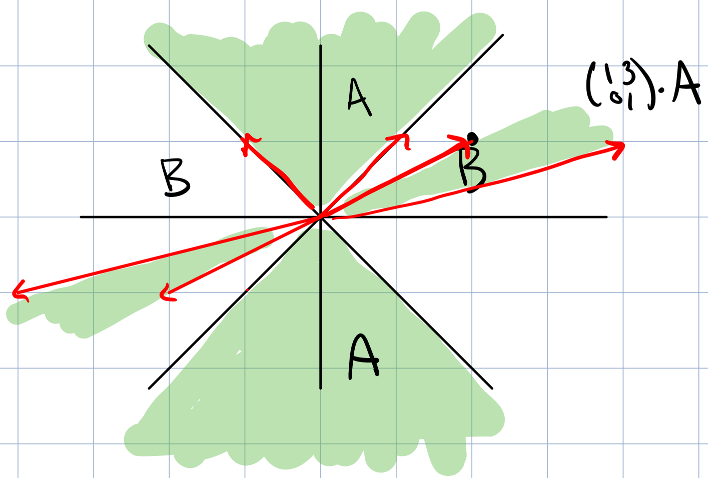


__Claim__

$$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 3n \\ 0 & 1 \end{pmatrix} \cdot A \subseteq B$$

$$\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ 3n & 1 \end{pmatrix} \cdot B \subseteq A$$

eg $\begin{pmatrix}1 & 3n\\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}3n+1\\1\end{pmatrix}$ $\begin{pmatrix}1 & 3n\\ 0 & 1\end{pmatrix}\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}-1+3n\\1\end{pmatrix}$



$n=1$ $\begin{pmatrix}-1+3\\1\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$

$\begin{pmatrix}3+1\\1\end{pmatrix} = \begin{matrix}4\\1\end{matrix}$