One more example of a group action

$SL(2,\mathbb{Z})$ again, but action on $\mathbb{H}^2$ instead of $\mathbb{R}^2$

$\mathbb{H}^2 =$ upper half-plane $= \{ z \mid \mathrm{Im}(z) > 0 \}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$ acts by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \dfrac{az+b}{cz+d}$

__Check__: $z = x+iy$, $y>0$

$$\Rightarrow \mathrm{Im}\left(\frac{az+b}{cz+d}\right) = \mathrm{Im}\left(\frac{a(x+iy)+b}{c(x+iy)+d}\right)$$

$$= \mathrm{Im}\left(\frac{ax+b+i\cdot ay}{cx+d+i\cdot cy}\right)\left(\frac{cx+d-i\cdot cy}{cx+d-i\cdot cy}\right)$$

denominator is real >0 $[(u+iv)(u-iv) = u^2+v^2]$
Imaginary part of numerator is

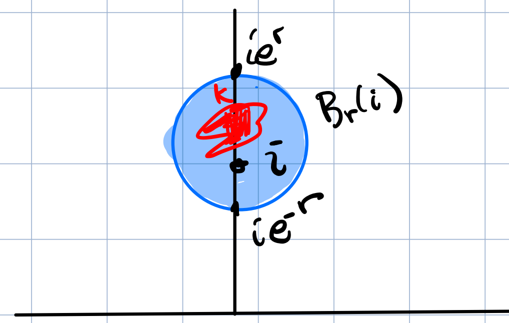
$$-(ax+b)cy + ay(cx+d) = (ad-bc)\cdot y = y > 0$$

This is one of the most famous group actions in mathematics

This action is not free: $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot i = \dfrac{0\cdot i+1}{-1\cdot i+0} = \dfrac{-1}{i} = i$

It is not faithful! $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \cdot z = \dfrac{-z}{-1} = z$

It is not cocompact

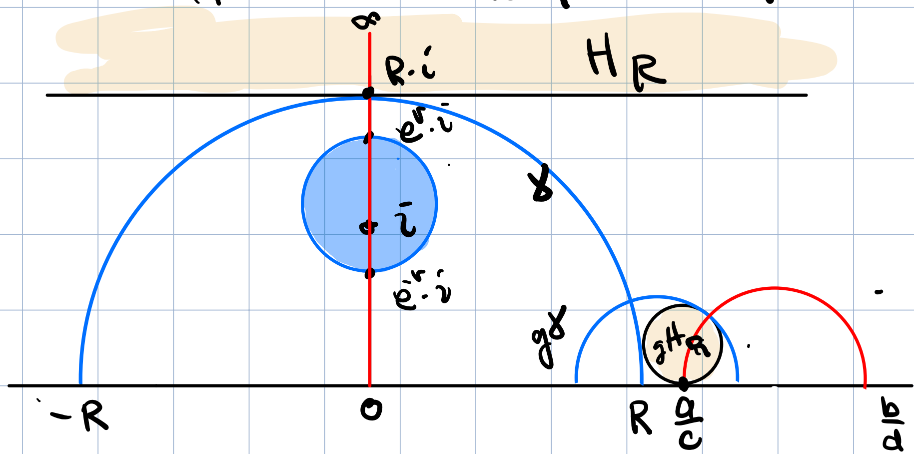


K compact $\subseteq B_r(i)$ for some r

So suffices to show translates $g \cdot B_r(i)$ don't cover $\mathbb{H}^2$.

The top of $B_r(i)$ is at $i \cdot e^r$, the bottom is at $i \cdot e^{-r}$

Look at the horoball $\{z \mid Im(z) > R\} = H_R$ for $R \gg e^r$



Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{Z}$.

If $c = 0$, $g^{-1} \cdot H_R = H_{R'}$ so $g^{-1} H_R \cap B_r(i) = \emptyset$

so $g H_R \cap B_r = \emptyset$ ✓

Now assume $c \neq 0$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \frac{1}{0} = \frac{a}{c}$

so $g \cdot H_R$ is a horosphere at $\frac{a}{c}$

H_R lies above the geodesic γ from $-R$ to R, so $g \cdot H_R$ lies <u>below</u> the geodesic from $g(-R)$ to gR.

<u>Claim</u> the highest point of $g \cdot \gamma$ has imaginary part

$$\frac{1}{c^2R - \frac{d^2}{R}}$$

<u>Proof of Claim</u>: We want to calculate the (Euclidean) radius of the geodesic, ie $\frac{1}{2}\left|g \cdot R - g\cdot(-R)\right|$

$$\frac{1}{2}\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}(-R) - \begin{pmatrix} a & b \\ c & d \end{pmatrix}(R)\right] = \left[\frac{a(-R)+b}{c(-R)+d} - \frac{aR+b}{cR+d}\right]\cdot\frac{1}{2}$$

$$= \frac{1}{2}\left[\frac{(b-aR)(d+cR) - (b+aR)(d-cR)}{-c^2e^{2r}+d^2}\right]$$

$$= \frac{1}{2}\left[\frac{\cancel{bd} + (bc-ad)R - \cancel{acR^2} - \cancel{bd} + (bc-ad)R + \cancel{acR^2}}{-c^2e^{2r}+d^2}\right]$$

$$= \frac{1}{2}\left[\frac{-2(ad-bc)R}{d^2-c^2R^2}\right] = \frac{R}{c^2R^2-d^2}$$

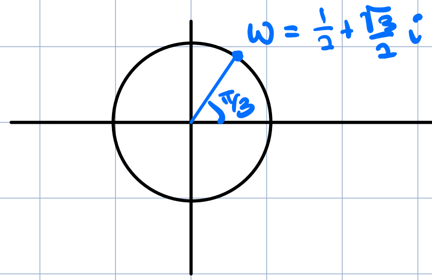
$$= \frac{1}{c^2R - \frac{d^2}{R}} \checkmark$$

Since $c^2 \geq 1$, if R is sufficiently large then $\dfrac{1}{c^2 R - \frac{d^2}{R}} < \dfrac{1}{e^r}$

So $B_r(i)$ does not intersect $g \cdot H_R$.

So $g^{-1} B_r(i)$ does not intersect H_R for <u>any</u> g, so translates of $B_r(i)$ do not cover $\mathbb{H}^2$.

The action $\underline{\text{is}}$ proper and stabilizers are small:



$\omega^6 = 1$

$G_z = 1$ unless $z = g \cdot i$ or $g \cdot \omega$

$\operatorname{stab}(i) = \left\langle \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle \cong \mathbb{Z}/4\mathbb{Z}$ $\quad \operatorname{stab}(\omega) = \left\langle \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle \cong \mathbb{Z}/6\mathbb{Z}$

<u>Aside</u>: If $G \curvearrowright X$ and $x \in X$, then

$$\operatorname{stab}(gx) = g(\operatorname{stab} x) g^{-1}.$$

<u>pf</u>: $\supseteq$ $h \in \operatorname{stab} x \Rightarrow ghg^{-1} \cdot (gx) = ghx = gx$,
i.e. $ghg^{-1} \in \operatorname{stab}(gx)$

$\subseteq$ $h \in \operatorname{stab} gx \Rightarrow hgx = gx \Rightarrow g^{-1}hgx = x$
$\Rightarrow g^{-1}hg \in \operatorname{stab} x$

✓

What makes this action especially interesting and useful is that it preserves the hyperbolic metric, ie acts by <u>isometries</u>

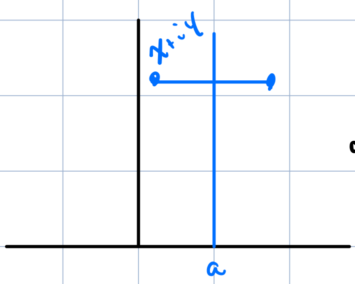
To prove this, you just need to check that generators for $SL(2, \mathbb{Z})$ act by isometries.

<u>Claim</u> $SL(2,\mathbb{Z})$ is generated by $\begin{pmatrix}1&1\\0&1\end{pmatrix}$ and $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$

(proof: Euclidean algorithm: - these matrices do row and column operations, which can reduce $\begin{pmatrix}a&b\\c&d\end{pmatrix}$ to $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ since $ad-bc=1$

$\Rightarrow$ $GCD(a,c) = GCD(b,d) = 1$.

<u>Recall</u>: • Geodesics in $\mathbb{H}^2$ are circles and lines perpendicular to the real axis

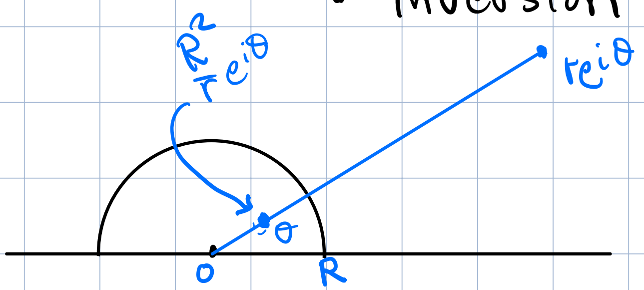


• Reflection in a vertical line is an isometry

$x+iy \longleftrightarrow -x+2a+iy$

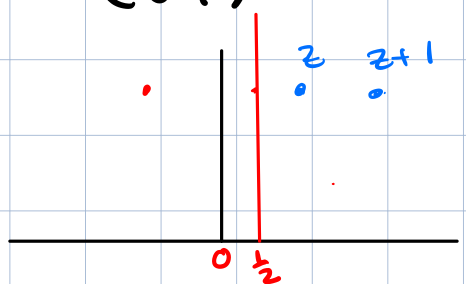
$z \longleftrightarrow -\bar{z}+2a$

• inversion in a circle is an isometry



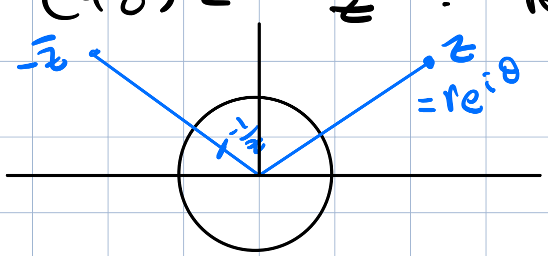
$re^{i\theta} \longleftrightarrow \frac{R^2}{r}e^{i\theta}$

$\begin{pmatrix}1&1\\0&1\end{pmatrix}z = z+1$: horizontal translation by 1.

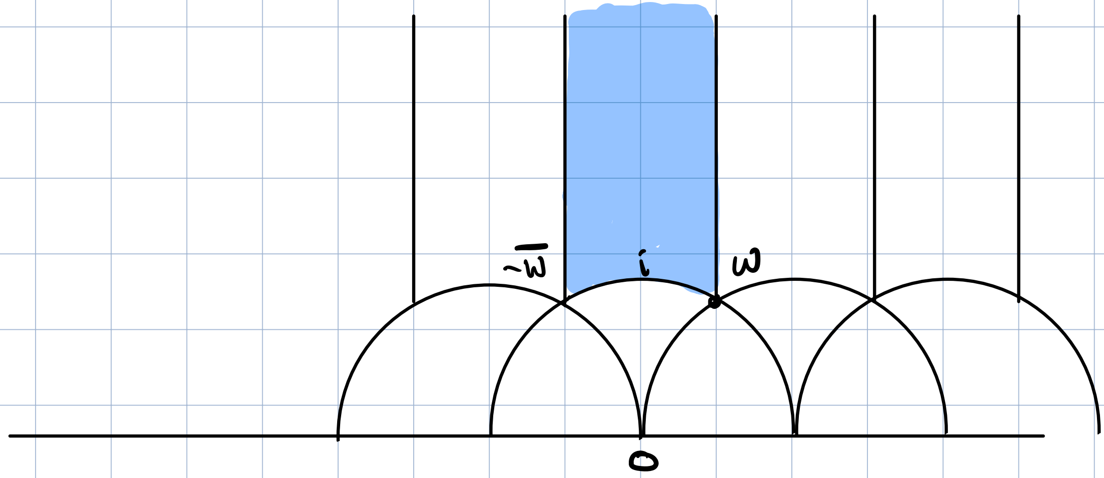


= product of reflections in imaginary axis and vertical line $Re(z) = \frac{1}{2}$

$\begin{pmatrix}0&1\\-1&0\end{pmatrix}z = -\frac{1}{z}$: reflection in imaginary axis followed by inversion in circle centered at 0 radius 1,



Translates of the blue area cover $\mathbb{H}^2$
(slightly tricky exercise).



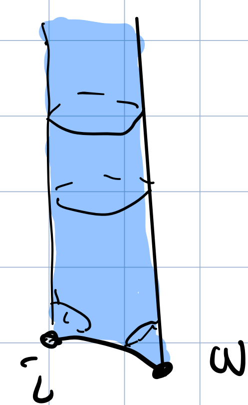
The left side is identified with the right by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

The circle arc from i to w is identified with the circle arc from i to $-\bar{w}$ by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

There are no other identifications

so the orbit space is

$$SL(2,\mathbb{Z}) \backslash \mathbb{H}^2 =$$



From now on we will be mostly interested in stuying finitely generated groups acting on metric spaces by isometries

If G is finitely generated by S, we can define a metric on G using S
then G acts on itself by multiplication, which is an isometry.

<u>Details</u>: For $g \in G$, define $l_S(g)$ = length of shortest word in S representing g. and $d_S(g,h) = l_S(g^{-1}h)$. We assume $1 \notin S$, so $l_S(1) = 0$.

<u>Check</u>: d is a metric

- $d_S(g,g) = l_S(g^{-1}g) = l_S(1) = 0$ ✓
- $d_S(g,h) = l_S(g^{-1}h) = l_S(h^{-1}g) = d_S(h,g)$ ✓
- $$\begin{aligned} d_S(g,h) + d_S(h,k) &= l_S(g^{-1}h) + l_S(h^{-1}k) \\ &\geq l_S(g^{-1}hh^{-1}k) = l_S(g^{-1}k) \\ &= d_S(g,k) \end{aligned}$$ ✓

<u>Action</u> $g \cdot h = gh$

The action preserves the metric:

$$d(gh, gk) = l(h^{-1}g^{-1}gk) = l(h^{-1}k) = d(h,k)$$ ✓

eg $G = \langle t \rangle$ $\quad t^{-1} \quad 1 \quad t \quad t^2 \quad t^3 \quad t^4$

$$t^k \cdot t^n = t^{k+n}$$

Note a different generating set S gives a <u>different</u> metric:

<u>eg</u>:
G is also generated by $S' = \{t^2, t^3\}$ $\quad (t^3 \cdot (t^2)^{-1} = t)$
so $d_{S'}(1, t^2) = d_{S'}\{1, t^3\} = 1$, $d_{S'}(t) = 2$.

Cayley Graphs

Let G be finitely generated by S $(1 \notin S)$

The <u>Cayley graph</u> $\mathcal{C}(G,S)$ is the graph

with
- vertices = elements of G
- an edge from g to gs for every $s \in S$.

<u>Notes</u>:
- $g \longrightarrow gs$ and $g \longleftarrow gs^{-1}$ are edges
- $1 \notin S \Rightarrow$ there are no loops

So all vertices have an incoming edge and outgoing edge for each $s \in S$.

<u>eg</u> $G = \langle t \rangle$, $S = \{t\}$

$$\mathcal{C}(G,S) = \cdots t^{-2} \xrightarrow{t} t^{-1} \xrightarrow{t} 1 \xrightarrow{t} t \xrightarrow{t} t^{2} \xrightarrow{t} t^{3} \cdots$$

G <u>acts</u> on $\mathcal{C}(G,S)$ (on the left) by:

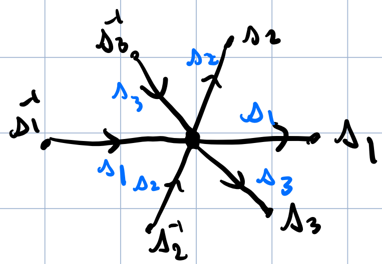
If $h \in G$ $h \cdot g = hg$, and if e is the edge $g \xrightarrow{s} gs$ then $h \cdot e$ is the edge $hg \xrightarrow{s} hgs$

This action is <u>free</u>: If g is a vertex and $h \in G_g$ then $hg = g$, which implies $h = 1$.

If e is an edge $g \xrightarrow{s} gs$ and $h \in \text{stab}(e)$ then $hg = g$ and $hgs = gs$, so again $h = 1$.

The action is <u>cocompact</u>:

Let $st(1) = 1$ plus all edges adjacent to 1

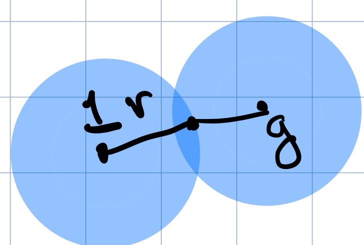


The translates of $st(1)$ cover $\mathcal{C}(G,S)$.

The <u>quotient</u> by the action is the rose with one petal for each generator

The action is <u>proper</u>: If $K \subset \mathcal{C}(G,S)$ is compact, $K \subset B_r(1)$ for some r

If $g B_r(1) \cap B_r(1)$ is not empty, then $g \cdot B_r(1)$ is centered at g, with $d(1,g) \leq 2r$



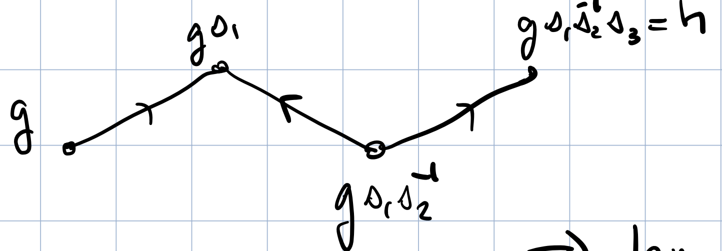
There are only finitely many such g, so $gB_r(1) \cap B_r(1) \neq \emptyset$ for only finitely many g.

To make $\mathcal{C}(G,S)$ a metric space,

- Define the distance between two vertices as the length of the shortest edge-path
- Make the interior of each edge isometric to $(0,1) \subseteq \mathbb{R}$.
- If $x \in e_x$ and $y \in e_y$,

  define $d(x,y)$ = length of shortest path from x to a vertex of e_x to a vertex of e_y to y

If g, h are vertices (= elements of G) then



$\Rightarrow s_1 s_2^{-1} s_3 = g^{-1}h$

$\Rightarrow$ length of shortest edge-path from g to h $= \ell_S(g^{-1}h) = d_S(g,h)$

$G \curvearrowright \mathcal{C}(G,S)$ preserves distance between vertices, and $\therefore$ length of paths, so distances between any points.

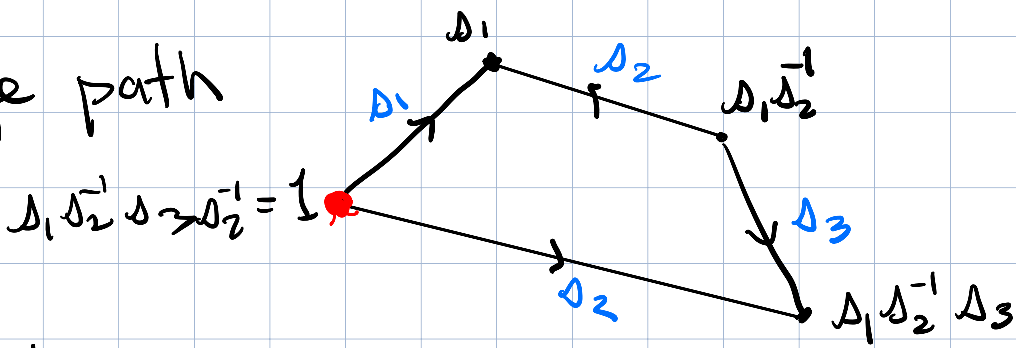
Suppose a product of generators and their inverses is the identity in G

$$s_1^{\varepsilon_1} s_2^{\varepsilon_2} \cdots s_k^{\varepsilon_k} = 1$$

Note $g \xleftarrow{s_1} g s_1^{-1}$ is an edge.

Eg suppose $s_1 s_2^{-1} s_3 s_2^{-1} = 1$

Then the edge path



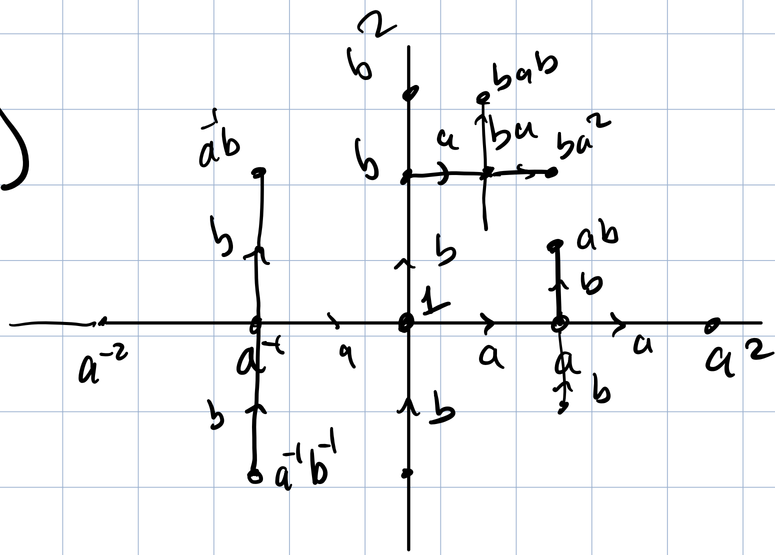
ends at 1,
so is a loop.

Loops in $\mathcal{C}(G,S)$ correspond exactly to elements in $\ker(F(S) \longrightarrow G)$

ie to <u>relations</u> in G wrt S.

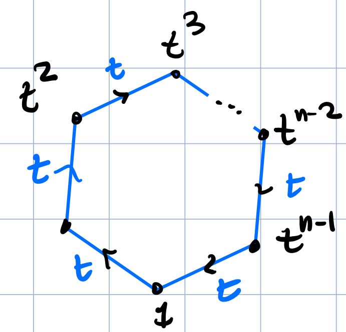
Some more examples:

eg $\mathcal{C}(F_2, \{a,b\})$

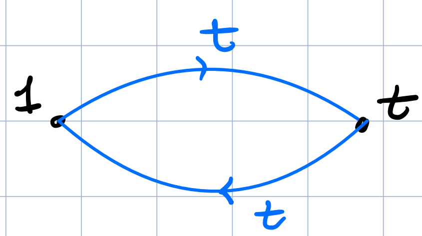


eg G = cyclic group of order n $= \langle t \mid t^n \rangle$, $S = \{t\}$

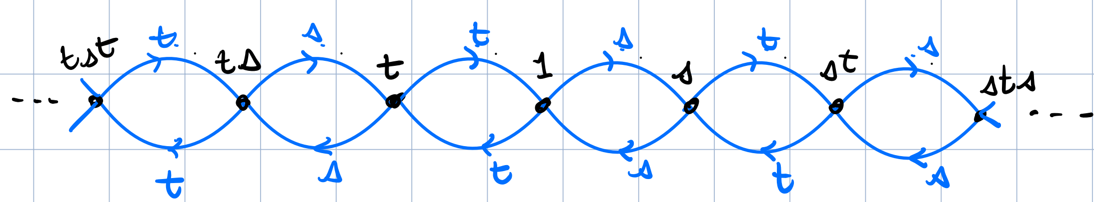
$\mathcal{C}(\langle t \mid t^n \rangle, \{t\}) =$



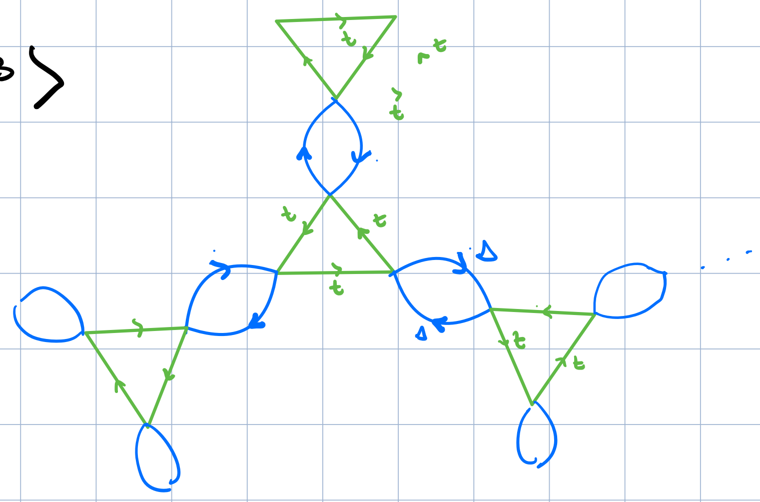
eg $\mathcal{C}(\langle t \mid t^2 \rangle, \{t\})$



eg Infinite dihedral group $\langle t, s \mid t^2, s^2 \rangle$
$S = \{s, t\}$

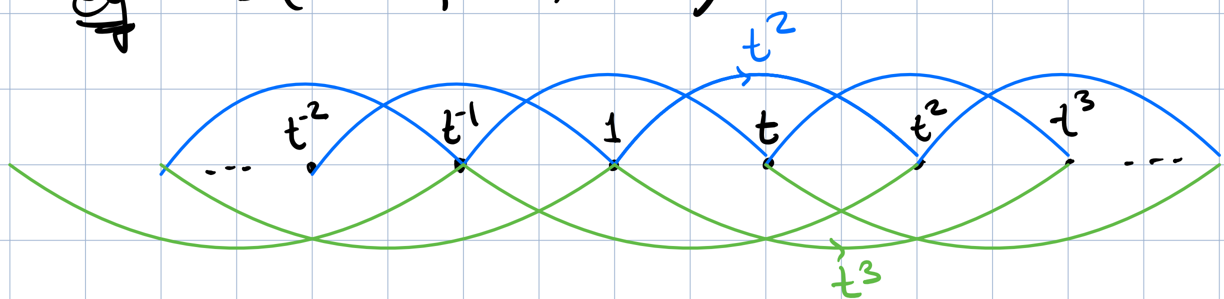


eg $\langle s, t \mid s^2, t^3 \rangle$



Different generating sets can produce non-isomorphic graphs

eg $\mathcal{C}(\langle t \rangle, \{t^2, t^3\})$



<u>and</u> non-isomorphic groups can have isomorphic <u>unlabeled</u> Cayley graphs.

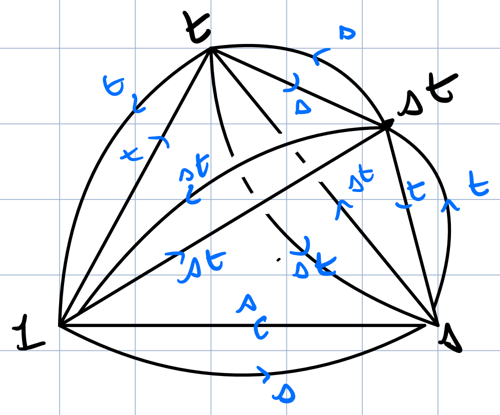
eg

$G = \mathbb{Z}/_2 \times \mathbb{Z}/_2$, $\mathbb{Z}/4$ both have 4 elements

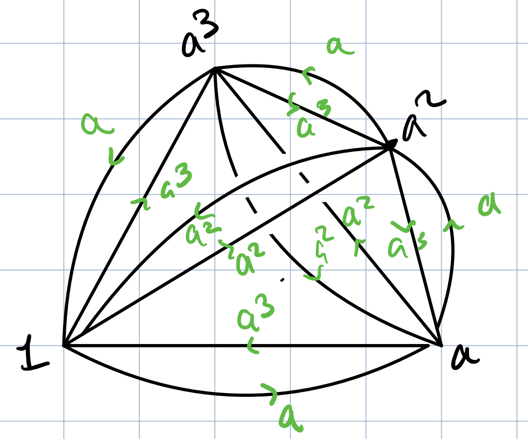
If you choose $S =$ <u>all</u> non-trivial elements of G

Then $\mathcal{C}(G, G)$ is the complete graph on 4 vertices, doubled.

$\mathbb{Z}/2 \times \mathbb{Z}/2 = \{1, s, t, st\}$



$\mathbb{Z}/4 = \{1, a, a^2, a^3\}$



But, if you include labels, the Cayley graph uniquely determines G.

<u>Prop</u> if there is a bijection $S \to S'$ and a (directed) graph isomorphism

$$\mathcal{C}(G,S) \to \mathcal{C}(G',S')$$

Then the map $S \to S'$ extends to an isomorphism $G \to G'$

<u>Proof</u> $G = \langle S \mid R \rangle$. Then

Elements of $R \longleftrightarrow$ loops in $\mathcal{C}(G,S)$

$\longleftrightarrow$ loops in $\mathcal{C}(G',S')$

$\longleftrightarrow$ words in $F(S')$ that are trivial in G',

$\therefore$ The map $G = F(S)/\langle\langle R\rangle\rangle \longrightarrow G'$

extends to a homomorphism (which is surjective since $S \twoheadrightarrow S'$.)

Similarly, get a surjective homomorphism

$$G' \longrightarrow G$$

Composition is identity (both ways) so both maps are also injective! ✓

We have

- $\mathcal{C}(G,S)$ is a metric space and $G \curvearrowright \mathcal{C}(G,S)$ by isometries : $d(gx,gy) = d(x,y)$

- $(G,d_s) \hookrightarrow \mathcal{C}(G,S)$ preserves distance $l_S(v^{-1}w) = d_S(v,w) = d_{\mathcal{C}}(v,w)$

Now we want to use the geometry of $\mathcal{C}(G,S)$ to study G - eg use $\mathcal{C}(G,S)$ as a ping-pong table to find free subgroups of G

<u>**Problem**</u>: Suppose G acts on two <u>different</u> metric spaces $G \curvearrowright X_1$, $G \curvearrowright X_2$. How is the geometry of X_1 related to the geometry of X_2?

Example 1 Let S, S' be two different finite generating sets for G

$$\left.\begin{array}{l} G \curvearrowright \mathcal{C}(G,S) \\ G \curvearrowright \mathcal{C}(G,S') \end{array}\right\} \text{ free, proper, cocompact}$$

Example 2 Let $G = \mathbb{Z}^2$, $S = \{e_1, e_2\}$

$$\left.\begin{array}{l} G \curvearrowright \mathcal{C}(G,S) = \text{unit grid} \\ G \curvearrowright \mathbb{R}^2 \quad \text{by translations} \end{array}\right\} \text{ free, proper, cocompact}$$

So X_1, X_2 are not necessarily isometric, But there's a coarser notion called quasi-isometry.

The Svarc-Milnor Lemma

(also called the Fundamental Theorem of Geometric Group Theory) says that if the actions are nice enough, then X_1 is quasi-isometric to X_2. Then quasi-isometry invariants are invariants of G.

To motivate the definition of quasi-isometry,

Consider the embedding $\mathcal{G}(\mathbb{Z}^2, \{e_1, e_2\}) \hookrightarrow \mathbb{R}^2$

This is not an isometry but.

① It is <u>almost</u> surjective

$$y \in \mathbb{R}^2.\ \exists\, x \in \mathcal{G} \text{ s.t. } d(y, f(x)) < \tfrac{1}{2}$$

② It <u>almost</u> preserves distance

$$\frac{1}{\sqrt{2}} d_{\mathcal{G}}(x,y) \leq d_{\mathbb{R}^2}(f(x), f(y)) \leq d_{\mathcal{G}}(x,y)$$

<u>Definition</u> X, Y metric spaces. A map $f: X \to Y$

- is <u>quasi-surjective</u> if $\exists\, K > 0$ s.t. $\forall y \in Y, \exists\, x \in X$ s.t. $d(y, f(x)) \leq K$.

(note K-quasi-surjective $\Rightarrow$ K' q-surj for any $K' > K$)

- is a <u>quasi-isometric embedding</u> if $\exists\, \lambda \geq 1, C \geq 0$ s.t. $\forall\, x, x' \in X$,

$$\frac{1}{\lambda} d(x,x') - C \leq d(f(x), f(x')) \leq \lambda d(x,x') + C$$

- is a <u>quasi-isometry</u> if it is a quasi-surjective quasi-isometric embedding. (or a (λ, C, K)-quasisometry)

Note: (λ, C)-qi $\Rightarrow$ (λ', C')-qi for any $\lambda' \geq \lambda, C' \geq C$

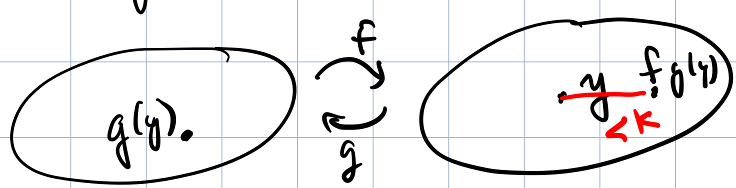
Lemma: Suppose $\exists f: X \to Y$, $g: Y \to X$ and $\lambda \geq 1$, $C \geq 0$, $K \geq 0$ st.

- $d(f(x), f(x')) \leq \lambda d(x,x') + C \quad \forall x, x' \in X$
- $d(g(y), g(y')) \leq \lambda d(y,y') + C \quad \forall y, y' \in Y$
- $d(x, gf(x)) \leq K \quad \forall x \in X$
- $d(y, fg(y)) \leq K \quad \forall y \in Y$

Then f and g are quasi-isometries.

Proof: Proof for f (proof for g is the same)

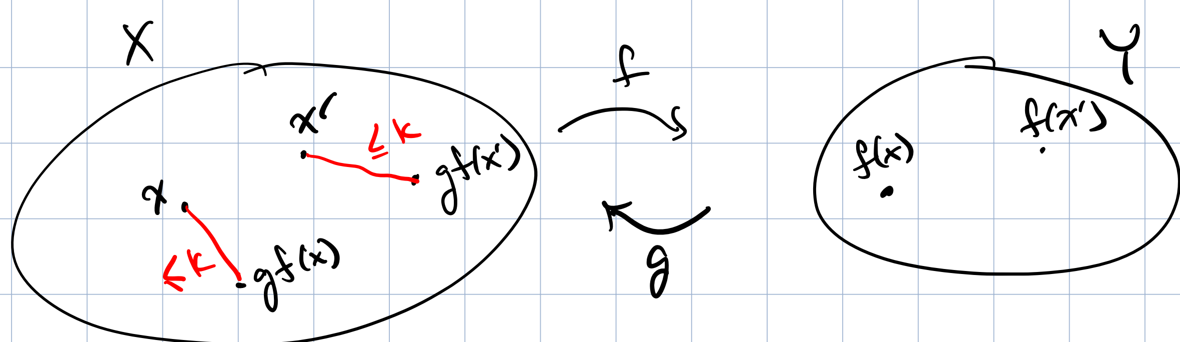
- f is quasi-surjective: $y \in Y$



and • f is a qi-embedding: need $\lambda' \geq \lambda$, $C' \geq C$ st

$$\frac{1}{\lambda'} d(x,x') - C' \leq d(f(x), f(x'))$$

ie $d(x,x') \leq \lambda d(f(x), f(x')) + C\lambda$



$$d(x,x') \le d(x, gf(x)) + d(gf(x), gf(x')) + d(gf(x'), x')$$
$$\le d(gf(x), gf(x')) + 2K$$
$$\le \lambda\, d(f(x), f(x')) + C + 2K$$

$\lambda' = \lambda$ $\qquad$ $C' = \max(C\lambda, C+2K)$

Cor S, S' two finite generating sets for G.

Then $\mathcal{C}(G,S)$ is qi to $\mathcal{C}(G,S')$

Pf Define $\mathcal{C}(G,S) \xrightarrow{f} \mathcal{C}(G,S')$

by:
- identity on vertices
- send $1 \overset{s}{\text{———}} s$ to a shortest path γ_s in $\mathcal{C}(G,S')$ connecting 1 to s.
- send $v \overset{s}{\text{———}} vs$ to $v \cdot \gamma_s$

Write each elt $s \in S$ as a word in $S' \cup S'^{-1}$

Define $\lambda = \max_{s \in S} \ell_{S'}(s)$

Then f stretches each edge by at most λ.

So $\quad d_{\mathcal{C}'}(f(x_1), f(x_2)) \le \lambda\, d_{\mathcal{C}}(x_1, x_2) + 0$

for any $x_1, x_2 \in \mathcal{C}(G,S)$

Similarly, define $h : \mathcal{C}(G,S') \longrightarrow \mathcal{C}(G,S)$ by sending each edge $v \overset{s'}{\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet} vs'$ to the v-translate of a shortest path in $\mathcal{C}(G,S)$ from 1 to s'

and let $\lambda' = \max\limits_{s' \in S'} \ell_S(s')$

$$d_{\mathcal{C}}(h(y_1), h(y_2)) \leq \lambda' d_{\mathcal{C}'}(y_1, y_2)$$

Then $d_{\mathcal{C}}(x, hf(x)) = \begin{cases} 0 & \text{if } x \text{ is a vertex} \\ \leq \lambda\lambda' & \text{otherwise} \end{cases}$

Prop Quasi-isometry is an equivalence relation

pf reflexive ✓, $X \xrightarrow{id} X$

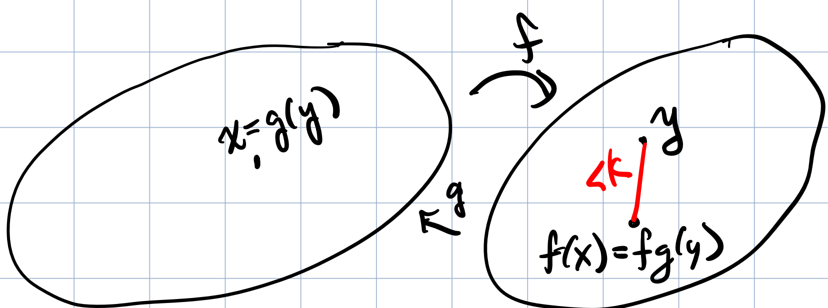
transitive ✓ $X \xrightarrow{f} Y \xrightarrow{g} Z$

(exercise)

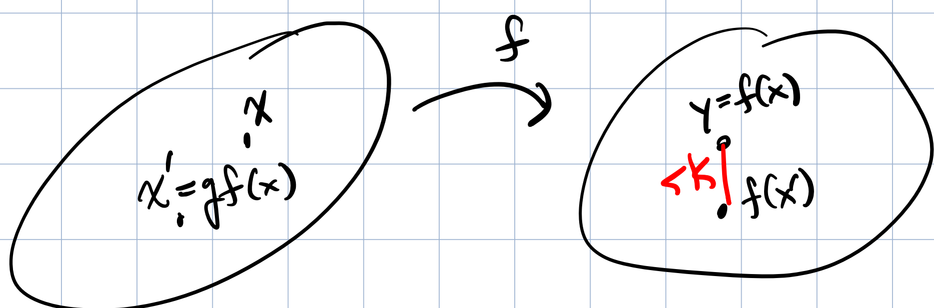
symmetric: Let $f: X \to Y$ be a (λ, C, K)-q.i.

Define $g: Y \to X$: For $y \in Y$, choose $x \in X$ w/ $d(f(x), y) < K$

Define $g(y) = x$.



Then $d(fg(y), y) = d(f(x), y) < K$ ✓



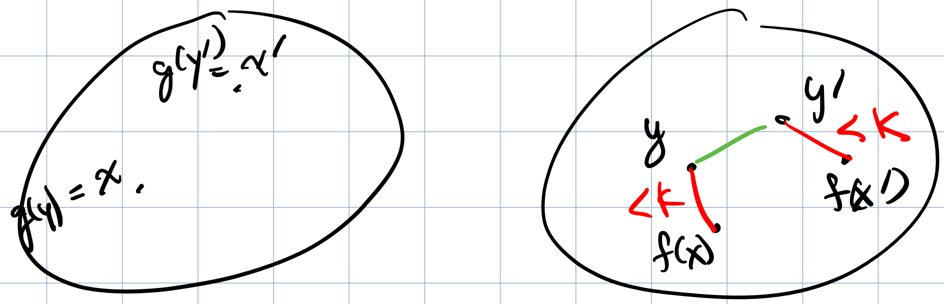
$$d(f(x), f(x')) \geq \frac{1}{\lambda} d(x, x') - c$$

$\Rightarrow$

$$d(gf(x), x) = d(x, x') \leq \lambda(d(f(x), f(x')) + c)$$

$$\leq \lambda K + c\lambda := K' \quad ✓$$

Now: $d(g(y), g(y')) < \lambda' d(y,y') + C$:



$$d(f(x), f(x')) \geq \frac{1}{\lambda} d(x,x') - C$$

$$\Rightarrow \lambda\left[d(f(x), f(x')) + C\right] \geq d(x,x')$$

So $d(g(y), g(y')) = d(x,x') \leq \lambda\left[d(f(x), f(x')) + C\right]$

$$\leq \lambda(2k + d(y,y') + C)$$

$$= \underbrace{\lambda}_{\lambda'} d(y,y') + \underbrace{2k\lambda + C\lambda}_{C'}$$

We want nice actions on nice metric spaces

Let X be a metric space

A path from x to y in X is a geodesic if its length is equal to $d(x,y)$.

(Geodesics are not necessarily unique!)

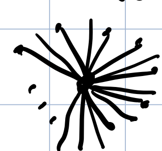
A metric space is geodesic if any two points are connected by a geodesic

A metric space is proper if closed balls of finite radius r are compact.

Proper metric space $\neq$ proper action!

non-examples:

- $\mathbb{R}^2 - \{(0,0)\}$ is not geodesic
- a graph with an infinite-valent vertex. is not proper



Ready to prove the Fundamental Theorem

Thm (Švarc-Milnor Lemma) (X, d_X) = proper geodesic metric space, $G \curvearrowright X$ a proper cocompact action. Then G is finitely generated and (G, d_S) is quasi-isometric to (X, d_X) for any finite generating set S.

Proof of Theorem: Choose any $x_0 \in X$ and define
$$f : G \longrightarrow X \text{ by } f(g) = g \cdot x_0$$

Claim 1 G is finitely generated.

Claim 2 For some (hence any) finite generating set S, $f : (G, d_S) \longrightarrow (X, d_X)$ is a quasi-isometry

Setup Choose $K \subset X$ compact s.t. $X = \bigcup_g gK$, with $x_0 \in K$.

Then $K \subset U$ for some open ball $U = B_r(x_0)$, with (compact) closure D.

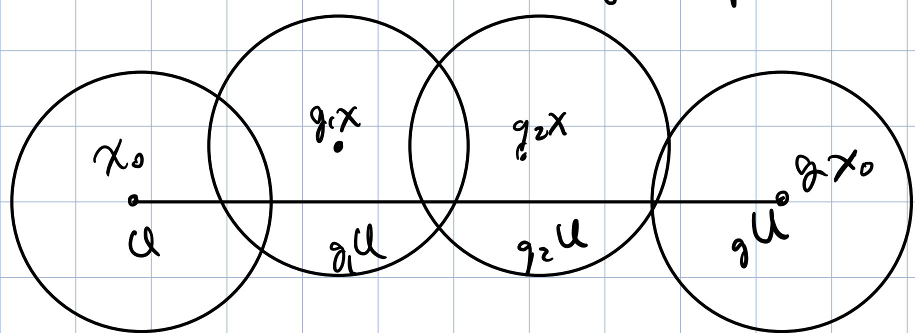
Now $X = \bigcup_{g \in G} gU$ and

$$S = \{ g \in G \mid gU \cap U \neq \emptyset \} \subseteq \{ g \in G \mid gD \cap D \neq \emptyset \},$$

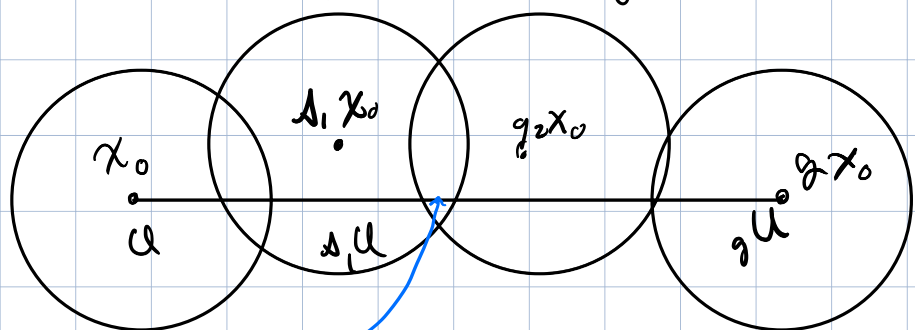
which is finite since the action is proper

To prove Claim 1 we will show S generates G.

<u>proof</u> For $g \in G$ let σ be a path from x_0 to gx_0. Cover σ by translates of U, then take a finite subcover $U, g_1U, g_2U, \dots, gU$



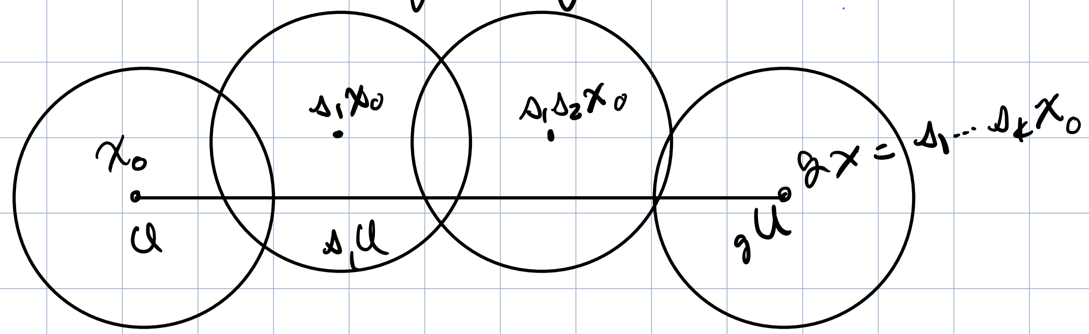
$U \cap g_1U \neq \emptyset \Rightarrow g_1 \in S$, say $g_1 = s_1$



$$s_1U \cap g_2U \neq \emptyset \Leftrightarrow U \cap s_1^{-1}g_2U \neq \emptyset$$

$$\Rightarrow s_1^{-1}g_2 = s_2 \in S \Rightarrow g_2 = s_1s_2$$

etc Continue, get $g = s_1 s_2 \cdots s_k$, $s_i \in S$



Claim 2: $f : (G, d_S) \longrightarrow (X, d_X)$ is a q.i.

$$g \longmapsto g x_0$$

We need to show

- f is quasi-surjective and
- $\exists\, \lambda \geq 1,\ c \geq 0$ s.t

$$(*)\begin{cases} \forall g_1, g_2 \in G, \\ \frac{1}{\lambda} d_S(g_1, g_2) - c \leq d_X(g_1 x_0, g_2 x_0) \leq \lambda d_S(g_1, g_2) + c \end{cases}$$

Now $d_S(g_1, g_2) = l_S(g_1^{-1} g_2) = d_S(1, g_1^{-1} g_2)$. Also, G acts by isometries, so $d_X(g_1 x, g_2 x) = d_X(1, g_1^{-1} g_2 h)$

Setting $g = g_1^{-1} g_2$, $(*)$ becomes the simpler-looking

$$(**)\begin{cases} \forall g \in G, \\ \frac{1}{\lambda} d_S(1, g) - c \underset{②}{\leq} d_X(x_0, g x_0) \underset{①}{\leq} \lambda d_S(1, g) + c \end{cases}$$

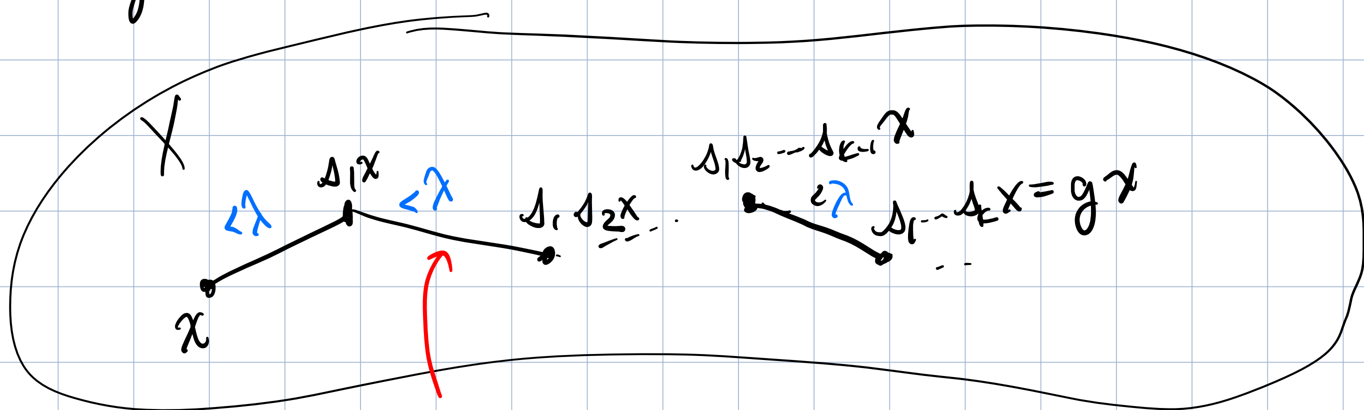
- f is quasi-surjective:
  The balls $g\,B_r(x) = B_r(gx)$ cover X,

  So every $x \in X$ is within r of some gx ✓

- ① Let S be <u>any</u> finite generating set.

  Suppose $g = s_1 \cdots s_k$ $\quad s_i \in S \cup S^{-1}$
  is a shortest word, ie $d_S(1,g) = k$.

Let $\lambda = \max\limits_{s \in S} d(x, sx)$. Then we can make a path in X from x_0 to gx_0 of length $\leq \lambda \cdot k$:



$d(s_1 x, s_1 s_2 x) = d(x, s_1 x) < \lambda$

So $d_X(x, gx) \leq \lambda\, d_S(1,g) + 0$

② It remains: $d_X(x, gx) \geq \frac{1}{\lambda} d_S(1,g) - c$

<u>ie</u> $d_S(1,g) \leq \lambda d_X(x,gx) + c\lambda$

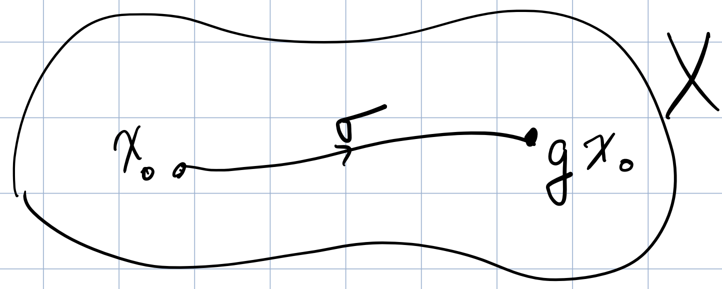
We have r s.t. $\{g \cdot B_r(x_0)\}$ cover X

Let $S' = \{g \mid g B_{3r}(x_0) \cap B_{3r}(x_0) \neq \emptyset\}$

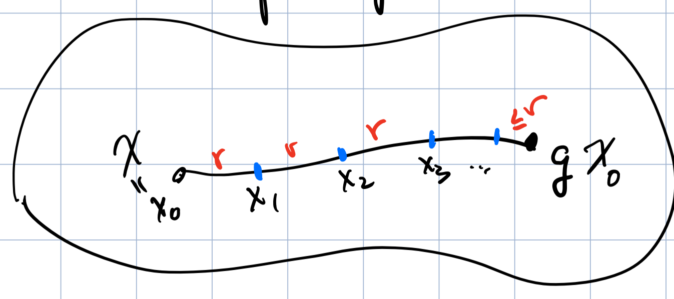
Since the translates of $B_r(x_0)$ cover X, so do the translates of $B_{3r}(x_0)$,

So the same arguments show S' is also a finite generating set, and ① holds

Now let σ be a geodesic from x to gx.

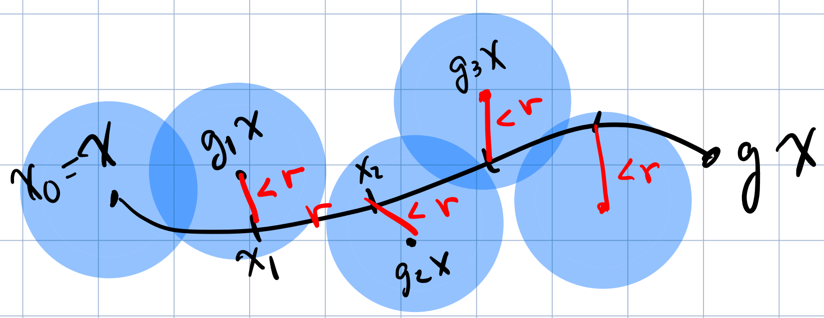


Divide σ into n pieces, each of length $= r$ except the last has size $r' \leq r$.



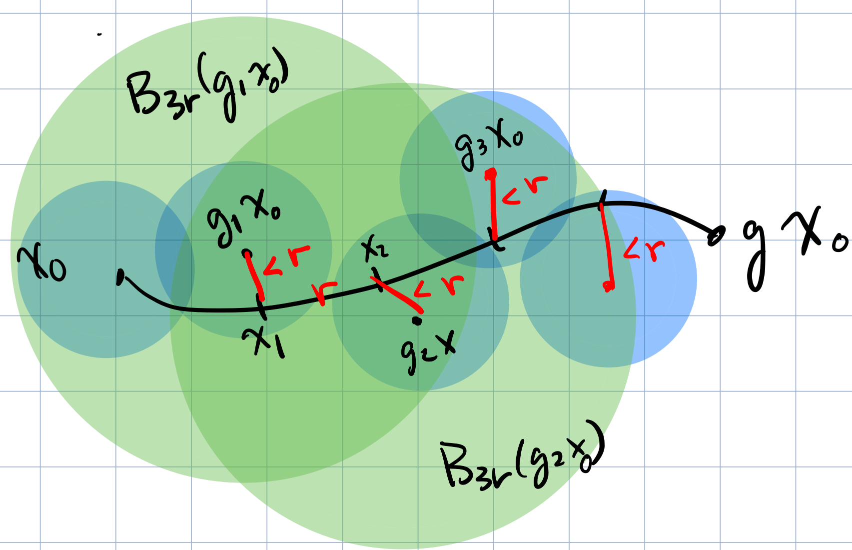
Now $d(x_0, gx_0) = \text{length}(\sigma) = (k-1)\cdot r + r'$

The translates of $B_r(x_0)$ cover X, so for each i, $\exists$ g_i with $d(x_i, g_i x) < r$



Then $d(g_i x_0, g_{i+1} x_0) < 3r$, so

$$B_{3r}(g_i x_0) \cap B_{3r}(g_{i+1} x_0) \neq \emptyset$$



so $B_{3r}(x) \cap g_i^{-1} g_{i+1} B_{3r}(x) \neq \emptyset$, and

$g_i^{-1} g_{i+1} \in S'$ for all i.

Write $g = (1 \cdot g_1)(g_1^{-1} g_2)(g_2^{-1} g_3) \cdots (g_{k-1}^{-1} g)$
$= \quad s_1 \quad s_2 \quad s_3 \quad \cdots \quad s_k$

So $d_{S'}(1, g) \leq k$

But $r(k-1) \leq d_X(x, gx) \Rightarrow k \leq \frac{1}{r} d_X(x, gx) + 1$

so $d_{S'}(1, g) \leq k \leq \frac{1}{r} d_X(x, gx) + 1$

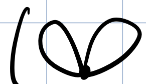
λ $\quad\quad$ $C \geq \lambda$ ✓

Definition A proper, cocompact action on a proper geodesic metric space will be called a ==**geometric**== action

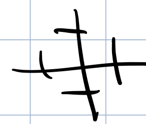
Examples:

1. $\mathbb{Z}^n \curvearrowright \mathbb{R}^n$ by linearly-independent translations is geometric

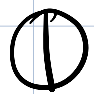
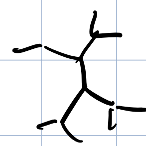
$$\Rightarrow \quad \mathbb{Z}^n \underset{qi}{\sim} \mathbb{R}^n.$$

2. $F_2 = \pi_1($  $)$ ↻ univ. cover of 

$= $ 4-valent tree T_4



$\Rightarrow F_2 \underset{qi}{\sim} T_4$

3. $F_2 = \pi_1($  $)$ ↻ univ cover of  $=$ 

$=$ trivalent tree T_3

$\Rightarrow F_2 \underset{qi}{\sim} T_3$

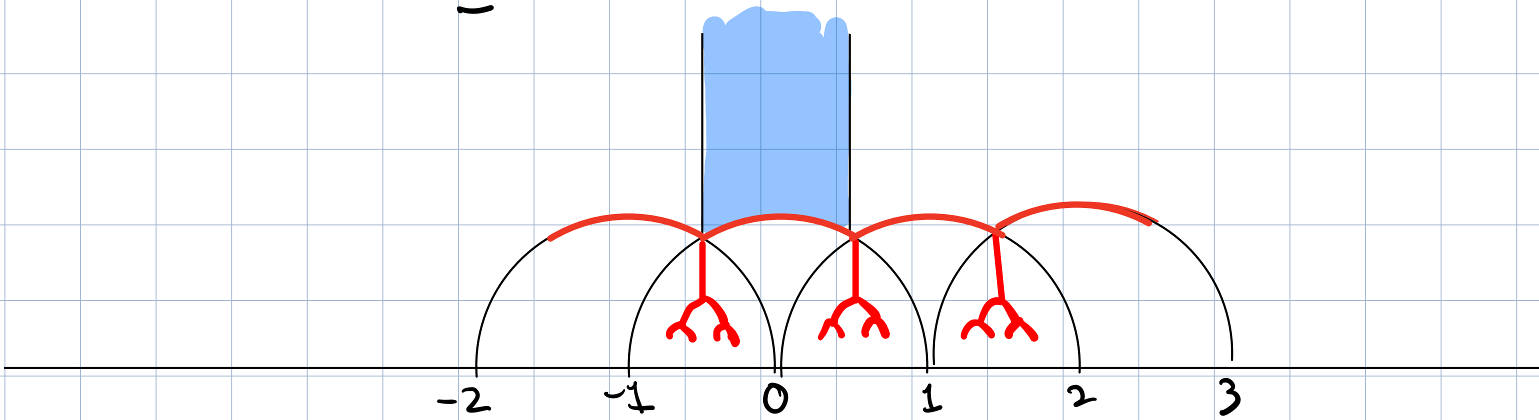
→ $T_4 \underset{qi}{\sim} F_2 \underset{qi}{\sim} T_3 \quad \Rightarrow T_1 \underset{qi}{\sim} T_3$

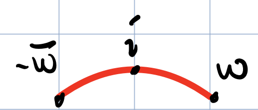
4. $F_3 = \pi_1$  ↻ T_3 = univ. cover of 

→ $F_3 \sim T_3 \sim F_2$

5 Action of $SL(2,\mathbb{Z})$ on $\mathbb{H}^2$ is proper but <u>not</u> cocompact, so you can't conclude $SL(2,\mathbb{Z}) \underset{qi}{\sim} \mathbb{H}^2$

But there's a trivalent tree T inside $\mathbb{H}^2$ that <u>is</u> invariant under the action.



T = union of images of 

$= \alpha$

$\mathrm{stab}(w)$ is generated by $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$.

$$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} z = \frac{1}{-z+1} \qquad \begin{aligned} 1 &\mapsto \infty \\ -1 &\mapsto \tfrac{1}{2} \end{aligned}$$

So sends the geodesic arc α to an arc in the vertical geodesic from $\frac{1}{2}$ to ∞

The action of $SL_2(\mathbb{Z})$ on T is cocompact,

so $SL(2,\mathbb{Z}) \underset{qi}{\sim} T_3 \sim F_2$

In fact, $SL(2,\mathbb{Z})$ has a finite-index subgroup which is a free group

Prop If G is finitely generated and $H < G$ has finite index, then H is quasi-isometric to G.

Proof The action of G on $\mathcal{C}(G,S)$

restricts to an action of H, which is still proper.

Let $g_1, \dots, g_k$ be coset representatives for H, $B_1(1)$ the ball of radius 1 in $\mathcal{C}(G,S)$ and $B = \bigcup_{g_i} g_i B_1(1)$

Then H-translates of B cover $\mathcal{C}(G,S)$, so

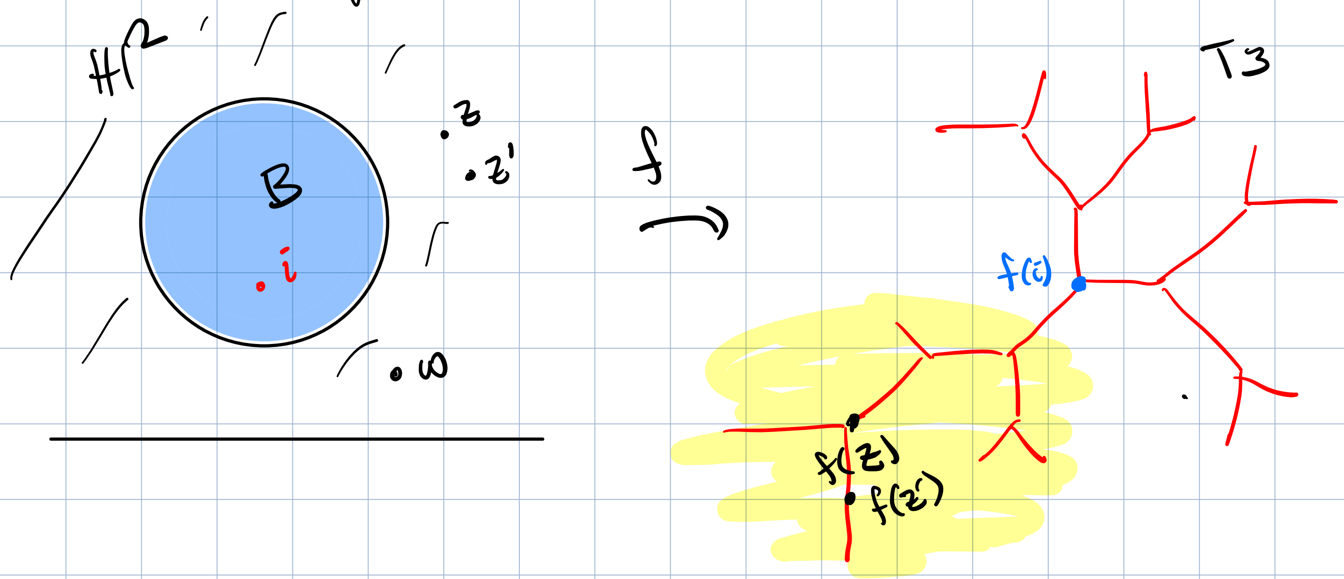
The action of H on $\mathcal{C}(G,S)$ is cocompact.

So $H \underset{qi}{\sim} \mathcal{C}(G,S) \underset{qi}{\sim} G$ ✓

Next question: Is $SL(2,\mathbb{Z})$ quasi-isometric to $\mathbb{H}^2$?
Well $SL(2,\mathbb{Z})$ is quasi-isometric to T_3, and

I claim T_3 is not qi to $\mathbb{H}^2$

Why? Suppose $f: T_3 \longrightarrow \mathbb{H}^2$ is a quasi-isometry.



$T_3 - f(i)$ has two or three components – call them "branches"

Idea

If z is far away from i, say outside some ball B then $f(z)$ is far away from $f(i)$

If z' is close to z, then $f(z')$ is close to $f(z)$ ... in particular it's in the same branch

I can get from any z outside B to any w outside B in small steps staying outside B... so $f(z)$ and $f(w)$ are in the same branch.

Every point in $f(B)$ is close to $f(c)$, so if y is far out in a <u>different</u> branch, there is no x with $f(x)$ close to y

Ie. f is <u>not</u> quasi-surjective. ✓

The difference between $\mathbb{H}^2$ and T_3 is that T_3 has many ends, $\mathbb{H}^2$ has only one end

Next we will make all of this precise.

$\boxed{\text{Ends}}$

We will define <u>ends</u> of a metric space, and show that if X and Y are quasi-isometric, then they have the same number of ends.

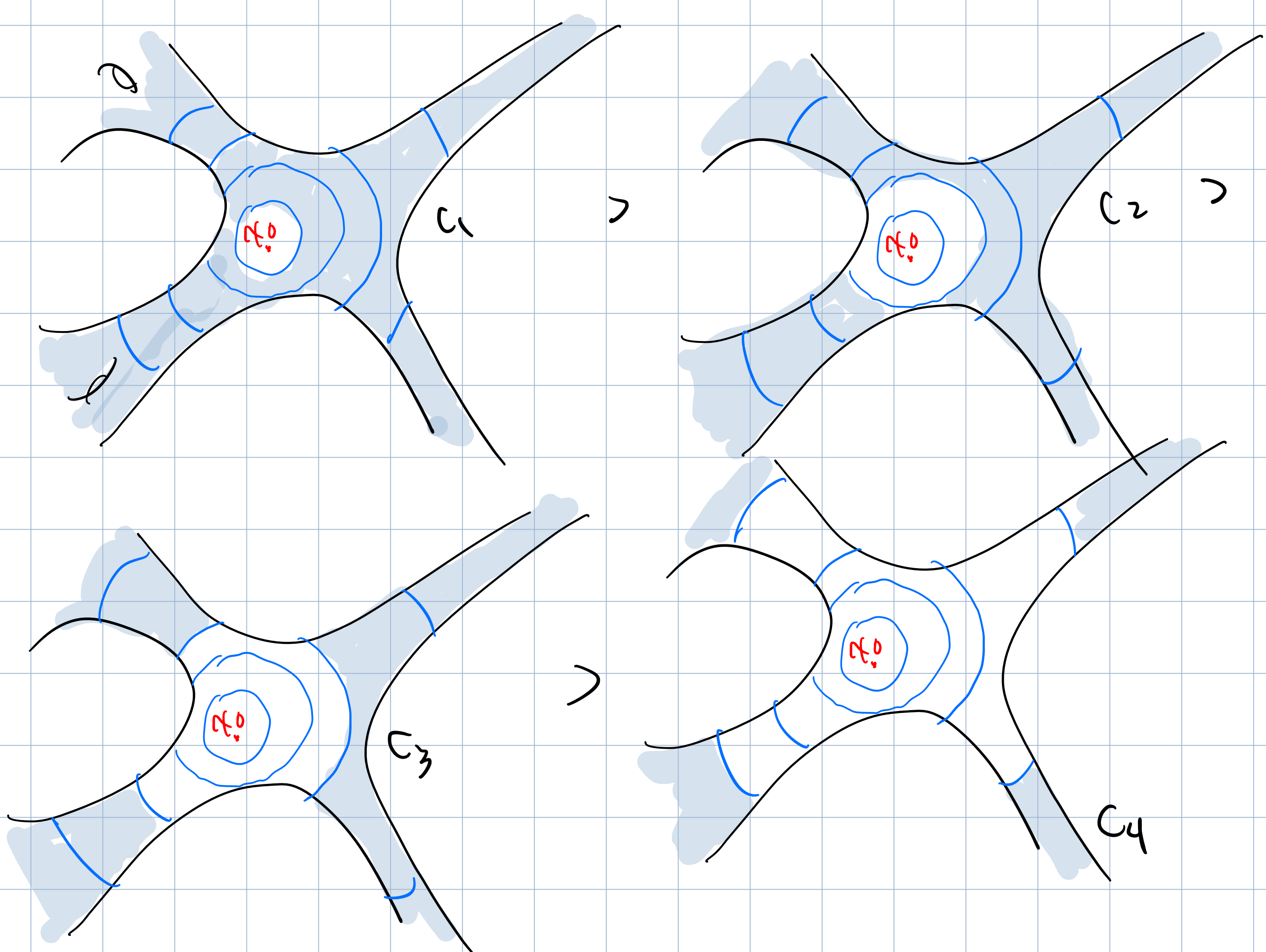
__Definition__ Let X be a proper geodesic metric space, $x_0 \in X$ and $B_n = B_n(x_0)$ the closed ball of radius n.

An __end__ of X is an infinite sequence

$$\varepsilon = C_1 \supseteq C_2 \supseteq C_3 \supseteq \cdots$$

where each C_n is a connected component of $X \setminus B_n$.

Note that C_n determines C_i for all $i \leq n$.



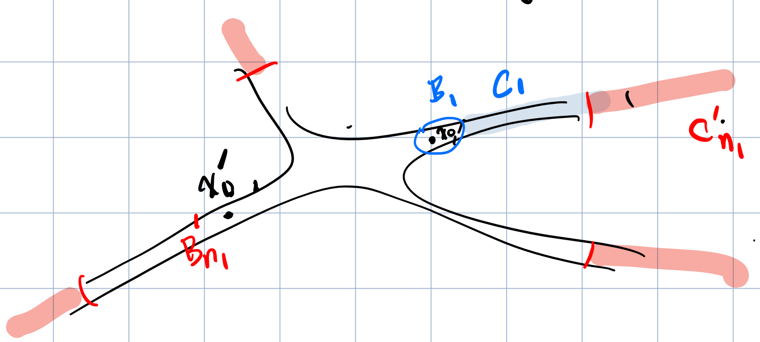
Prop: If x_0' is another point in X, a sequence $\mathcal{E}' = C_1' \supseteq C_2' \supseteq C_3' \supset$ of components of $X \setminus B_n(x_0')$ determines a unique end of X

pf Set $B_i = B_i(x_0)$, $B_i' = B_i(x_0')$

For any i, $\exists\, n_i$ s.t. $B_i \subset B_{n_i}'$,

so $X \setminus B_i \supseteq X \setminus B_{n_i}' \supset C_{n_i}'$

C_{n_i}' is connected, so is in a unique component C_i of $X \setminus B_i$.



Thus the C_{n_i}' determine the chain $C_1 \supseteq C_2 \supseteq \cdots$

Corollary The number of ends of X does not depend on the choice of x_0

Prop If X is quasi-isometric to Y, then X and Y have the same number of ends.

The proof is basically the same as the one I informally explained for $X = \mathbb{H}^2$ and $Y = T_3$.

Proof Let $f: X \to Y$ be a quasi-isometry, w/

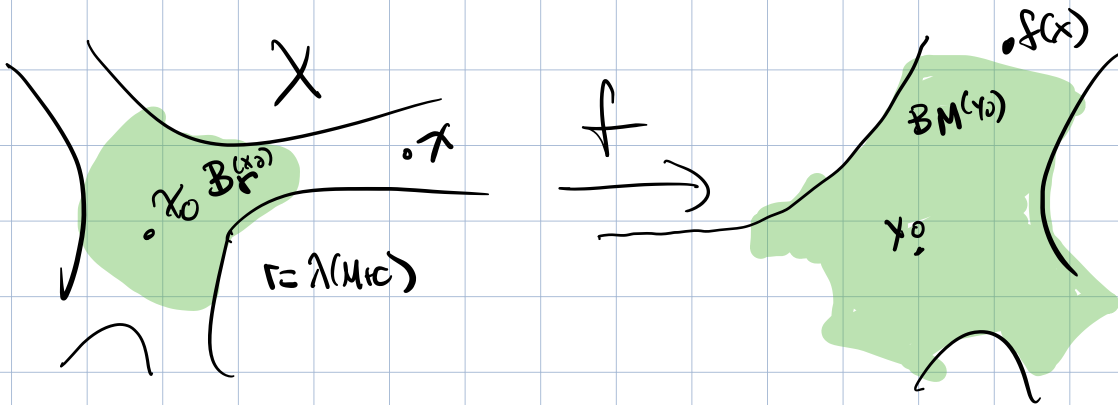
$$\frac{1}{\lambda} d(x,x') - C \le d(f(x), f(x')) \le \lambda d(x,x') + C$$

Fix $N > 0$ and $M > N + C$

pick $x_0 \in X$, let $y_0 = f(x_0)$

Then ① $d(x, x_0) > \lambda(M+C)$

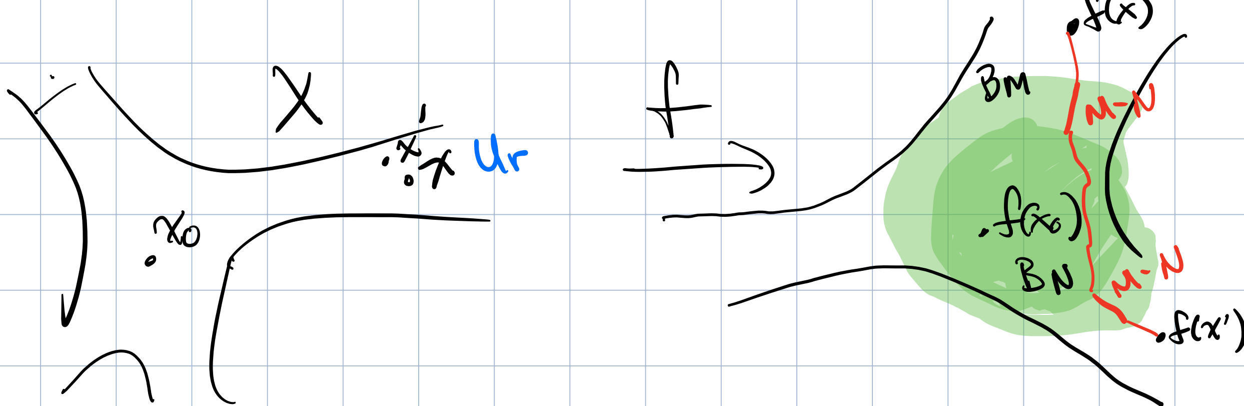
$$\Rightarrow d(f(x), y_0) \ge \frac{1}{\lambda} d(x, x_0) - C > \frac{1}{\lambda}(\lambda(M+C)) - C = M$$



and ② $d(x, x') < \frac{M-N-C}{\lambda} := \varepsilon$

$$\Rightarrow d(f(x), f(x')) < \lambda d(x,x') + C < \lambda \frac{(M-N-C)}{\lambda} + C = M - N$$

Now suppose x, x' are outside $B_r(x_0)$ and $d(x, x') < \varepsilon$. If $f(x), f(x')$ are in different components of $Y \setminus B_N(y_0)$, then $d(f(x), f(x')) > 2(M-N)$ $\divideontimes$



So: If x, x' are in the same component C_r of $X \setminus B_r$, connect them by a path, divide the path into pieces of size $< \varepsilon$, conclude $f(x), f(x')$ are in the same component of $Y \setminus B_N$.

Now $f(B_r) \subset B_{\lambda r + c}(y_0)$, so we can conclude

$\text{ends}(X) \longrightarrow \text{ends}(Y)$ is surjective:

If not, can find a point y in some end of Y that is arbitrarily far from $f(X)$, contradicting the fact that f is quasi-surjective.

Similarly, $\text{ends}(Y) \longrightarrow \text{ends}(X)$ is surjective, so either $\text{ends}(X) = \text{ends}(Y) < \infty$ or they are both infinite.

So $\mathbb{R}$ is not qi to $\mathbb{R}^2$ (or any $\mathbb{R}^n$, $n \geq 2$)
$\mathbb{Z}$ is not qi to $\mathbb{Z}^2$ (or $\mathbb{Z}^n$, $n \geq 2$)

$\mathbb{R}$ is not qi to T_3 (or any T_n, $n \geq 3$)
$\mathbb{Z}$ is not qi to F_2 (or F_n, $n \geq 2$

Q: Is $\mathbb{R}^2$ q.i. to $\mathbb{R}^3$?

A: No, but you have to use more sophisticated qi invariants.

Finite groups have 0 ends.

$\mathbb{Z}$ has 2 ends

$\mathbb{Z}^n$ has 1 end.

F_2 has ∞ many ends

Q: Is there a group with 3 ends?

A: No!

<u>Theorem</u> A finitely-generated group G has 0, 1, 2 or infinitely many ends.

<u>proof</u>: Suppose $\mathcal{C} = \mathcal{C}(G,S)$ has e ends, $2 < e < \infty$

$g \in G \Rightarrow g: \mathcal{C} \to \mathcal{C}$ is an isometry,

so permutes the finite number e of ends of $\mathcal{C}$

Let $N = \mathrm{Ker}\,(G \longrightarrow \mathrm{Sym}_e)$
$$g \longmapsto \sigma_g$$

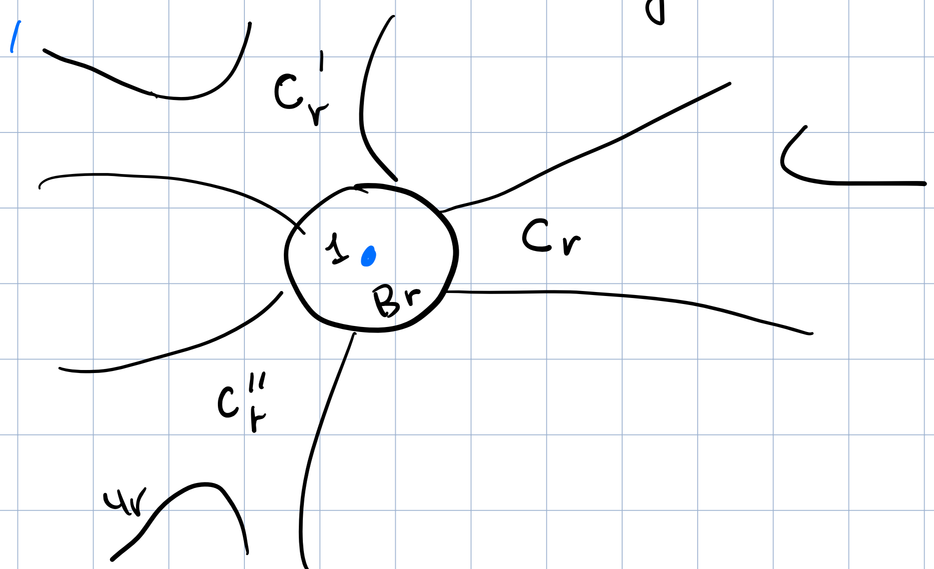
Since $e < \infty$, N has finite index in G, so is qi to G so has the same number of ends

So, replacing G by N, we may assume G fixes the ends of $\mathcal{G}$.

Now:

Let $\mathcal{E} = C_1 \supset C_2 \supset \cdots$
$\mathcal{E}' = C_1' \supset C_2' \supset \cdots$
$\mathcal{E}'' = C_1'' \supset C_2'' \supset \cdots$
be 3 different ends; (C_i, C_i', C_i'' = components of $\mathcal{G} \setminus B_i(1)$)

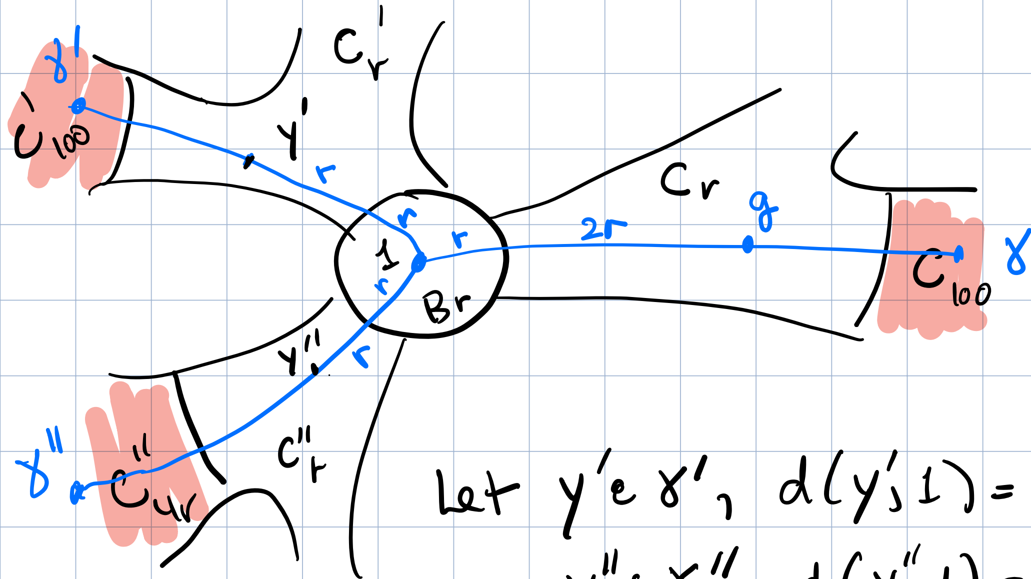
Then for some r, C_r, C_r' and C_r'' are different components of $X \setminus B_r(1)$



Let γ be a geodesic from 1 to a vertex in C_{100}

γ' — 1 to a vertex in C'_{100}

γ'' — 1 to a vertex in C''_{100}



Let $y' \in \gamma'$, $d(y',1) = 2r$

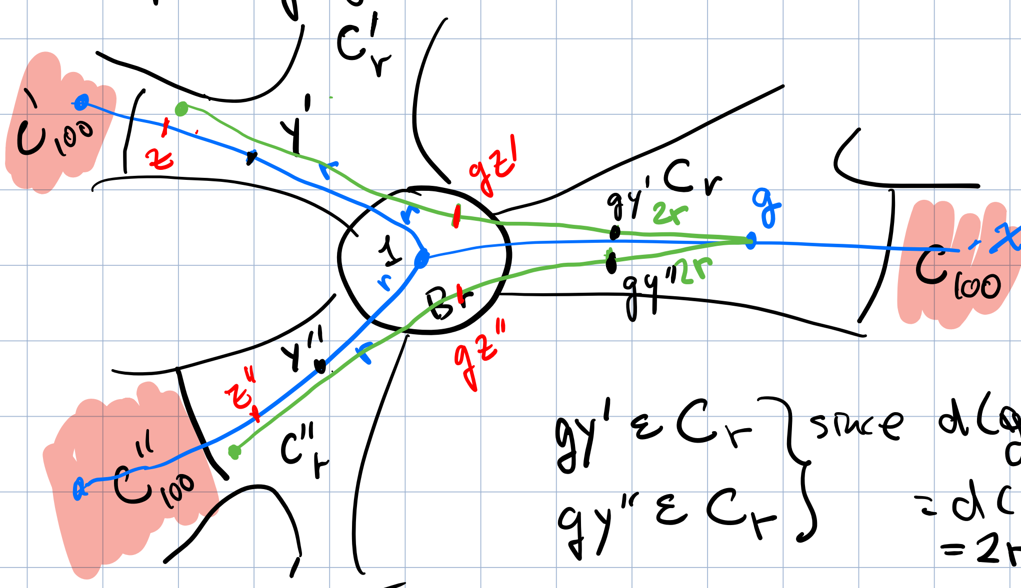
$y'' \in \gamma''$, $d(y'',1) = 2r$

Then $\left.\begin{array}{l} z' \in \gamma', \ d(z',1) > 2r \\ z' \in \gamma'', \ d(z'',1) > 2r \end{array}\right\} \Rightarrow d(z',z'') > 2r$

Now let $g \in \gamma$, $d(1,g) = 3r$

Act on this picture by g: Since g fixes ends

- the endpoint of $g\gamma'$ is still in C'_r
- the end point of $g\gamma''$ is still in C_r



$\left.\begin{array}{l} gy' \in C_r \\ gy'' \in C_r \end{array}\right\}$ since $d(gy',g) = d(y',1) = 2r$

So $\exists z'$ on γ', z'' on γ''
with gz', gz'' both in $B_r(i)$
so $d(gz', gz'') < 2r$

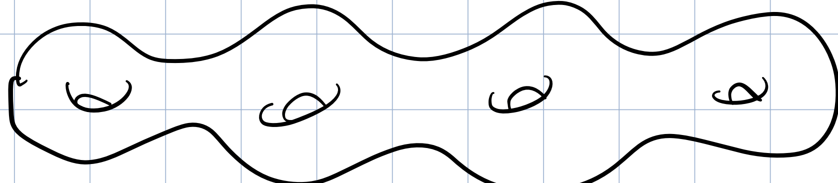
but $d(gz', gz'') = d(z, z') > 2r$ since
$d(z', B_r)$, $d(z'', B_r) > r$. $\divideontimes$

Next question: $\mathbb{H}^2$ has one end.
We showed $SL(2,\mathbb{Z})$ has ∞ many ends, so
is not q.i. to $\mathbb{H}^2$

Is there a group quasi-isometric
to $\mathbb{H}^2$?

(A: yes)

<u>Definition</u> A <u>surface group</u> is the fundamental
group of a closed orientable surface
of genus $g \geq 2$.



$\mathbb{H} = \{z \mid \mathrm{Im}(z) > 0\}$ with the metric $\frac{ds}{y} = \sqrt{dx^2+dy^2}/y$ is one model for the hyperbolic plane.

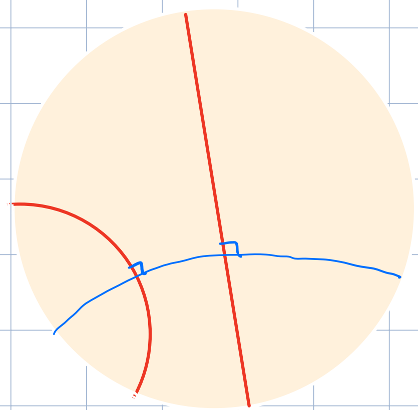
Another model is the Poincaré disk model $\mathbb{D}$. This is the interior of the unit disk in $\mathbb{R}^2$ with the metric $\frac{2ds}{1-r^2}$.

The map

$$f : \left(\mathbb{H}, \frac{ds}{s}\right) \longrightarrow \left(\mathbb{D}, \frac{ds}{1-r^2}\right)$$
$$z \longmapsto \frac{z-i}{z+i}$$

is an isometry.

Geodesics in $\mathbb{D}$ are circle-arcs $\perp$ to S^1 and straight lines through the origin



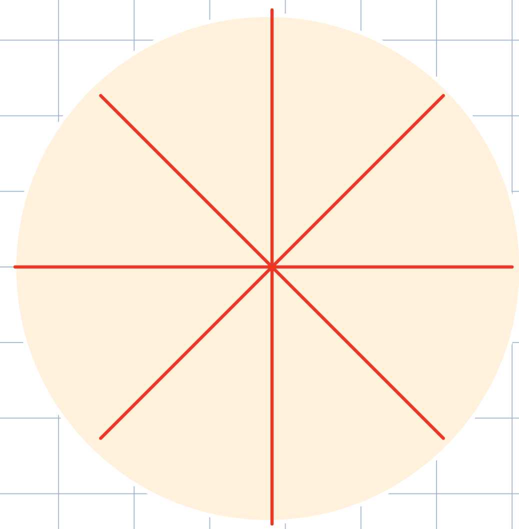
As before, isometries are generated by inversions in circles, and there is an orientation-preserving isometry taking any geodesic to any other geodesic, and any point to any other point.

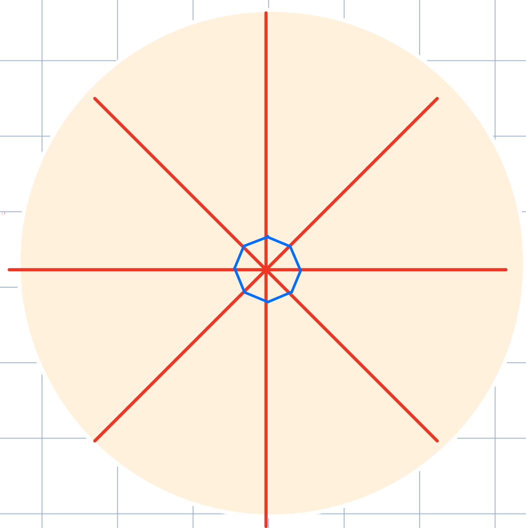
(= translation along the unique geodesic orthogonal to both)

It is easy to see that for $n \geq 2$ $\mathbb{D}$ contains a regular $4n$-gon with all angles $\frac{\pi}{n}$.

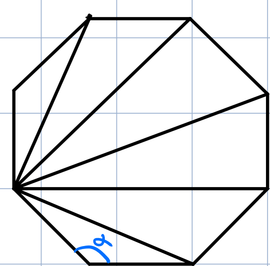
Picture for $n=4$:

take points on these rays at distance r from O

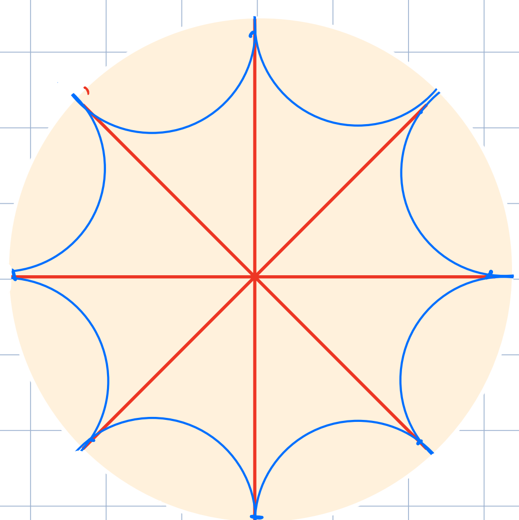




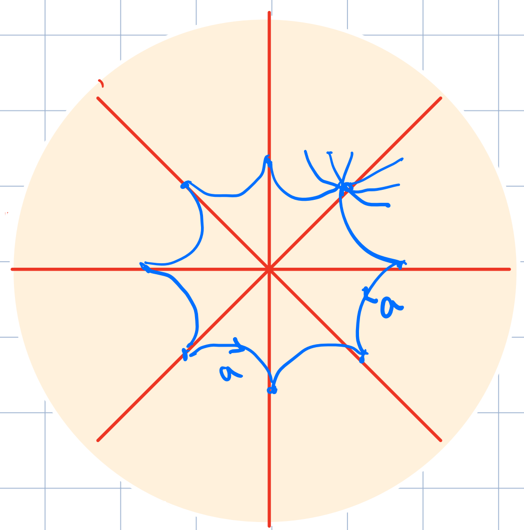
If r is small, $\frac{ds}{1-r^2}$ is almost ds, the octagon is almost Euclidean, so has angles almost $\frac{3\pi}{4}$



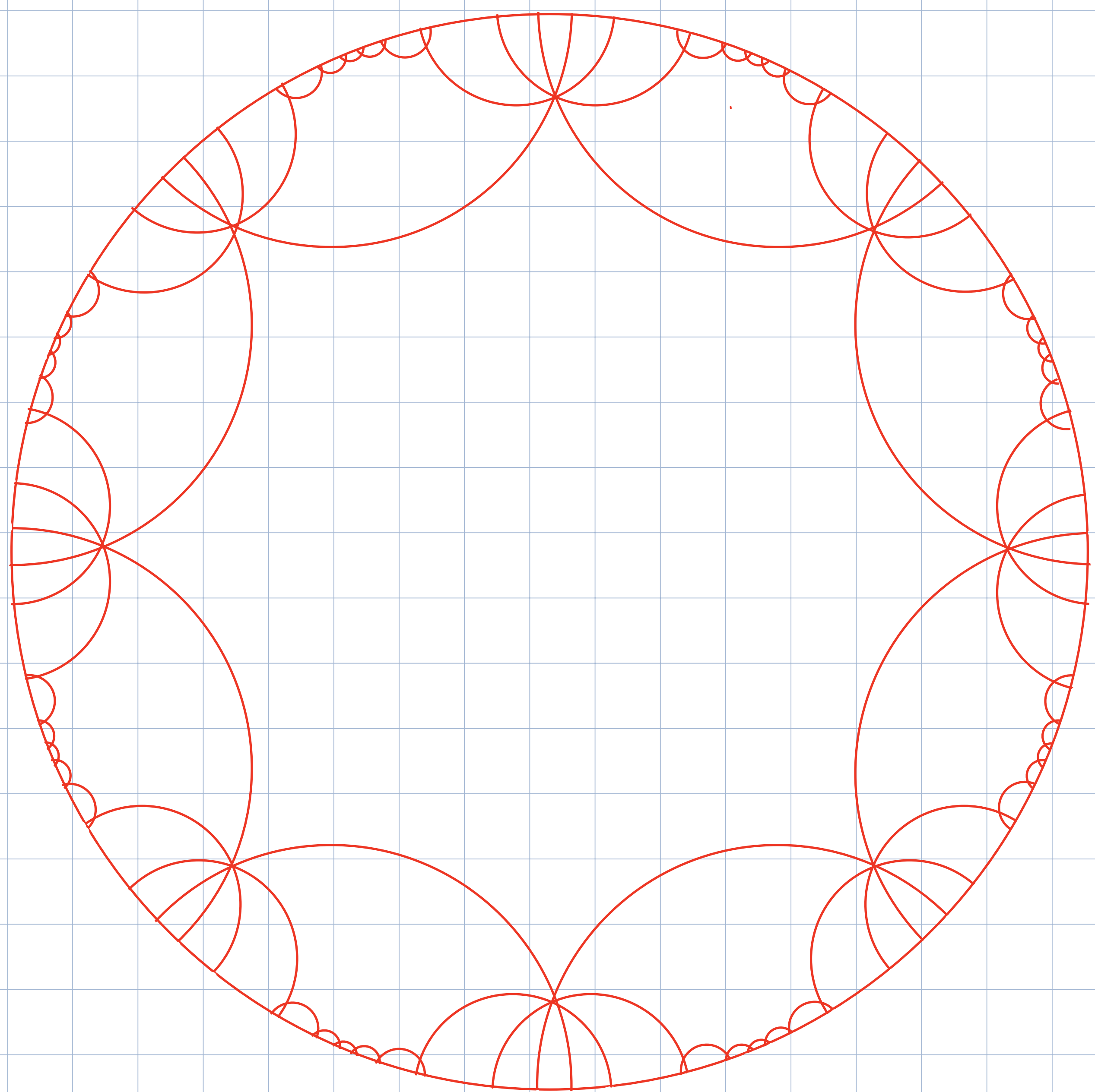
(in $\mathbb{R}^2$, $6 \cdot \pi = 8 \cdot \alpha$, so $\alpha = \frac{3\pi}{4}$)



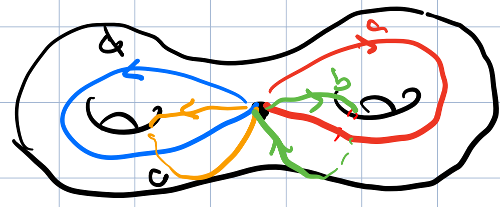
If r is close to 1, the angles are almost 0



So somewhere in between they are $\frac{\pi}{4}$. So 8 octagons fit around each vertex:

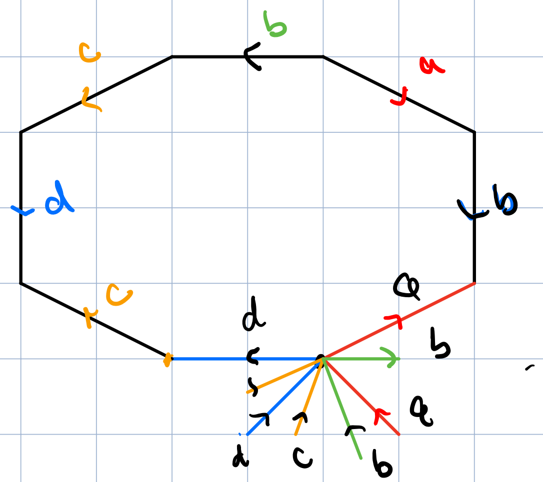


Let Σ_2 be the closed orientable surface of genus 2:



The loops a, b, c, d generate $\pi_1(\Sigma_2, b)$

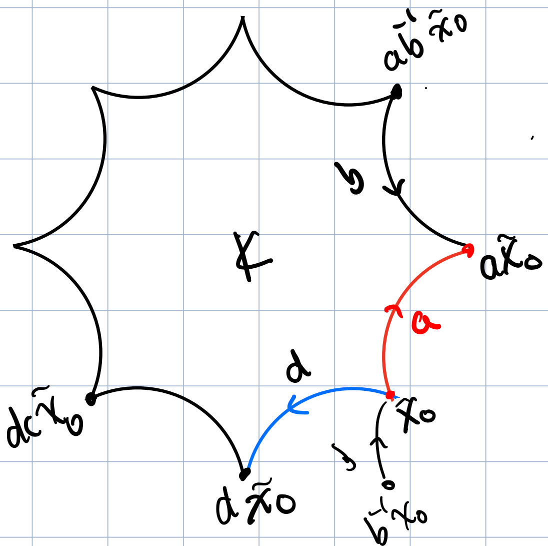
The universal cover of Σ_2 is a non-compact simply-connected surface, so homeomorphic to the plane; the pre-image of the colored loops cut the surface into octagons, 8 around each vertex



$\pi_1(\Sigma_2)$ acts freely on $\tilde{\Sigma}_2$ by deck transformations

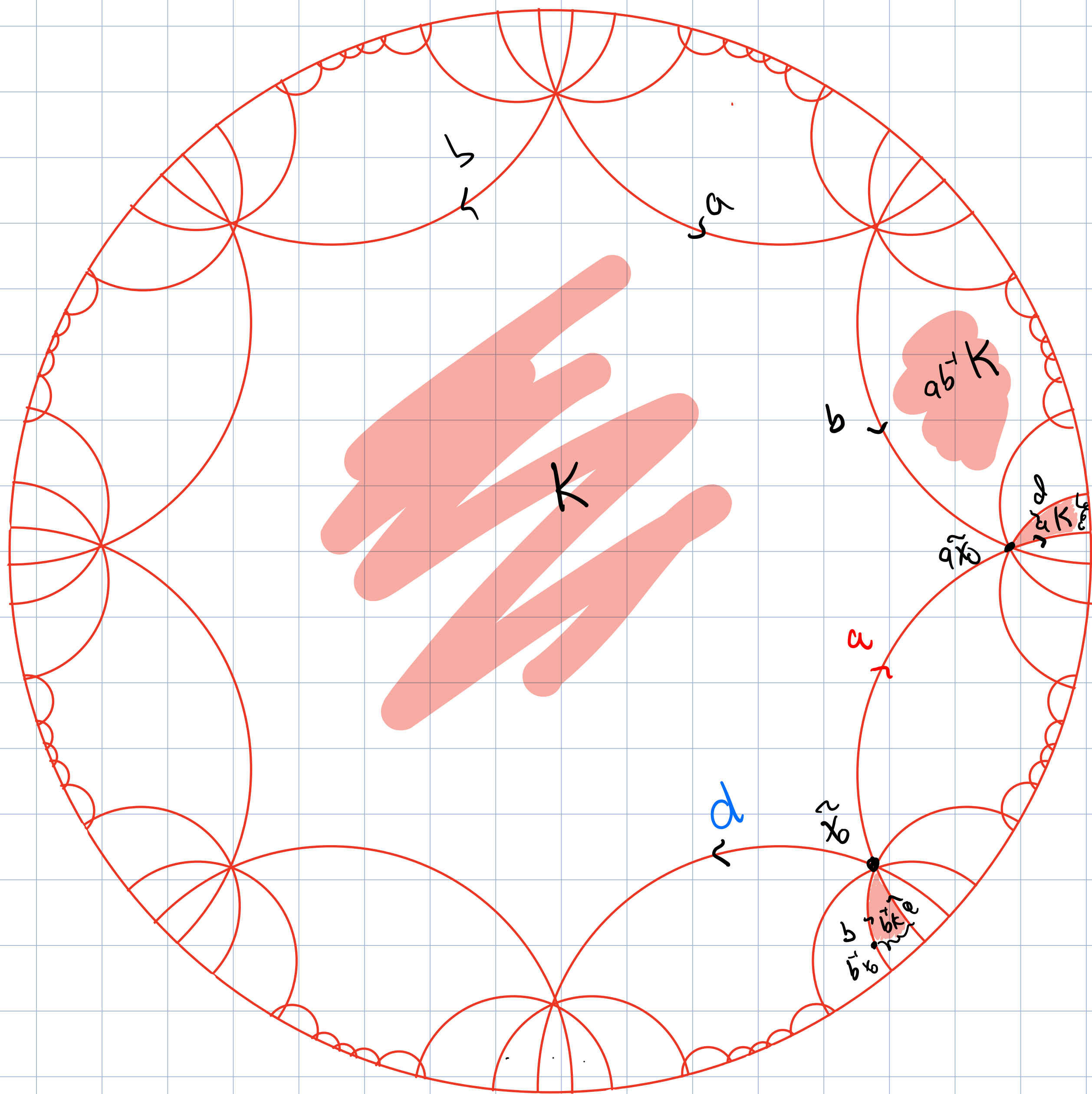
We saw we can tile the hyperbolic plane $\mathbb{D}$ by regular octagons with dihedral angle $\frac{\pi}{4}$, So we can identify $\tilde{\Sigma}_2$ with $\mathbb{D}$, and

Each generator of $\pi_1 \Sigma_2$ acts by an isometry



so all of $\pi_1 \Sigma_2$ acts by isometries

So $\pi_1 \Sigma_2$ acts freely and cocompactly on $\mathbb{D}$ by isometries.



There's nothing special about genus 2 –
You can find a regular 4g-gon in
$\mathbb{D}$ with all angles $\frac{2\pi}{4g}$, and identify $\tilde{\Sigma}_g$
with $\mathbb{D}$ tiled by these.

so $\pi_1(\Sigma_2) \underset{qi}{\sim} \mathbb{D} \underset{qi}{\sim} \pi_1(\Sigma_3) \underset{qi}{\sim} \pi_1(\Sigma_4) \sim \ldots$

Max Dehn noticed this around 1910. He used the geometry of $\mathbb{D}$ to prove interesting facts about fundamental groups of surfaces ... eg they contain free subgroups, and never contain free abelian subgroups of rank 2.