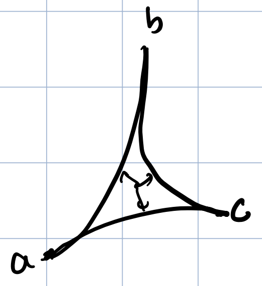
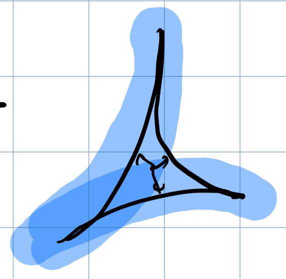
Inspired by Dehn's work on surface groups using their action on the Poincaré disk $\mathbb{D}$, Gromov singled out certain geometric features of $\mathbb{D}$ and defined a metric space with these features to be hyperbolic. He called groups that act properly & cocompactly on a hyperbolic metric space hyperbolic groups and showed they have many of the same algebraic properties as surface groups.

The relevant features of $\mathbb{D}$ include:

① geodesic triangles are "thin"

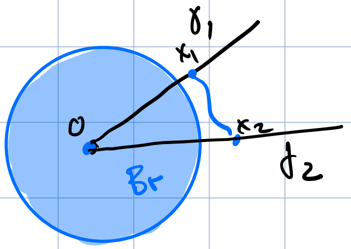


There are many ways to say this. One way: there is a constant δ $(= \log 3)$ such that each side is contained in a δ-neighborhood of the other 2 sides:



the triangle is "δ-thin"

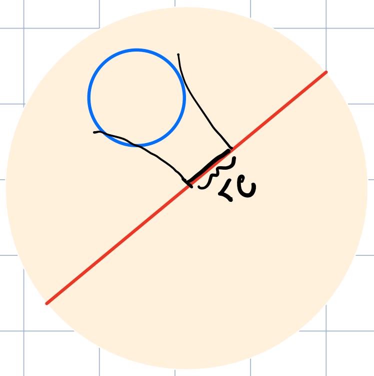
② Geodesics diverge exponentially fest, ie.



There is a constant $C>0$ satisfying:
If $x_i \in \gamma_i$, $d(x_i,0)>r$ let $d_r(x_1,x_2)$ be the length of the shortest path from x_1 to x_2 that stays outside $B_r(0)$.

then $d_r(x_1,x_2) \geq C^r$

③ There is a constant $C>0$ satisfying:
If a ball in $\mathbb{D}$ is disjoint from a geodesic, its projection onto the geodesic has length $\leq C$.



It turns out that any proper geodesic metric space satisfying ① also satisfies ② and ③, so we will use ① to define the notion of "hyperbolic metric space".

Fix $\delta > 0$. A geodesic triangle in a metric space X is <u>δ-thin</u> if each side is contained in a δ-neighborhood of the other two sides.

<u>Definition</u> A geodesic metric space X is <u>δ-hyperbolic</u> if every geodesic triangle is δ-thin

X is (Gromov) <u>hyperbolic</u> if it is δ-hyperbolic for some δ

<u>Definition</u> A group G is <u>hyperbolic</u> if it acts properly and cocompactly on a proper hyperbolic metric space

It turns out (as we will see) that being hyperbolic implies many algebraic properties of G.

Recall that any finitely generated group G acts properly and cocompactly on any Cayley graph $\mathcal{C}(G,S)$. So if $\mathcal{C}(G,S)$ is hyperbolic, then G is hyperbolic

Claim: G is hyperbolic if and only if $\mathcal{C}(G,S)$ is hyperbolic

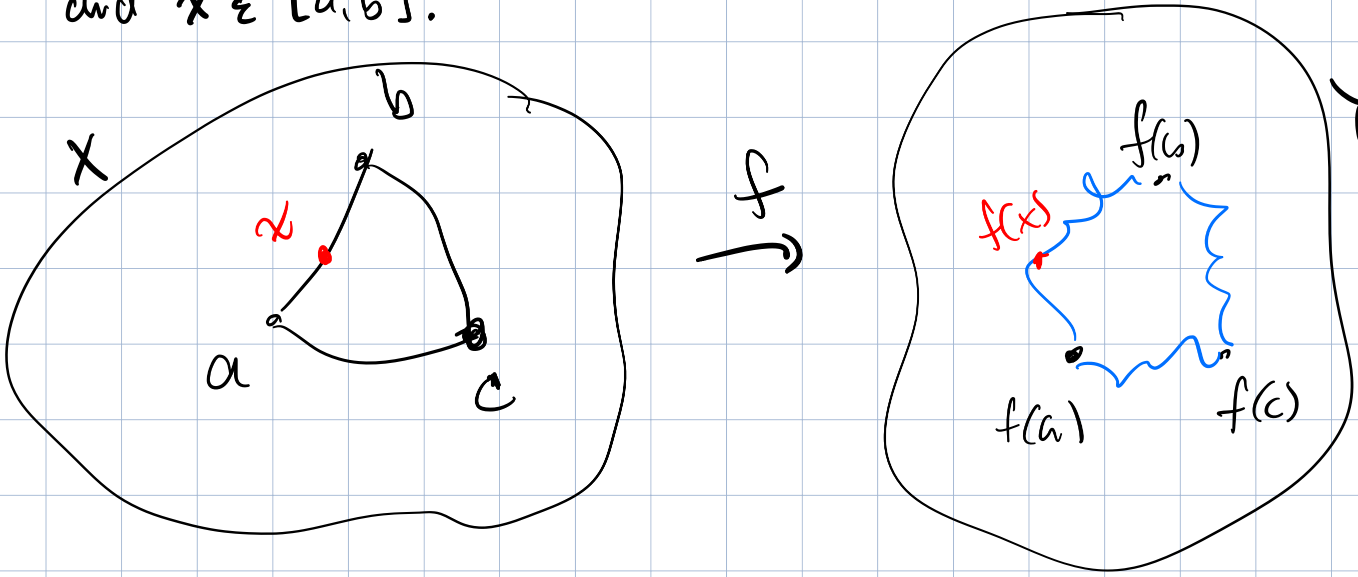
By the Svarc-Milnor lemma, if G acts geometrically on a hyperbolic space X, then G is quasi-isometric to X.

Since G is also quasi-isometric to $\mathcal{C}(G,S)$, it suffices to show that hyperbolicity is a quasi-isometry invariant for proper geodesic metric spaces:

Theorem Let $f: X \to Y$ be a quasi-isometry between proper geodesic metric spaces. If Y is hyperbolic, then X is hyperbolic.

Proof: Suppose f is a (λ, C)-quasi-isometry and Y is δ-hyperbolic.

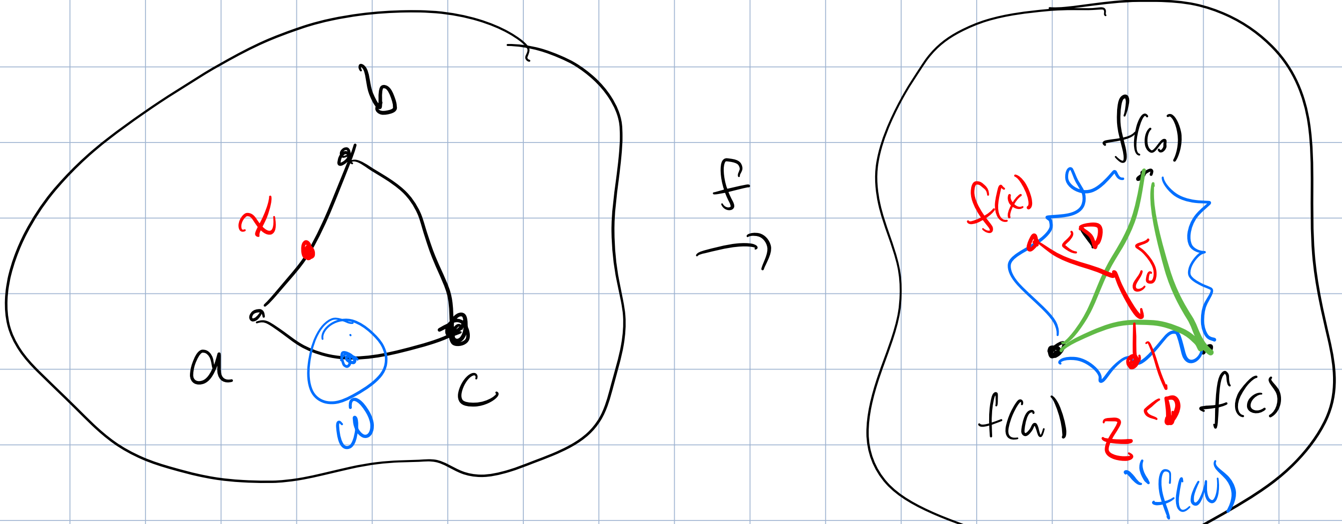
Let $\Delta = \Delta(a,b,c)$ be a geodesic triangle in X, and $x \in [a,b]$.



We want to find $\delta' > 0$ depending only on λ, C and δ and a point $w \in [a,c]$ or $[b,c]$ with $d(x,y) < \delta'$

Suppose we can find $D = D(\lambda, C, \delta)$ s.t.

$$(\ast) \quad \boxed{\begin{array}{c} f[a,b] \subset N_D[f(a), f(b)] \\ \text{and } [f(a), f(b)] \subset N_D f[a,b] \end{array}}$$



Then:

$\exists\, z \in f[a,c]$ or $f[b,c]$ with $d(f(x),z) < 2D+\delta$

say $z \in f[a,c]$.

Then $z = f(w)$ for some $w \in [a,c]$

so $d(f(x),z) = d(f(x),f(w))$

$$\geq \frac{1}{\lambda} d(x,w) - c$$

so $$d(x,w) \leq \lambda(d(f(x),z) + c)$$

$$\leq \underbrace{\lambda(2D+\delta)+c}_{=\delta'}$$

✓

So we need to prove $(*)$

$f[a,b]$ is not a geodesic, but is the image of a geodesic under a quasi-isometry .... it's a quasi-geodesic

so we want to understand these.

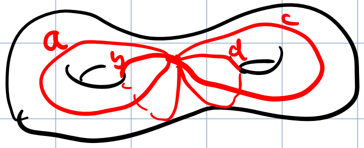
Let $I \subset \mathbb{R}$ be a closed interval.

A <u>Quasi-geodesic</u> in X is
a quasi-geodesic embedding $\alpha : I \longrightarrow X$

ie $\exists\, \lambda, C$ s.t. $\frac{1}{\lambda} \cdot d(s,t) - C \leq d(\alpha(s), \alpha(t)) \leq \lambda\, d(s,t) + C$

ie $\frac{1}{\lambda}|s-t| - C \leq d(\alpha(s), \alpha(t)) \leq \lambda |s-t| + C$

<u>eg</u> $G = \pi_1(S_2) = \langle \overbrace{a, b, c, d}^{S} \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} \rangle$



$i : \mathcal{C}(G,S) \hookrightarrow \mathbb{D} \ni x_0$

$\underset{g}{\bullet}\!\!\longrightarrow\!\!\underset{gs}{\bullet} \mapsto$ (geodesic from gx_0 to gsx_0)

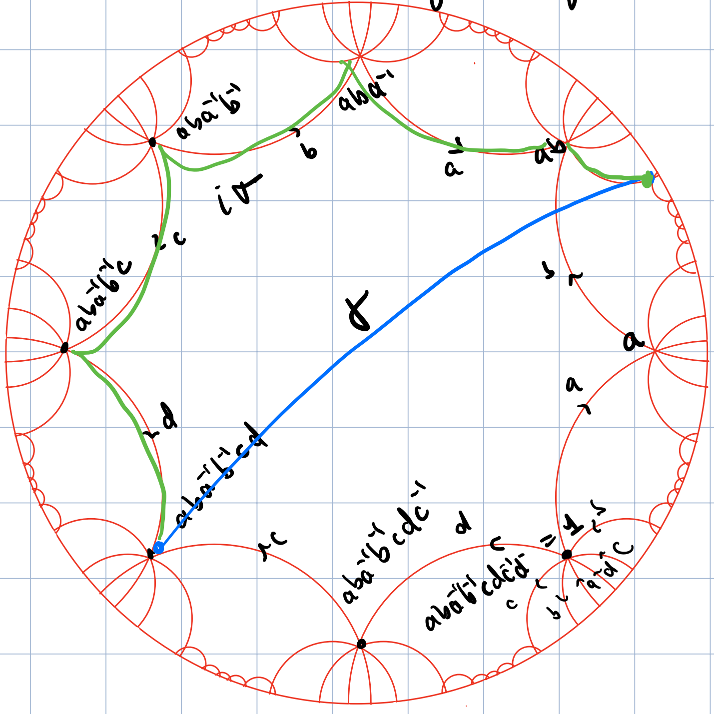
is a quasi-isometry

If σ is a geodesic in $\mathcal{C}$ from 1 to g, with $|g| = n$
paramatrize σ by arc length, $\sigma : [0,n] \longrightarrow \mathcal{C}$.
Then $i \circ \sigma : [0,n] \xrightarrow{\sigma} \mathcal{C} \xrightarrow{i} \mathbb{D}$ is a quasi-geodesic.

<u>claim</u> it stays close to the (unique) geodesic from x_0 to gx_0

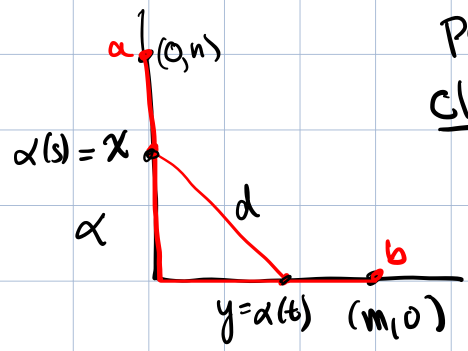
<u>ie</u> $\exists M$ st.

$$\boxed{\begin{array}{l} \gamma \subset N_M(i\sigma) \\ \text{and } i\sigma \subset N_M(\gamma) \end{array}}$$



eg

Let α be the path below from $a=(0,n)$ to $b=(m,0)$, paramaterized by arc length



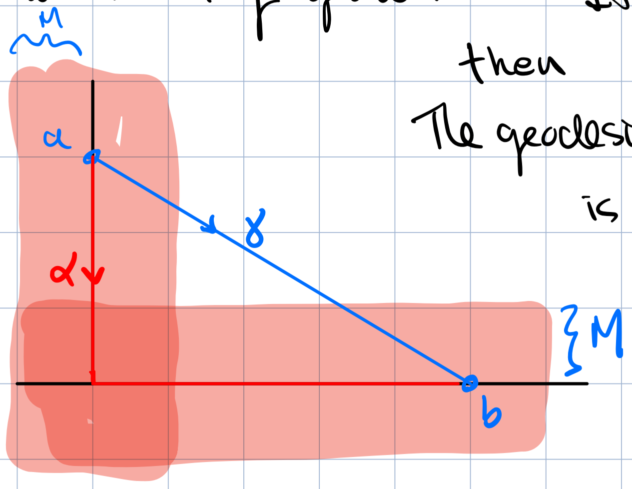
Claim: α is a $(\sqrt{2},0)$-q.geodesic

$$\frac{1}{\sqrt{2}}(x+y) - 0 \le d(x,y) \overset{\checkmark}{\le} (x+y)+0$$

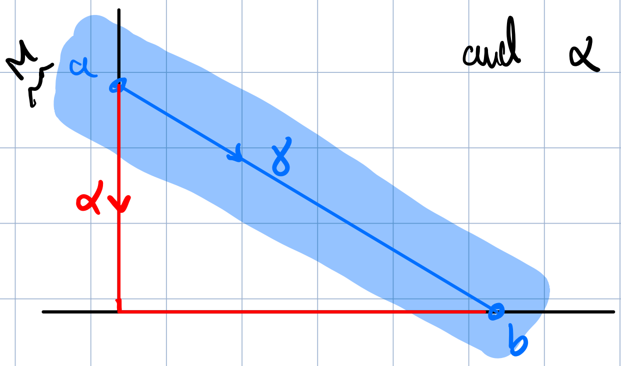
ie $x+y \overset{\checkmark}{\le} \sqrt{2}\sqrt{x^2+y^2}$

Note there is no constant M st quasi-geodesics stay within M of geodesics: If a and b are >> M, then

The geodesic γ from a to b is not within M of the quasi-geodesic α.



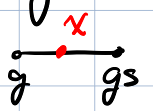
and α is not within M of γ



eg The map $f: \mathcal{C}(G,S) \longrightarrow \mathcal{C}(G,S)$

sending $g \longmapsto g$

and



$\longmapsto g$

is a quasi-isometry.

So if $\alpha: I \longrightarrow \mathcal{C}(G,S)$ is a geodesic,

then $f\circ\alpha: I \longrightarrow \mathcal{C}(G,S)$ is a quasi-geodesic (with $\lambda = 1, c = 2$)

Note it is <u>not</u> a continuous map!

But it will be convenient for proofs to just use <u>continuous</u> quasigeodesics.

Lemma: Given $\alpha: [0,\ell] \longrightarrow X$ a (λ, c)-quasi-geodesic. Then there is a continuous $(\lambda, 2(\lambda+c))$-quasi-geodesic $\beta: [0,\ell] \longrightarrow X$ s.t.

① $\beta(0) = \alpha(0)$, $\beta(\ell) = \alpha(\ell)$

② $\beta \subset N_{\lambda+c}(\alpha)$, $\alpha \subset N_{\lambda+c}(\beta)$

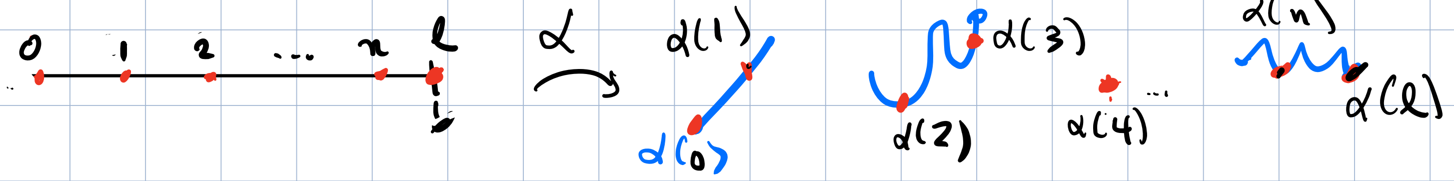
and

③ If $s,t \in [0,\ell]$ then

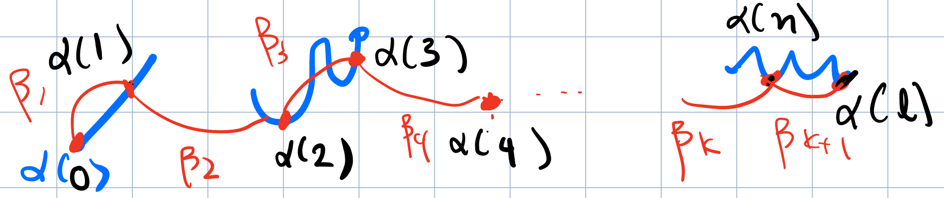
$$\ell(\beta|_{[s,t]}) \leq \lambda' d(\beta(s), \beta(t)) + c'$$

for constants λ', c' depending on δ, λ, c

Proof: let $0, 1, \dots n = [\ell]$ be the integer points in $[0,\ell]$:



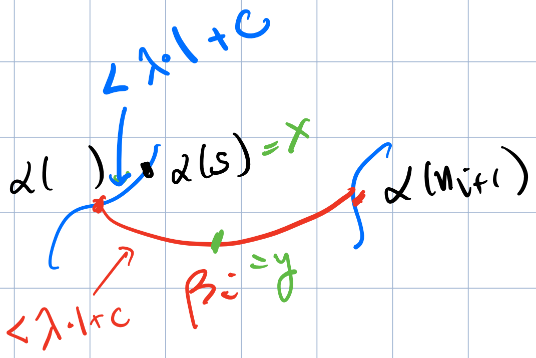
Connect $\alpha(n)$ to $\alpha(\ell)$ and $\alpha(i)$ to $\alpha(i+1)$ for all i by geodesics $\beta_1, \dots, \beta_{k+1}$:



Then $\ell(\beta_i) \le d(\alpha(i-1), \alpha(i)) \le \lambda \cdot 1 + C$ for all i.

So $\beta \subseteq N_{\lambda+C}(\alpha)$

and $\alpha \subseteq N_{\lambda+C}(\beta)$



Paramaterize β so that $\beta(i) = \alpha(i)$ for all i and $\beta(\ell) = \alpha(\ell)$

Then $d(\alpha(s), \beta(s)) \le 2(\lambda + C)\ \forall s$

To see that β is a quasi-geodesic, check the inequalities ie $\exists\, \lambda', C'$ st $\frac{1}{\lambda'}|s-t| - C' \underset{②}{\le} d(\beta(s),\beta(t)) \underset{①}{\le} \lambda'|s-t| + C'$

① $d(\beta(s),\beta(t)) \le d(\beta(s),\alpha(s)) + d(\alpha(s),\alpha(t)) + d(\alpha(t),\beta(t))$

$\le 4(\lambda+c) + d(\alpha(s),\alpha(t))$

$\le 4(\lambda+c) + \lambda|s-t| + C = \lambda|s-t| + C'$

② $|s-t| \le \lambda\, d(\alpha(s),\alpha(t)) + C\lambda$

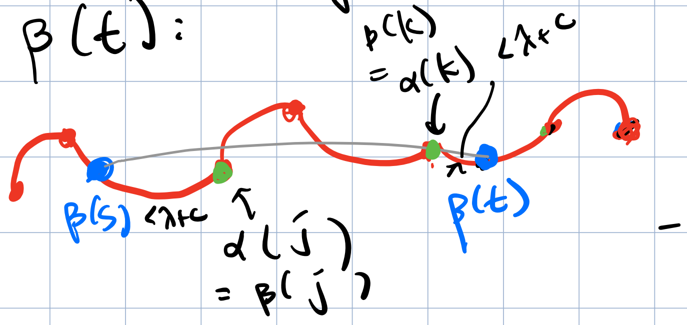
$\le \lambda\left[d(\alpha(s),\beta(s)) + d(\beta(s),\beta(t)) + d(\beta(t),\alpha(t))\right] + C\lambda$

$\le \lambda\left(4(\lambda+c) + d(\beta(s),\beta(t))\right) + C\lambda$

$= \lambda\, d(\beta(s),\beta(t)) + \underline{C\lambda + 4\lambda(\lambda+c)}$

ie $d(\beta(s),\beta(t)) \geq = \frac{1}{\lambda}|s-t| - C''$ ✓

It remains to prove ③, ie we want to bound the length of β from $\beta(s)$ to $\beta(t)$ in terms of the distance from $\beta(s)$ to $\beta(t)$:



$$l(\beta|_{[s,t]}) \le (\lambda+c) + (k-j)\cdot(\lambda+c) + (\lambda+c)$$

$$= (k-j+2)(\lambda+c) \le (s-t)(\lambda+c)$$

$$\le \lambda\,(d(\beta(s),\beta(t)) + \lambda C''$$ ✓

OK, so we can replace our arbitrary quasi-geodesic by a continuous quasi-geodesic, just have to show continuous quasi-geodesics stay close to geodesics.

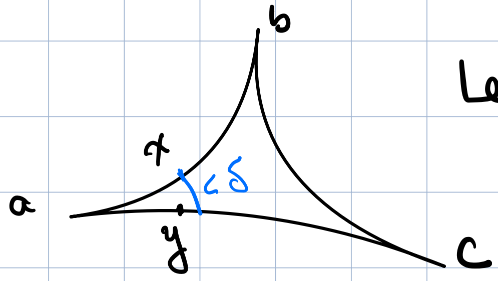
Notation: $[x,y]$ means a geodesic from x to y

First: look more closely at geodesic triangles!

Geodesic triangles in a δ-hyperbolic space

Let Δ be a geodesic triangle with vertices a, b, c

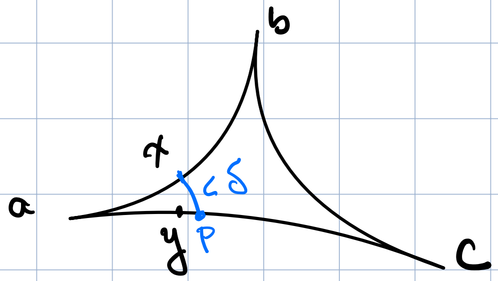
① Let $x \in [a,b]$. Then $d(x,[a,c]) < \delta$ or $d(x,[b,c]) < \delta$; assume the former.



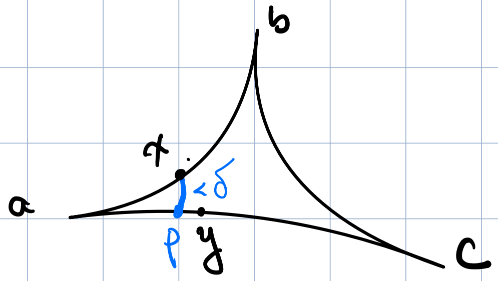
Let $y \in [a,c]$ with $d(a,x) = d(a,y)$

Then $d(x,y) < 2\delta$

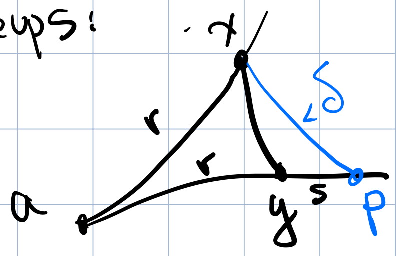
proof: 2 cases



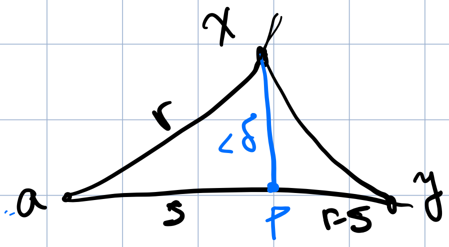
and



closeups:



and



$r+s \le r+\delta \Rightarrow s \le \delta$

$\Rightarrow d(x,y) \le s+\delta \le 2\delta$ ✓

$r \le s+\delta \Rightarrow r-s \le \delta$

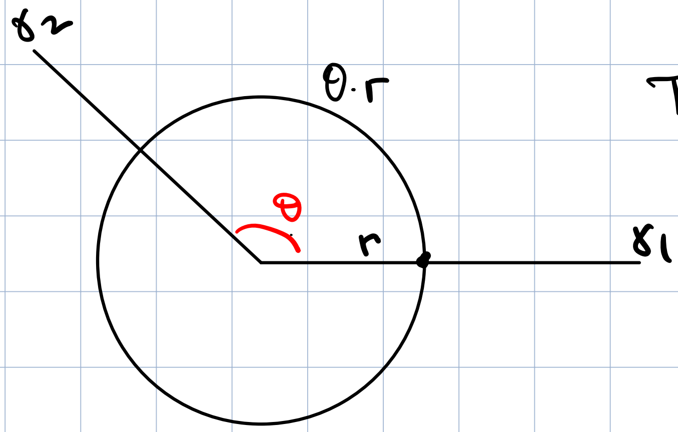
$\Rightarrow d(x,y) \le \delta + (r-s) \le 2\delta$ ✓

Now use this to show that geodesics starting at the same point diverge exponentially fast.

What does this mean?

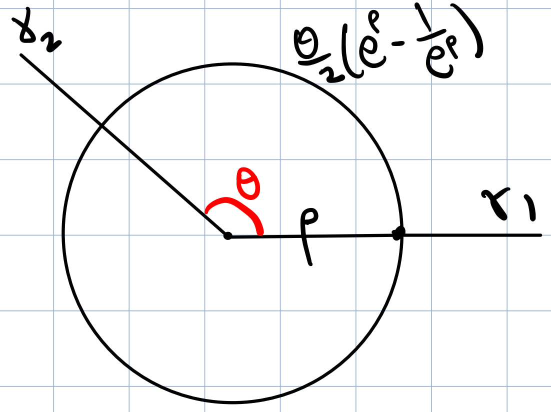
In $\mathbb{R}^2$ the circumference of a circle is $2\pi r$

Two geodesics emanating from the center of the circle at angle θ cut an arc of length θr on the circle



The length of this arc grows linearly with r

In $\mathbb{D}$ the circumference of a circle of (hyperbolic) radius ρ is $\pi\cdot(e^{\rho}-\frac{1}{e^{\rho}})$



(see notes on the Poincaré disk model $\mathbb{D}$, posted on Moodle)

The length of the arc subtended by θ grows exponentially with ρ

We say the geodesics γ_1 and γ_2 diverge exponentially.

If X is a metric space with more than one end, then geodesics that go out different ends can't be connected at all outside some B_r; we also say they diverge exponentially

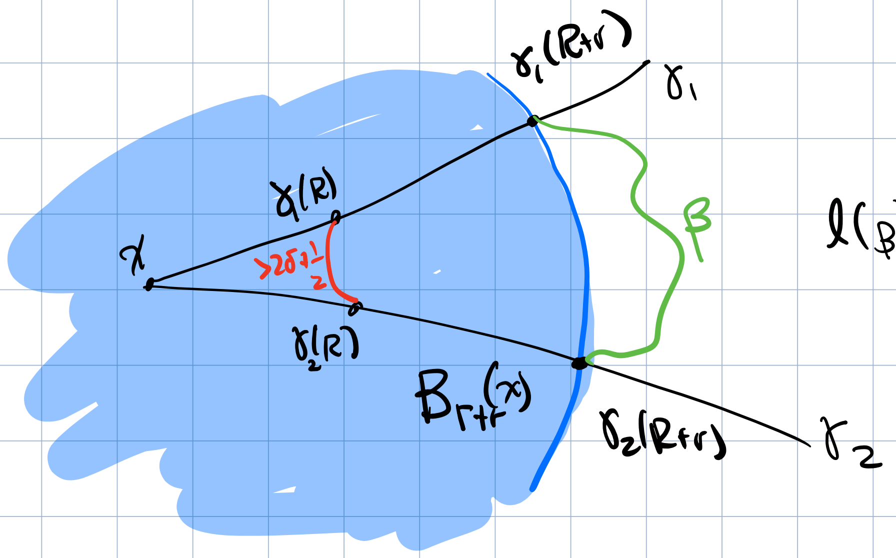
Theorem Let X be δ-hyperbolic, $x \in X$ and γ_1, γ_2 geodesics starting at x, paramaterized by arc length,

Suppose that $d(\gamma_1(R), \gamma_2(R)) \geqslant 2\delta + \frac{1}{2}$ for some $R > 0$.

Then there are constants $K, \mu > 0$ such that for all $r > 0$, If β is a path from $\gamma_1(R+r)$ to $\gamma_2(R+r)$ that stays outside $B_{R+r}(x)$, then

$$\ell(\beta) > K e^{\mu \cdot r}$$

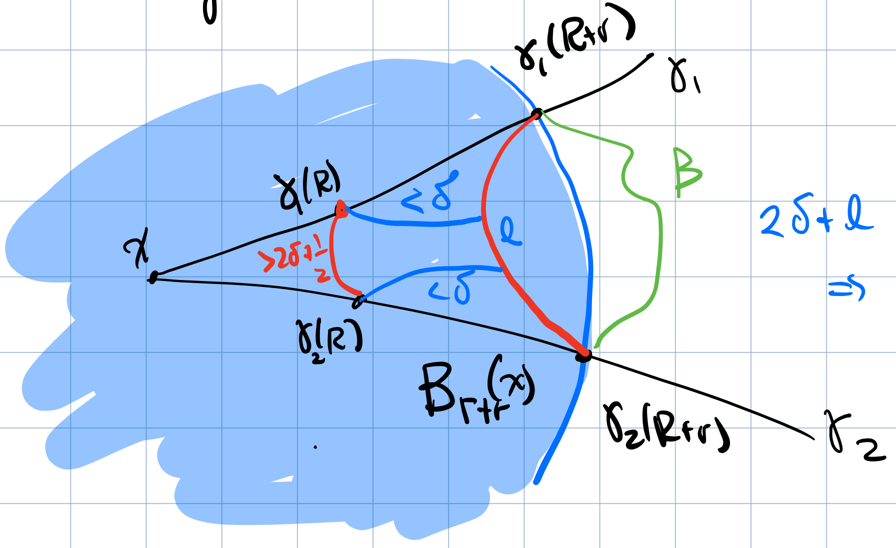
Picture:



$\ell(\beta) > K e^{\mu r}$.

Proof:

Since $d(\gamma_1(R), \gamma_2(R)) > 2\delta$, we know both $\gamma_1(R)$ and $\gamma_2(R)$ are within δ of the geodesic from $\gamma_1(R+r)$ to $\gamma_2(R+r)$



$2\delta + \ell \geq 2\delta + 1/2$

$\Rightarrow \ell \geq 1/2$

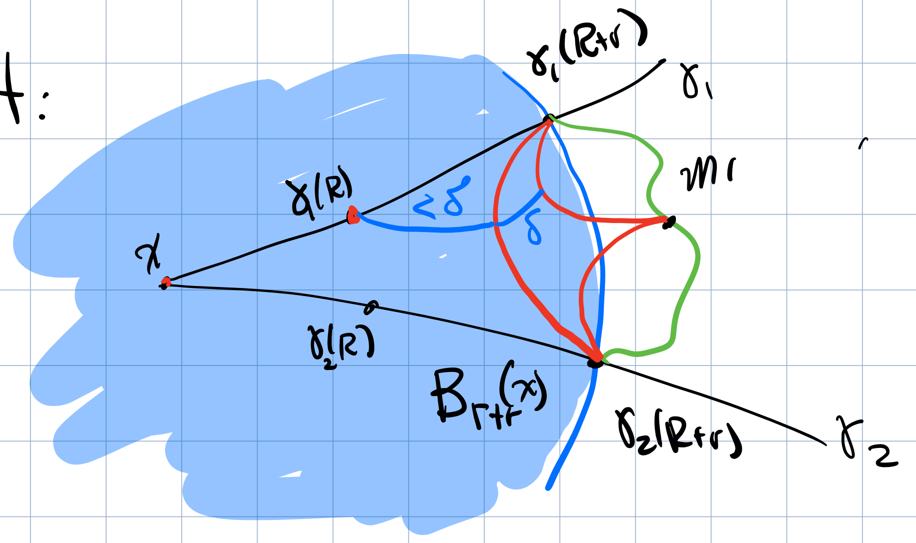
So $\quad d(\gamma_1(R+r), \gamma_2(R+r)) \geq \frac{1}{2}$ for any r

So $\quad \ell(\beta) \geq \frac{1}{2}$ $\quad \therefore$ There is some $n \geq 0$ with

$$2^{n-1} \leq \ell(\beta) < 2^n.$$

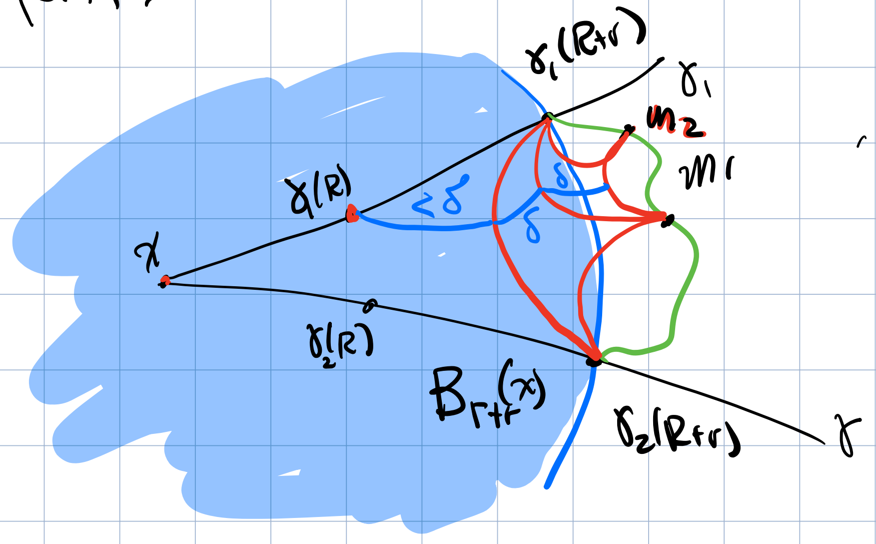
If we take midpoints n times we cut β into 2^n pieces, each of length between $\frac{1}{2}$ and 1. We will use the fact that all the midpoints are outside $B_{R+r}(x)$ to estimate n, and therefore $\ell(\beta)$, in terms of r.

First midpoint:



$$R+r < d(x, m_1) < R + 2\delta + d(\gamma_1(R+r), m_1)$$
$$\leq R + 2\delta + \frac{\ell(\beta)}{2}$$

Next midpoint:



$$R+r < d(x, m_2) < R + 3\delta + d(m_1, m_2) \leq R + 3\delta + \frac{\ell(\beta)}{4}$$

etc. after n times get

$$\cancel{R}+r < d(x, m_n) < R + (n+1)\delta + \frac{\ell(\beta)}{2^n}$$
$$< \cancel{R} + (n+1)\delta + 1$$

$$r < (n+1)\delta + 1$$

$$\Leftrightarrow \frac{r-1}{\delta} < n+1$$

$$\Leftrightarrow n-1 > \frac{r}{\delta} - \frac{1}{\delta} - 2$$

So $\ell(\beta) \geq 2^{n-1} \geq \underbrace{2^{\frac{r}{\delta}}}_{e^{\mu \cdot r}} \cdot \underbrace{2^{-\frac{1}{\delta}-2}}_{K} = e^{\mu \cdot r} \cdot K$ ✓

Now that we understand both quasi-geodesics and geodesics better, we can prove our theorem:

Theorem: Let X be a δ-hyperbolic space,

$x, y \in X$,

$\alpha : [a,b] \to X$ a (λ, c)-quasi-geodesic with $\alpha(a) = x$, $\alpha(b) = y$,

γ a geodesic from x to y

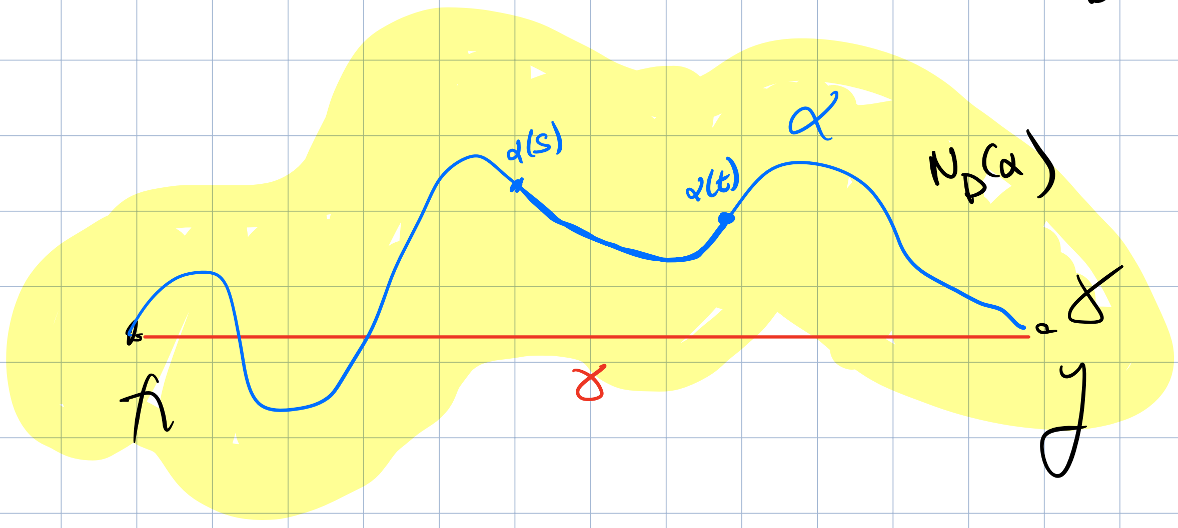
Then there is $D = D(\delta, \lambda, c)$ s.t.

① $\gamma \subset N_D(\alpha)$ and ② $\alpha \subset N_D(\gamma)$

Proof The lemma we proved earlier shows we may assume α is continuous and

$$\ell(\alpha|_{[s t]}) \le \lambda d_X(\alpha(s),\alpha(t)) + C.$$

First show ①: There is D s.t. $\gamma \subset N_D(\alpha)$



Idea: Find $z \in \gamma$ as far as possible from α, say distance r, show r is bounded by a constant $D = D(\delta,\lambda,C)$

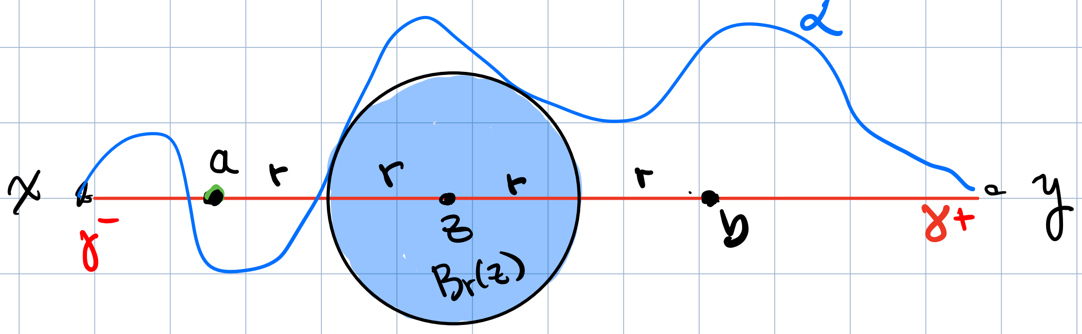
Details:

Let $z \in \alpha$, with $d(z,\gamma) = r$ maximal.

Then the ball $B_r(z)$ is disjoint from α. Divide γ into two geodesics $\gamma^- = [x,z]$, $\gamma^+ = [z,y]$

Let $a \in \gamma^-$, $d(a,z) = 2r$

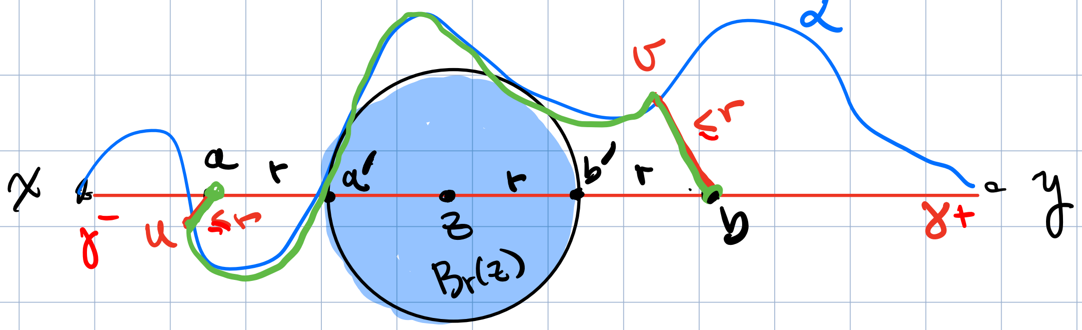
$b \in \gamma^+$, $d(z,b) = 2r$



Pick $u, v \in \alpha$ with $d(u,a) \le r$ and $d(v,b) \le r$

Join a to b by a (green) path that

- follows a geodesic from a to u
- follows α from u to v
- follows a geodesic from v to b:



This path stays outside $B_r(z)$ and has length

$$L \le r + (\lambda d_X(u,v) + c) + r \quad \text{(by lemma)}$$

$$\le 2r + \lambda(6r) + c$$

Now a and b are on geodesics that start at the same point z, so if $r > 2\delta + \frac{1}{2}$ any path joining them and staying outside $B_r(z)$ has length

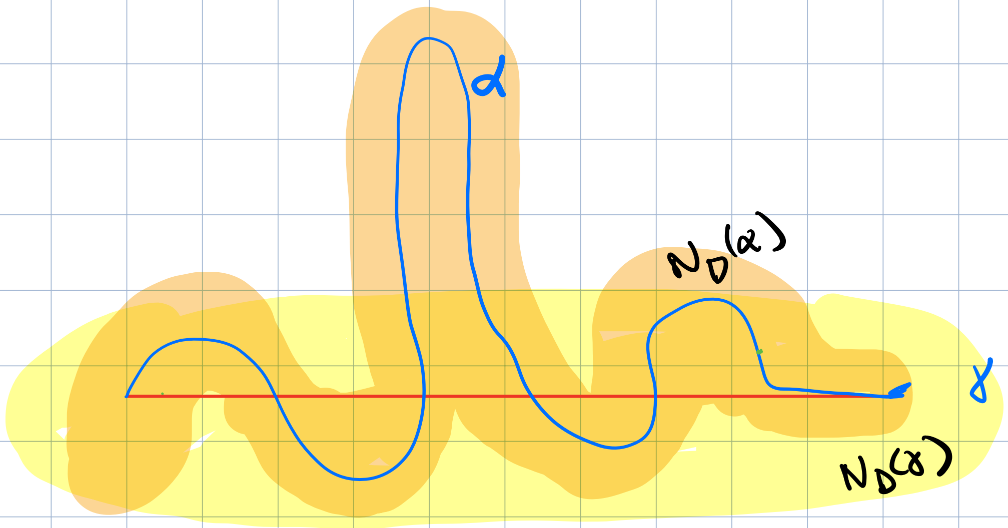
$L > Ke^{r\mu}$ for some constants K, μ (depending on δ, λ and C)

$$\underset{\substack{\uparrow \\ \text{exponential} \\ \text{function in } r}}{Ke^{r\mu}} < L < \underset{\substack{\uparrow \\ \text{Linear function of } r}}{r(2+6\lambda)+C}$$

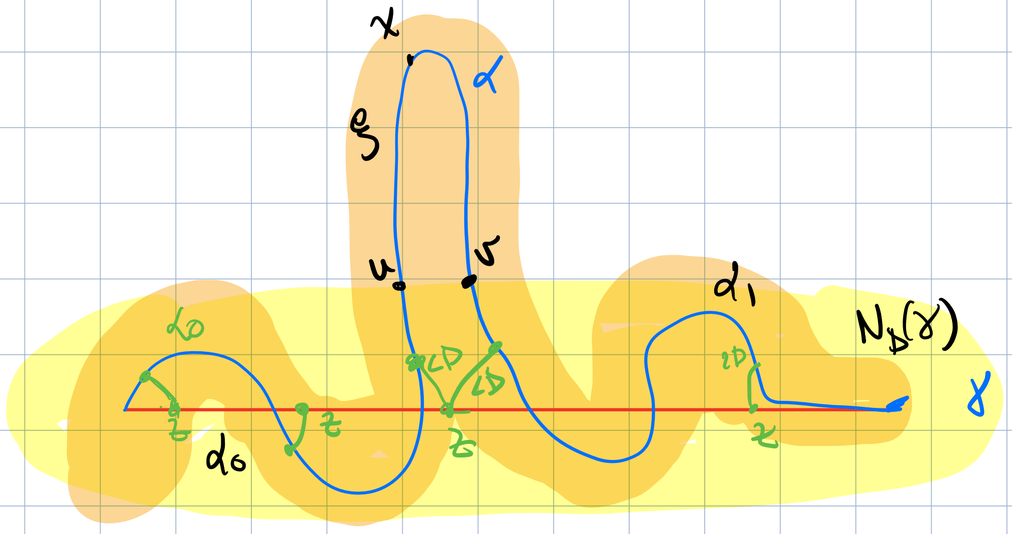
Exponential functions grow faster than linear ones, so there is some D s.t. $r < D$ ✓

② $\exists D'$ s.t. $\alpha \subset N_{D'}(\gamma)$ (then take $D = \max(D, D')$)

We know $\gamma \subset N_D(\alpha)$. Suppose α <u>leaves</u> $N_D(\gamma)$:



Let u, v be endpoints of the first interval ξ on α that leaves $N_D(\gamma)$:



Decompose $\alpha = \alpha_0 \cup \xi \cup \alpha_1$

If $z \in \gamma$, it is not within D of anything in ξ. But it is within D of <u>some</u> point of α, which must therefore be on α_0 or on α_1

γ starts close to α_0 on the left, then is close to α_1 on the right. Since α and γ are continuous, it is close to <u>both</u> at some z in between:

$$\exists\ u_1 = \alpha(s) \in \alpha_0, \quad v_1 = \alpha(t) \in \alpha_1,$$

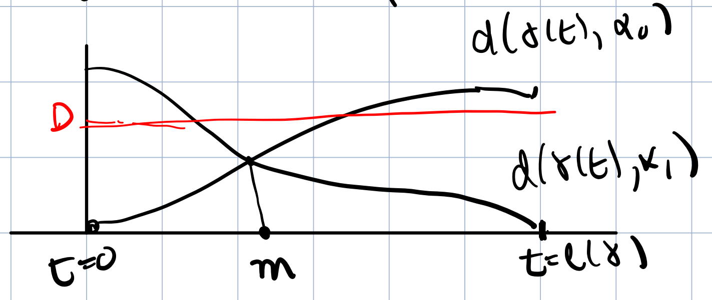
$$d(z, \alpha(s)),\ d(z, \alpha(t)) < D$$

(proof: α_0, γ continuous $\Rightarrow d(\gamma(t), \alpha_0)$ continuous

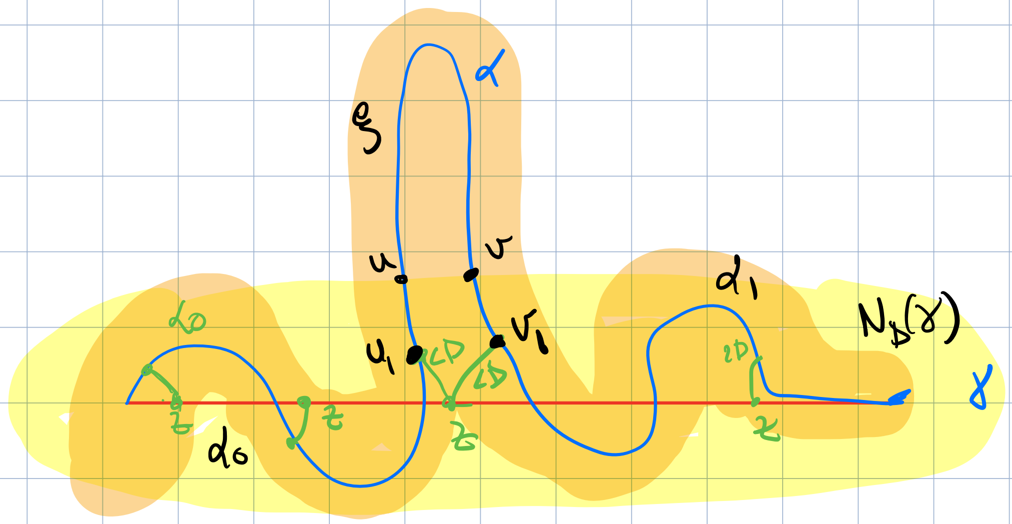
α_1, γ continuous $\Rightarrow d(\gamma(t), \alpha_1)$ continuous

$t=0 \Rightarrow d(\gamma(t),\alpha_0)=0$ $\quad$ $t=\ell(\gamma) \Rightarrow d(\gamma(t),\alpha_1)=0$

So the graphs of $t \mapsto d(\gamma(t), \alpha_i)$ cross at some point $t=m$ (possibly several),



)



Let $z=\gamma(m)$, $u_1=\alpha(s)\in\alpha_1$, $u_2=\alpha(t)\in\alpha_2$

with $d(z,u_1)=d(z,u_2)\leq D$

$\ell(\alpha|_{[s,t]}) \leq \lambda d_X(\alpha(s),\alpha(t))+C \leq \lambda 2D+C.$

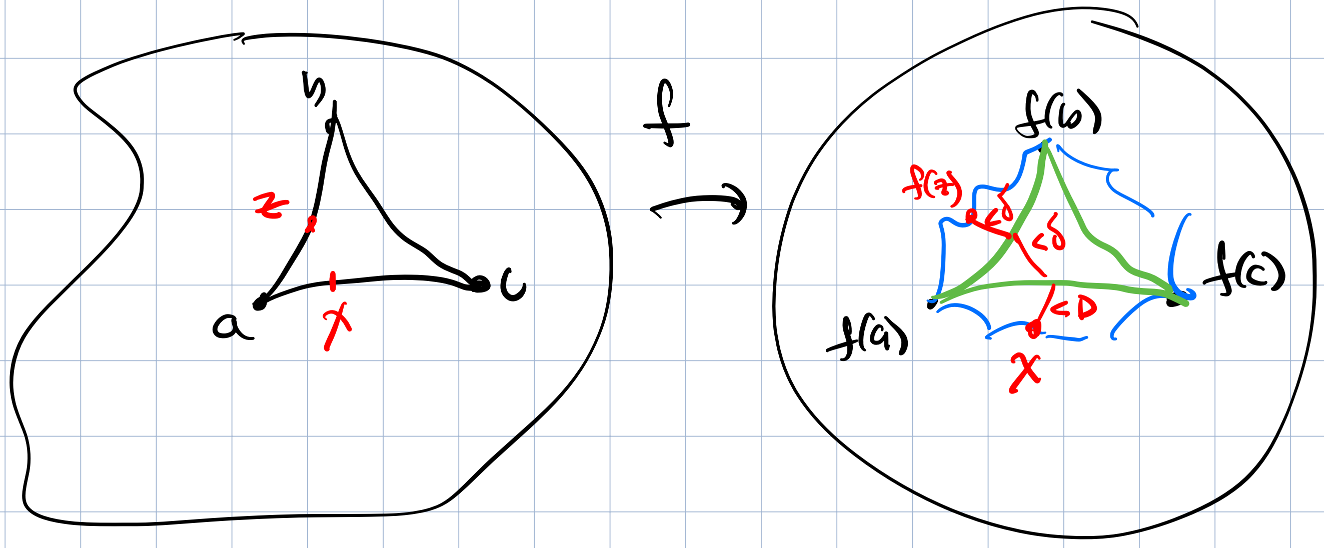
$\Rightarrow$ every point of ξ is within $D' = D + \lambda D + C/2$ from a point of γ

ie $\alpha \subseteq N_{D'}(\gamma)$ ✓

This completes the proof that hyperbolicity is a qi invariant. Here's a <u>recap</u>:

Suppose $f : X \to Y$ is a quasi isometry.

Then Y hyperbolic $\Rightarrow$ X hyperbolic



$$\lambda d(x,z) - C < d(f(x), f(z)) < 2D + \delta$$

$$\Rightarrow d(x,z) < \underbrace{\frac{1}{\lambda}\left(2D + \delta + c\right)}_{= \delta'}$$

There are a lot of hyperbolic groups:

- If you write down a random group presentation, (where "random" is appropriately defined) then the group is hyperbolic.

- If X is a compact manifold of negative Riemannian curvature, then $\tilde{X}$ has thin triangles (so is hyperbolic), and $\pi_1 X$ acts properly and cocompactly on $\tilde{X}$, so $\pi_1 X$ is hyperbolic

How can you use the geometry of the Cayley graph to study G?

Svarc-Milnor says a hyperbolic group is finitely generated (it acts properly and cocompactly on a hyperbolic metric space).

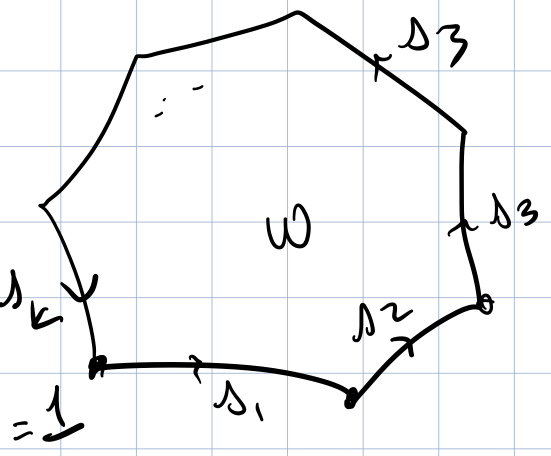
But in fact you can say something stronger

<u>Theorem</u> If G is hyperbolic, then it has a finite presentation.

Idea: Let S be a finite generating set for G. We want to find $R \subseteq F(S)$ such that any word w in the generators that evaluates to the identy in G is a product of conjugates of elements of R

Since $w = 1$ in G, it gives a loop in $\mathcal{C} = \mathcal{C}(G,S)$.

$w = s_1 s_2 s_3 \dots s_k$



$s_1 \dots s_k = 1$

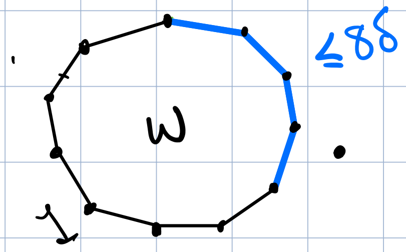
Key point: In a hyperbolic space, long loops have short segments that are not geodesics, where "short" depends only on δ.

[Not true if $\mathcal{C}(G,S)$ is not hyperbolic, eg Look at $\mathcal{C}(\mathbb{Z}^2, (1,0),(0,1))$. In a $k \times k$ square, every segment of length $\leq k$ is a shortest path, ie a geodesic.]

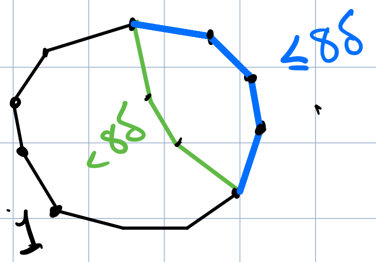
Here's a precise statement.

Proposition If w is any loop in a δ-hyperbolic Cayley graph $\mathcal{G}$, it has a segment of length $\leq 8\delta$ that is not a geodesic.

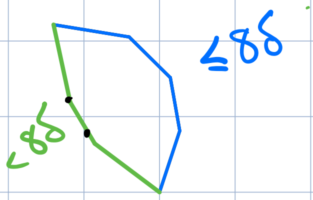
The theorem follows: Let R be the set of all words in S that evaluate to 1 in G and have length $< 16\delta$. (This is a finite set!) Then w contains more than half of a relator. If $\ell(w) < 16\delta$ it Is a relater. If $\ell(w) \geq 16\delta$



. Draw a geodesic

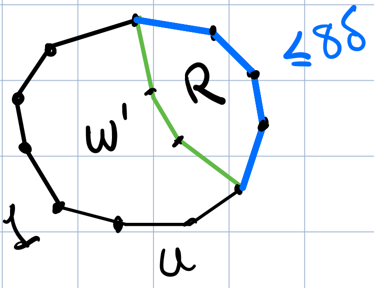


Then



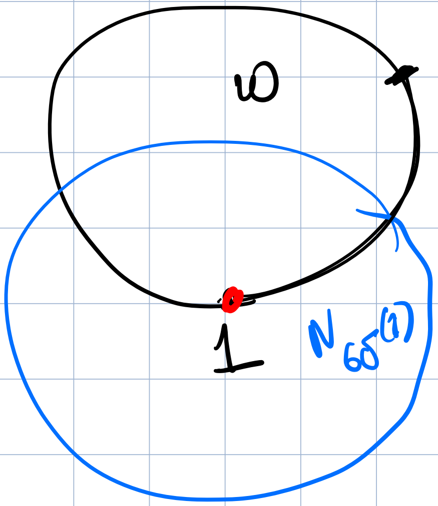
has length $< 16\delta$, so is in R.

Then $w = (uRu^{-1})w'$:



with w' shorter than w, and we can continue until w is a product of conjugates of relators.

To prove a loop ω of length $> 16\delta$ has a segment of length $\leq 8\delta$ that is not a geodesic, suppose this is false, and show this $\Rightarrow$ $\omega \subseteq N_{6\delta}(1)$.

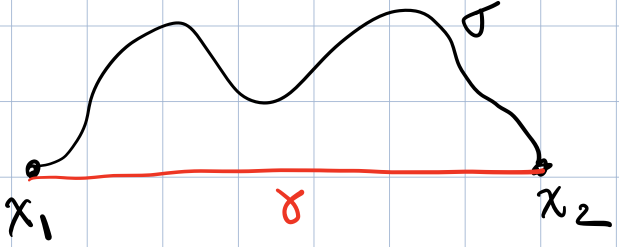


$\ell(\omega) \geq 16\delta \Rightarrow$ has a segment starting at 1 of length $\geq 8\delta$.

If this is a geodesic, then $\omega \nsubseteq N_{6\delta}(1)$!

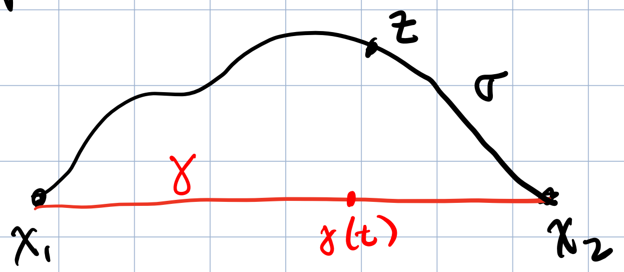
We'll actually prove a more general statement — for any path σ instead of just a loop.

Prop Suppose all segments of a path σ that have length $\leq 8\delta$ are geodesics.
Let γ be a geodesic between the endpoints x_1 and x_2 of σ.



Then $\sigma \subseteq N_{6\delta}(\gamma)$

Proof Let z be a point on σ.

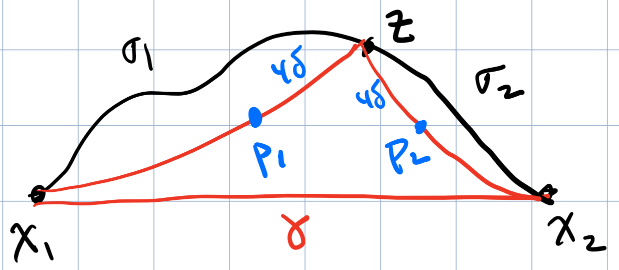


If σ is a geodesic, then $d(z,\gamma) < \delta$

(consider the triangle with vertices x_1, x_2, x_2 and sides σ, γ and x_2)

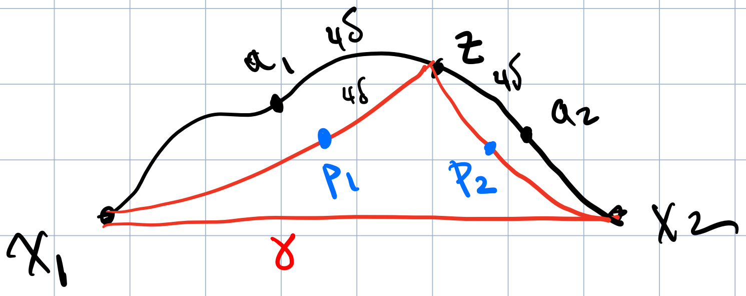
Furthermore, if $d(z,x_1) = d(\gamma(t), x_1)$ then $d(z,\gamma(t)) < 2\delta$.

If σ is not a geodesic, draw geodesics $[x_i, z]$ and mark points p_1, p_2 on them at distance 4δ from z:



z cuts σ into two pieces, σ_1 from x to z and σ_2 from z to y

Mark points a_i on σ_i at distance 4δ from z along σ (so the segment a_1 to a_2 is geodesic.)



Claim $d(p_i, a_i) < 2\delta$. Assuming this, we have

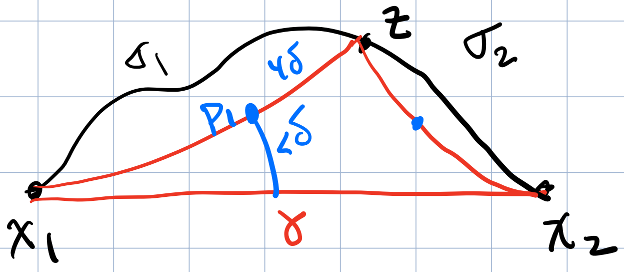
$$d(a_1,p_1) + d(p_1,p_2) + d(p_2,a_2) \geq d(a_1,a_2) = 8\delta$$

so $d(p_1,p_2) \geq d(a_1,a_2) - d(a_1,p_1) - d(a_2,p_2)$

$$\geq 8\delta - 2\delta - 2\delta = 4\delta$$

$\therefore$ $d(p_1,p_2) > 2\delta$, so $d(p_1,\gamma) < \delta$, so

$d(z,\gamma) < 5\delta$ and we are done.

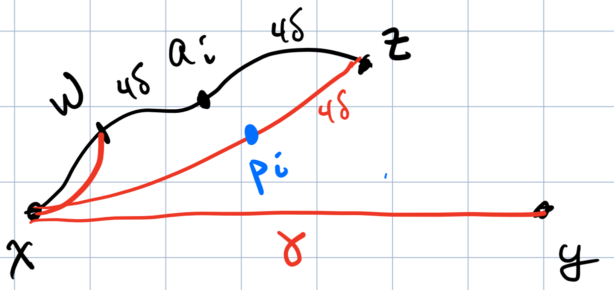


To prove the claim, we'll induct on the length of the σ_i

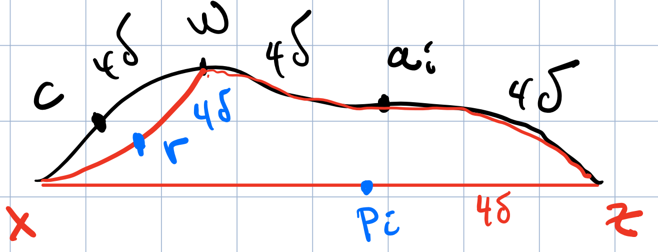
If σ_i has length $\leq 8\delta$ it is geodesic

so $d(p_i a_i) < 2\delta$

If σ_i is not a geodesic, cut off another 4δ-piece at a point w:



This is the same picture we had before, but with shorter pieces (and σ from w to z is already geodesic):



so by induction $d(r,c) < 2\delta$

So $d(a_i, r) \geqslant d(a_i,c) - d(r,c)$

$\geqslant 8\delta - 2\delta = 6\delta$

So $d(a_i,[x,z]) < \delta$, and $d(a_i, p_i) < 2\delta$. ∎

The proposition we just proved can also be used to show that a hyperbolic group has only finitely many conjugacy classes of finite-order elements (See Exercise sheet)

Next we want to show that a hyperbolic group G cannot contain a free abelian subgroup $\mathbb{Z}^n$.

The centralizer of any $h \in \mathbb{Z}^n$ (ie the set of elements of G that commute with h) contains all of $\mathbb{Z}^n$, so

it suffices to show that the centralizer of any infinite-order element of a hyperbolic group is basically cyclic; more precisely, it has a finite-index subgroup that is cyclic (note all finite-index subgroups of $\mathbb{Z}^n$ are isomorphic to $\mathbb{Z}^n$).

To do this we will need to understand the behavior of powers of g in the Cayley graph

Proposition Suppose $g \in G$ has infinite order. Then the map $\mathbb{N} \longrightarrow G$ sending $n \longmapsto g^n$ is a quasi-isometric embedding

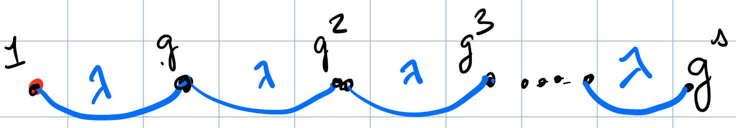
Proof: Since g has infinite order, $\{g^i\}$ leaves every ball around 1 in the Cayley graph $\mathcal{C}$.

We need to find λ, C s.t.

$$\frac{s}{\lambda} - C \le d_{\mathcal{C}}(1, g^s) \le \lambda s + C$$

The RH inequality is just the triangle inequality:

Let $\lambda = d(1,g)$

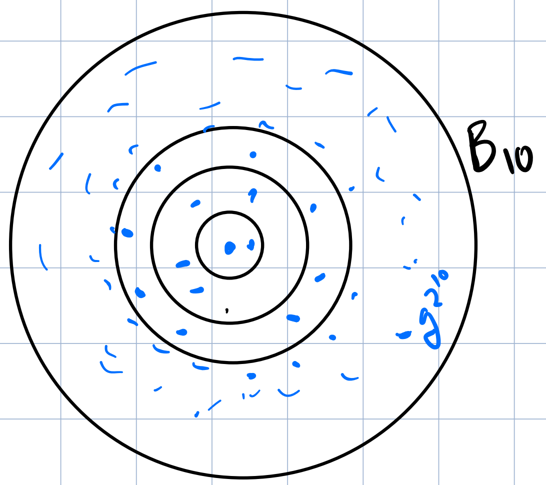


Then $d(1,g^s) \le \lambda s + 0$ ✓

The hard part is the LH inequality: $d(1,g^s) \geq \frac{1}{\lambda} s - C$

Balls in $\mathcal{C} = \mathcal{C}(G,S)$ are finite, but the number of vertices in $B_n(1)$ probably grows exponentially fast, so you might worry that the number of g^i inside $B_n(1)$ is exponential in n, so $d(1,g^s)$ does not grow linearly with s.

Claim: This doesn't happen ie



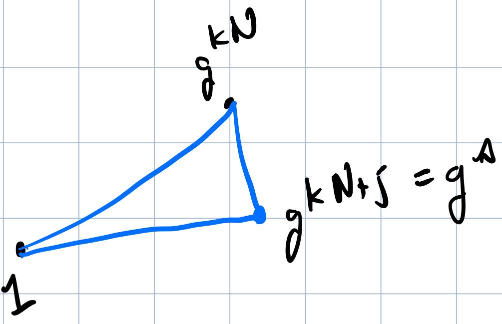
(*) There is N such that for any $k > 0$,

$$d(1, g^{Nk}) \geq k.$$

Assuming (*) we get, for any s

let k be such that $kN \leq s \leq (k+1)N$

and write $s = kN + j$ with $j < N$ $\left(\Leftrightarrow \frac{s}{N} - 1 \leq k \right)$

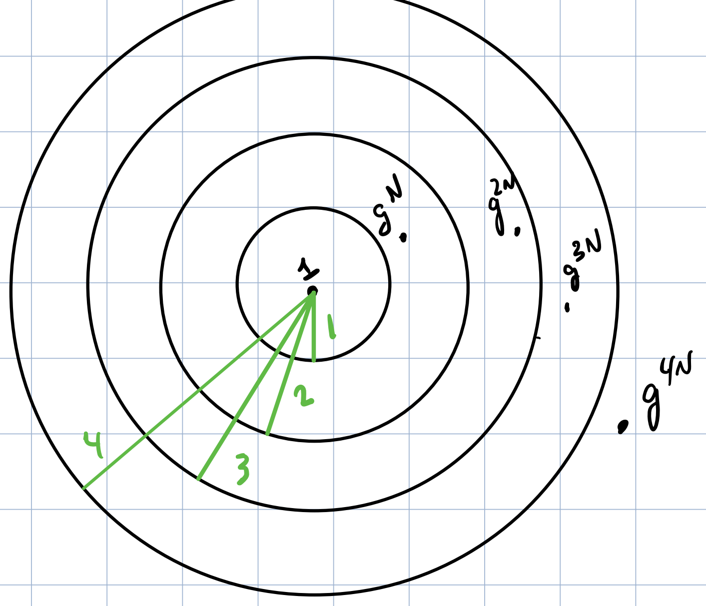


Then

$$k \leq d(1, g^{kN}) \leq d(1, g^s) + d(g^{kN+j}, g^{kN})$$
$$= d(1, g^s) + d(g^j, 1)$$

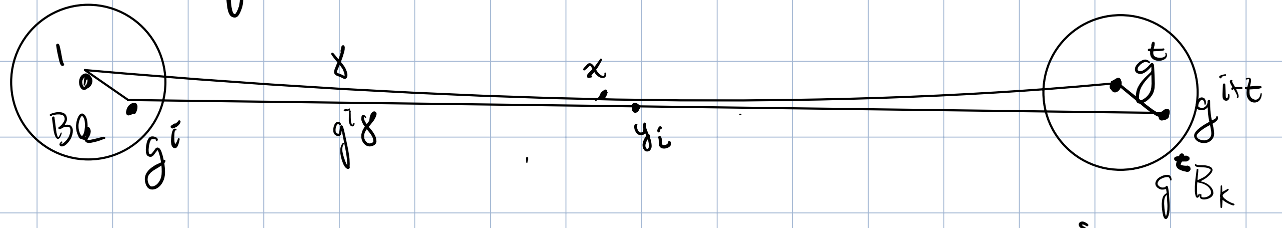
Let $C = \max_{j \leq N} d(g^j, 1)$

Then $d(1, g^s) \geq k - C \geq \frac{s}{N} - 1 - C$ ✓



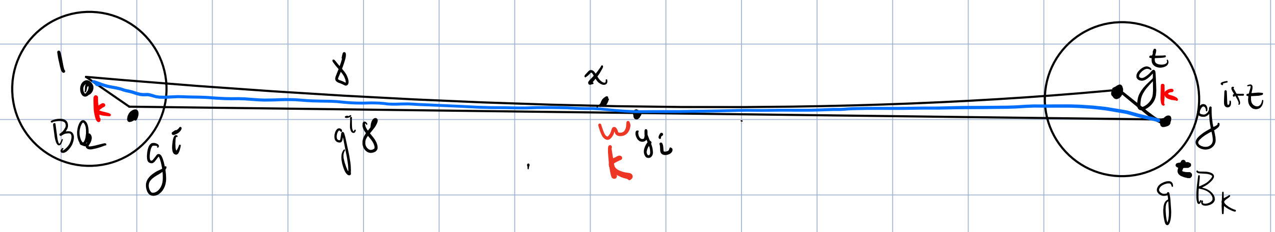
To prove (*) we will first bound the number of g^i that are in $B_k(1)$

Intuition: Suppose $g^i \in B_\ell$. For t large, a geodesic quadrilateral with vertices $1, g^i, g^t, g^{t+i}$ is very thin at the center



The midpoints y_i of all $g^i\gamma$ with $g^i \in B_K(1)$ are distinct, since the action of G on its Cayley graph is free and isometric

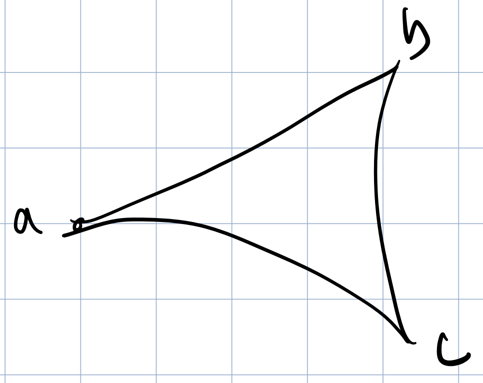
For t sufficiently large, x and y_i are both very close to the geodesic from 1 to g^{t+i}, but possibly shifted by k.



So $d(x, y_i) \leq k$ x some fudge factor C depending on δ

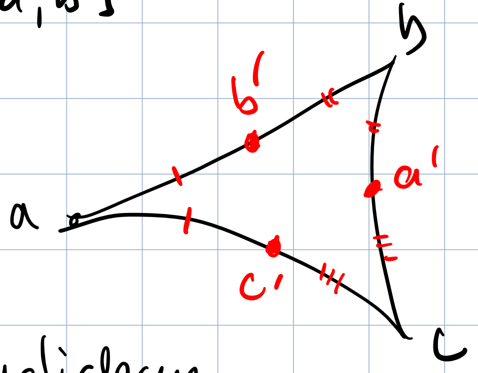
Since there are at most $k \cdot C$ midpoints y_i within k of x, there are at most $k \cdot C$ elements g^i in $B_1(k)$ ✓

To justify this intuition, first think a little more about triangles in a hyperbolic space X:

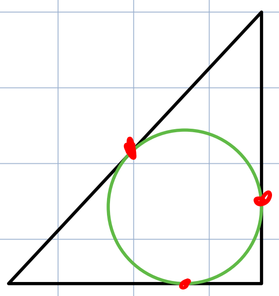


The <u>incenters</u> of $\Delta(a,b,c)$ are points
$a' \in [b,c]$
$b' \in [a,c]$
$c' \in [a,b]$

s.t. $d(a,b') = d(a,c')$
$d(b,a') = d(b,c')$
$d(c,a') = d(c,b')$

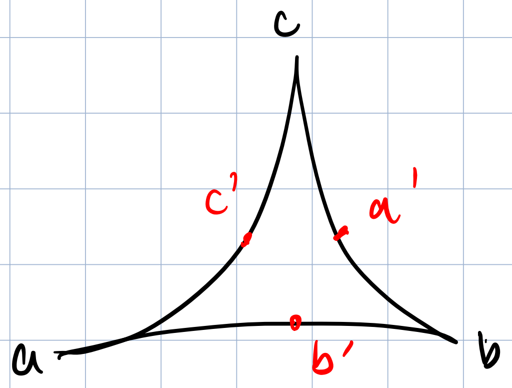


(To see they exist, consider a Euclidean triangle with the same side lengths:



Look at the largest possible inscribed circle)

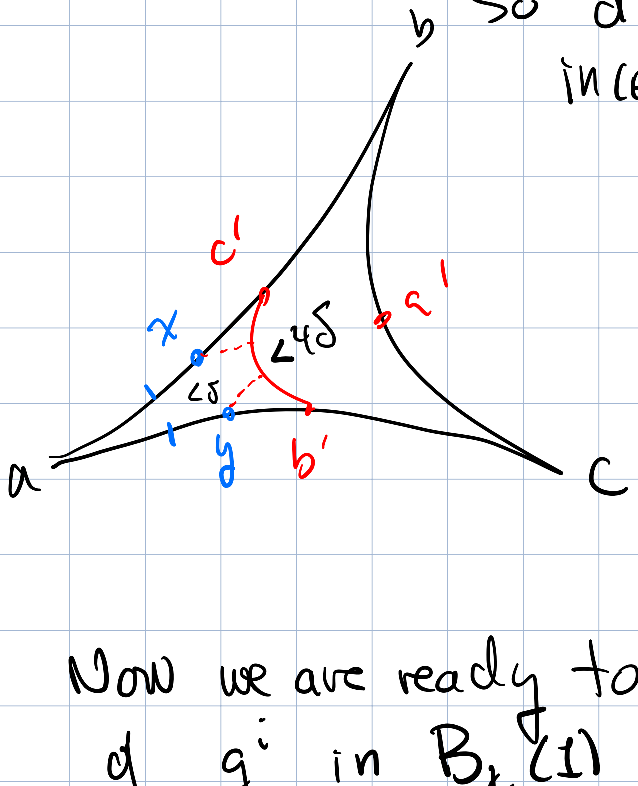
In a hyperbolic triangle



a' is δ-close to either [ab] or [a,c], so is within 2δ of either b' or c'

Similarly for b' and c'

So distance between any two incenters is < 4δ.

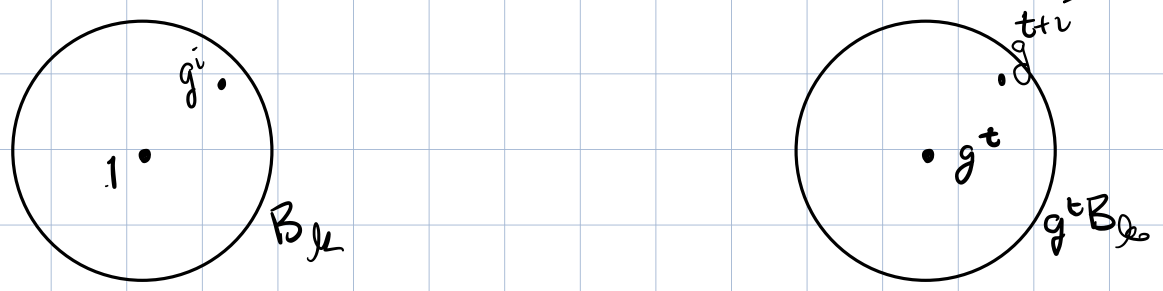


x on $[a,c']$, y on $[a,b']$, $d(a,x) = d(a,y)$

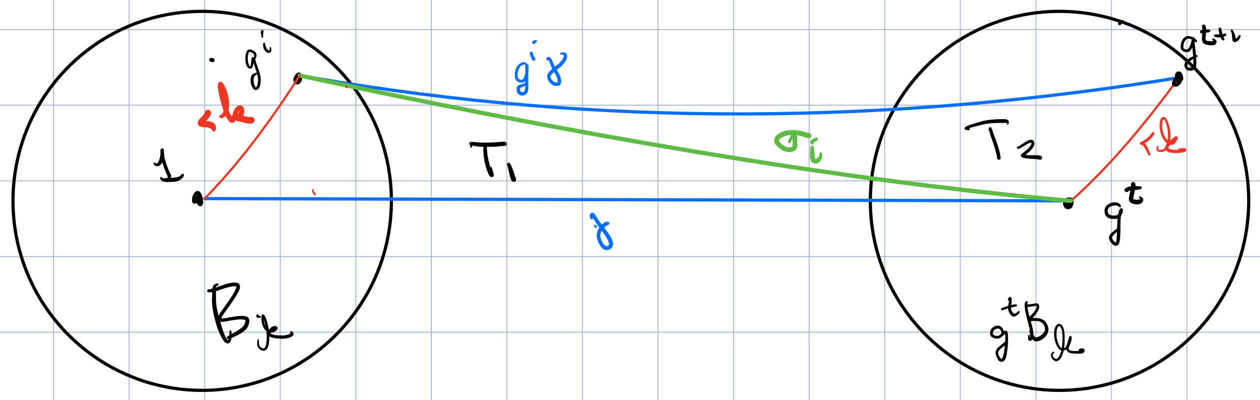
$\Rightarrow d(x,y) < 6\delta$

Now we are ready to estimate the number of g^i in $B_r(1)$ rigorously:

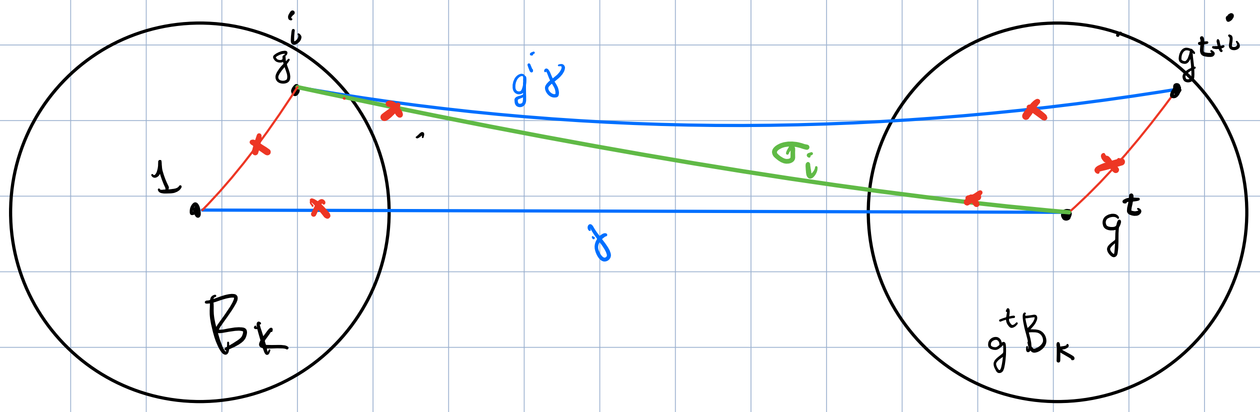
Fix k, and choose t so that $d(1, g^t) > 8k + 12\delta$



Draw geodesics γ from 1 to g^i, σ_i from g^i to g^t so that $\gamma, \sigma,$ and $g^i\gamma$, together with geodesics $[1, g^i]$ and $[g^t, g^{t+i}]$ form two triangles T_1 and T_2:



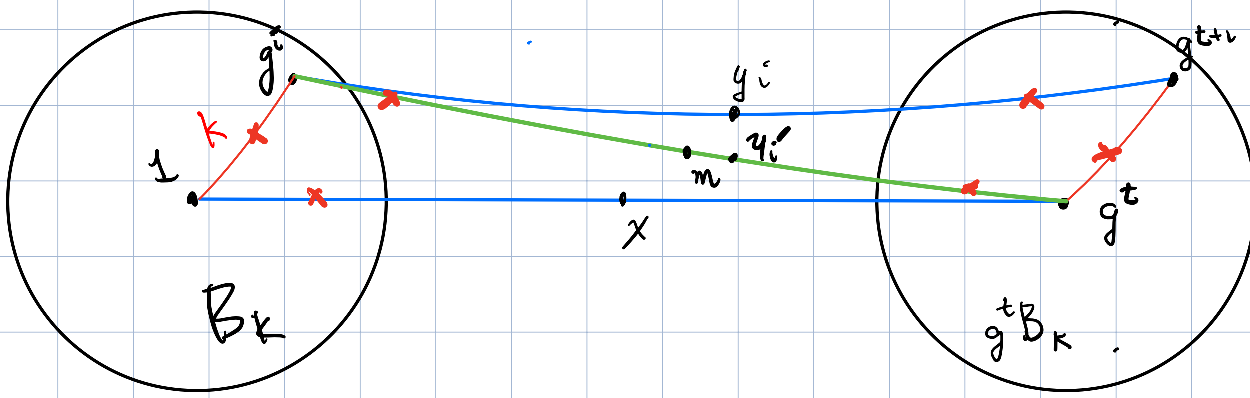
Since $d(1, g^t)$ is much larger than k, the incenters of T_1 are close to $B_k(1)$ and the incenters of T_2 are close to $B_k(g^t)$.



So points near the middle of γ and σ_i that are the same distance from g^t have distance $< 6\delta$.

Similarly, points near the middle of σ_i and $g^i\gamma$ that are the same distance from g^i are at most 6δ apart.

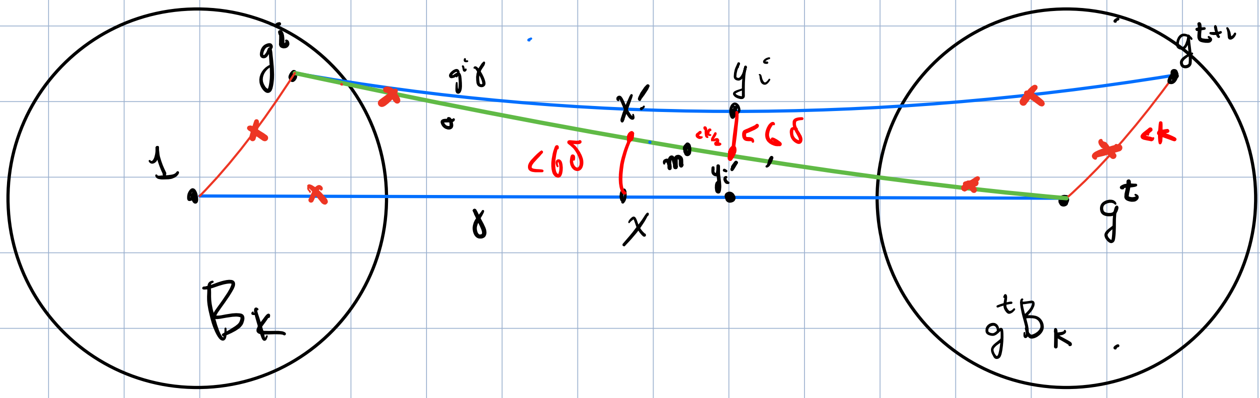
Let x = midpoint of γ, m_i = midpoint of σ_i and y_i = midpt of $g^i\gamma$



Let $y_i' \in \sigma_i$, $d(y_i', g^i) = d(y_i, g^i)$

Then $d(y_i', y_i) < 6\delta$ and

$$d(y_i', m) = \tfrac{1}{2} d(g^i, g^{t+i}) - \tfrac{1}{2} d(g^i, g^t)$$
$$= \tfrac{1}{2}\left(d(g^i, g^{t+i}) - d(g^i, g^t)\right)$$
$$\leq \tfrac{1}{2} k \quad \text{since } d(g^t, g^{t+i}) \leq k$$

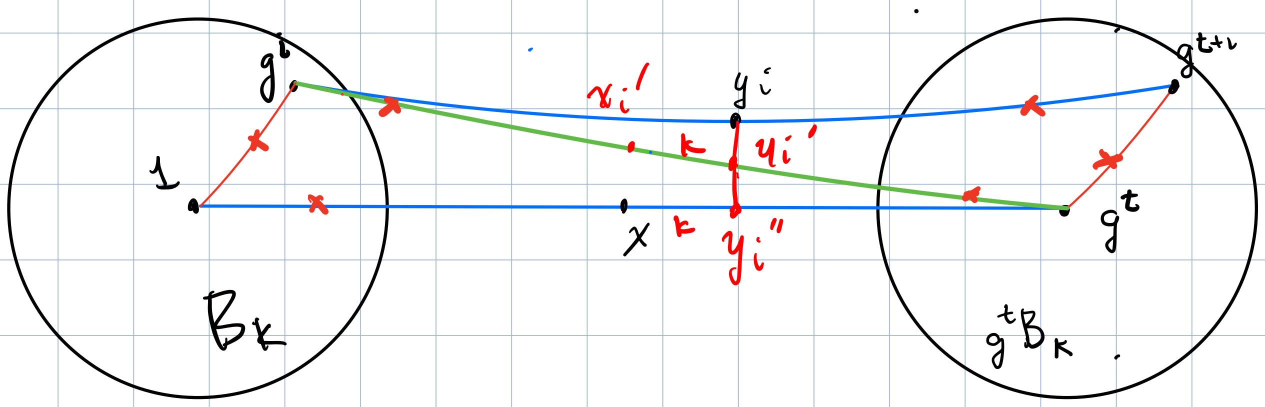


Similarly $d(x', m) \le \frac{1}{2}k$.

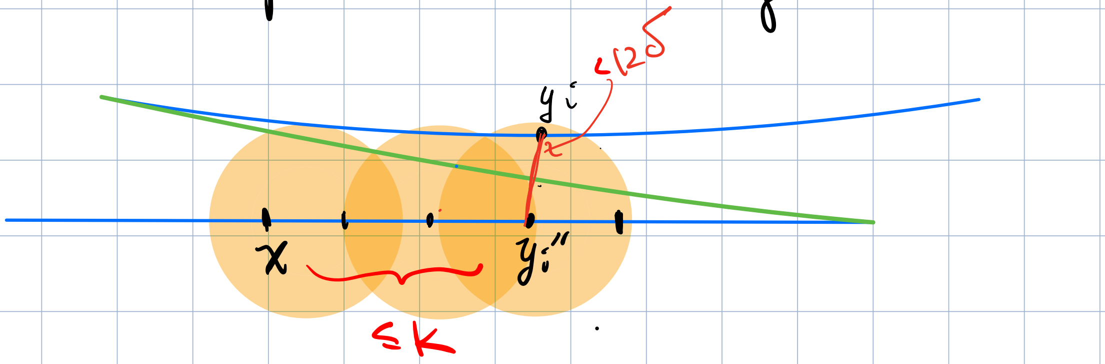
so $d(y_i', x') \le k$

Now find $y_i'' \in \gamma$ with $d(y_i', g^t) = d(y'', g^t)$

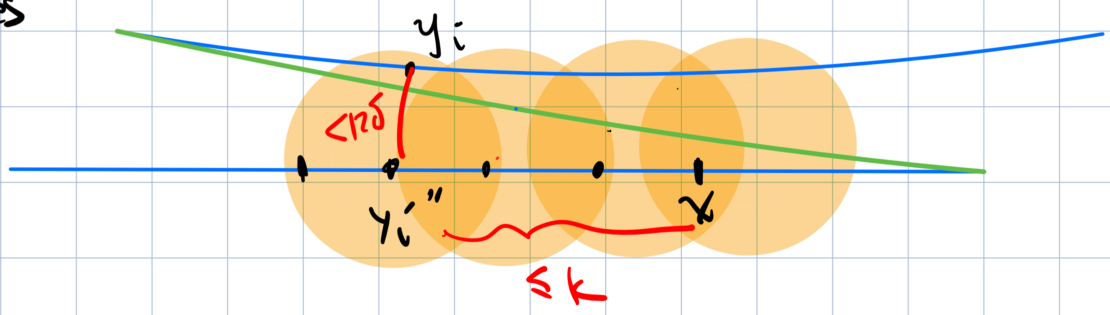
Then $d(x, y_i'') \le k$:



Cover $[x, y_i'']$ with $\le k$ balls of radius 12δ:



Then y_i is in the union of these balls,
If $B_{12\delta}(1)$ contains C vertices,
Then y_i is one of $k \cdot C$ possibilities
If y_i'' is on the other side of x, there are another $k \cdot C$ possibilities



ie y_i is in the union of $2k$ balls of radius $m = \max(1, 12\delta)$

let C = # of vertices in $B_m(1)$

Then y_i is one of $(2k+1)\cdot C < 3kC$ points

Since the $y_i = g^i x$ are all distinct, there are at most $3C\cdot k$ elements g^i with $g^i \in B_k(1)$.

So g^i escapes $B_k(1)$ for some $i = e(k) \leq 3Ck$.

We also know g^i <u>can't</u> escape $B_k(1)$ if $d(1,g)\cdot i \leq k$. Set $\lambda = d(1,g)$, so $e(k) \geq \frac{k}{\lambda}$.

$$\boxed{\frac{k}{\lambda} \leq e(k) \leq 3Ck}$$

<u>Claim</u> For all k, $d(1, g^{3Ck}) > k$.

<u>Proof</u> Suppose not, ie there is some k_0 with

$$d(1, g^{3Ck_0}) = k_0 - \varepsilon, \quad \text{with } \varepsilon \geq 1$$

For any k, write $e(k) = r\cdot(3Ck_0) + j$ with $j < 3Ck_0$

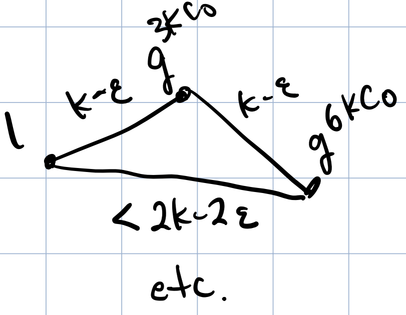
Since $e(k) < 3Ck$, we get

So $k_0 r + \frac{j}{3C} \leq k$

So $k_0 r \leq k$.

Then $k < d(1, g^{e(k)}) \leq d(1, g^{r\cdot(3Ck_0)}) + d(g^{r(3Ck_0)}, g^{r(3Ck_0)+j})$

$= d(1, g^{r(3Ck_0)}) + d(1, g^{j})$, $j < 3Ck_0$



$\leq r(k_0 - \varepsilon) + M = \max_{j<3Ck_0} d(1, g^{j})$

$< k_0 r - r\varepsilon + M$

Since $e(k) > \frac{k}{d(1,g)}$ we can make $e(k)$ (and therefore r) arbitrarily large, to get

$r\varepsilon > M$, so

$k < d(1, g^{e(k)}) < k_0 r - r\varepsilon + M < k_0 r < k$.

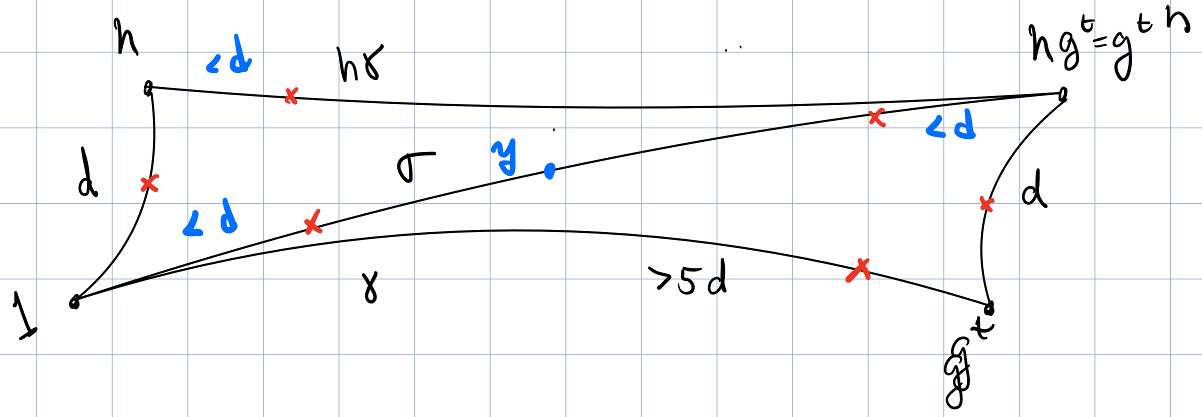
$\times$

Proposition If $g \in G$ has infinite order, then $\langle g \rangle$ has finite index in its centralizer $C\langle g \rangle$

Proof Let $h \in C(g)$, ie $hg = gh$

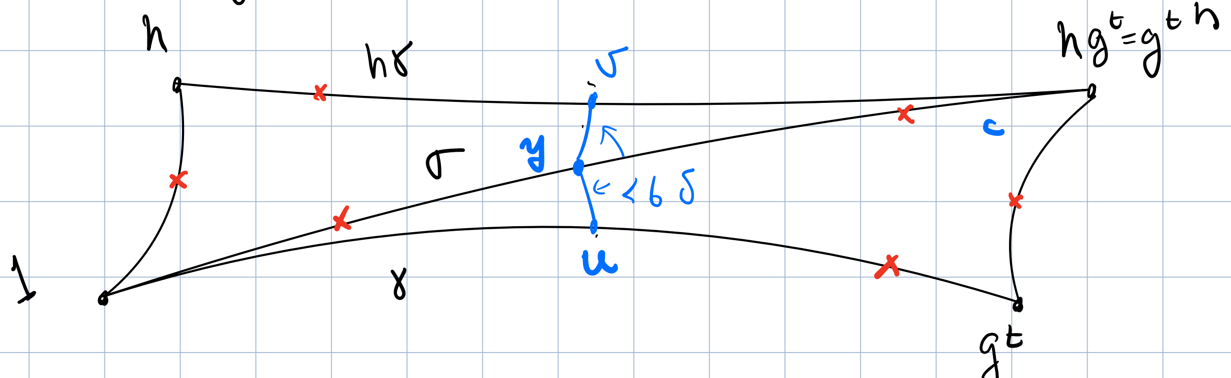
Choose t large enough so that $d(1, g^t) > 5 \cdot d(1, h)$

Let γ be a geodesic 1 to g^t, σ a geodesic 1 to $g^t h = h g^t$, y = midpoint of σ.



Since $\ell(\sigma) > 4d$, y is not close to the interior points of either triangle,

So there are $u \in \gamma$, $v \in h\gamma$ s.t. $d(u,y)$, $d(v,y) < 6\delta$, so $d(u,v) < 12\delta$

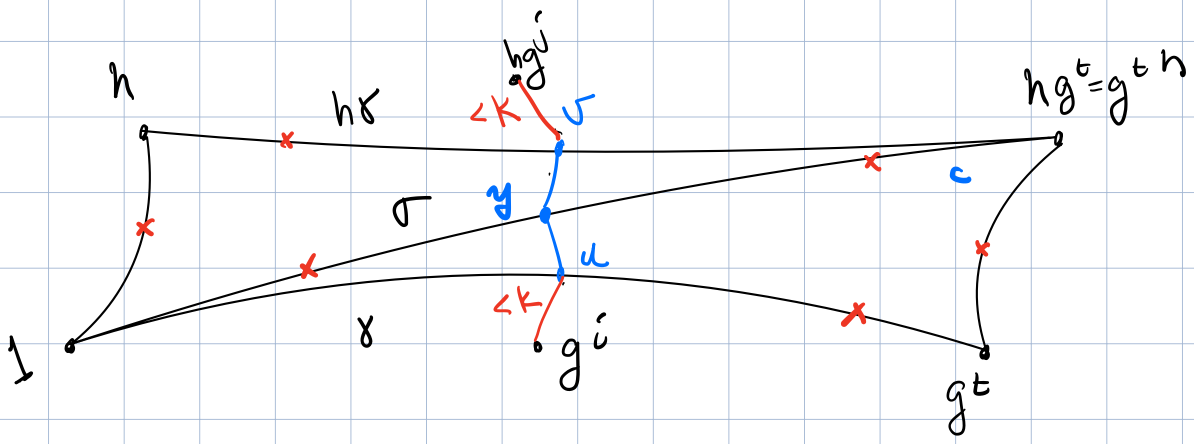


Since $i \longmapsto g^i$ is a quasi-geodesic,

$\{g^i \mid 0 \leq i \leq t\}$ stays a bounded distance from γ, ie $\exists K, i, j$ st.

$$d(g^i, u) < K$$

and $d(hg^j, v) < K$



So $d(g^i, hg^j) < 2K + 12\delta$

so $d(1, hg^{j-i}) < 2K + 12\delta$

So the coset $h\langle g\rangle$ enters $B_{2K+12\delta}(1)$

But $B_{2K+12\delta}(1)$ is a finite ball, and the cosets of $\langle g\rangle$ in $C\langle g\rangle$ are disjoint, so only finitely many of them can intersect $B_{2K+12\delta}$

We just proved they *all* do!

So there are only finitely many of them ✓

Corollary: If G is hyperbolic, it does not contain $\mathbb{Z}^2$.

Proof: the centralizer of any $g \in \mathbb{Z}^2$ is at least all of $\mathbb{Z}^2$, and $\langle g \rangle$ does not have finite index in $\mathbb{Z}^2$.

Here is one more theorem that uses the geometry of hyperbolic Cayley graphs to prove algebraic facts about hyperbolic groups

There exist infinite, finitely generated groups, all of whose elements have finite order

The most famous of these are the free Burnside groups $B(m,n)$ — m generators, exponent n:
If $m \geqslant 2$ and $n \gg 1$, they have infinite order. This is a famous theorem of Adian for n odd > 461, later S. Ivanov proved it for n even, $n > 2^{48}$, $2^6 \mid n$ ... etc.

It is an open problem whether there is a finitely <u>presented</u> group, all of whose elements have finite order. (Common wisdom is there should be one.)

Today we will show it can't be hyperbolic:

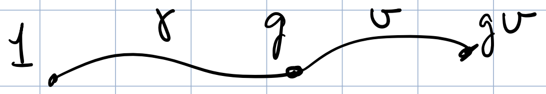
Theorem If a hyperbolic group G is infinite, it contains an element of infinite order

To prove this, we use the concept of cone type of an element in a Cayley graph.

Definition Let $\mathcal{C} = \mathcal{C}(G,S)$ be a Cayley graph for G, and $g \in G$. The cone type $c(g)$ is the set of words v such that

$$d(1, gv) = d(1,g) + \ell(v)$$

Ie if γ is a geodesic path from 1 to g, the cone type $c(g)$ is the set of paths starting at g such that the concatenation $\gamma \circ v$ is a geodesic.

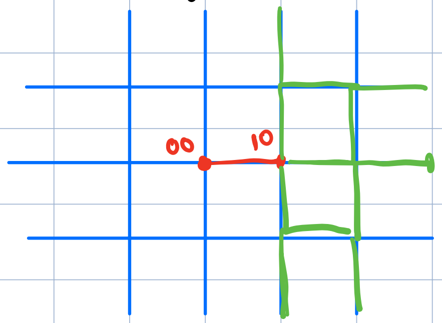


Examples

1 $\mathbb{Z}^2$, $(1,0)$, $(0,1)$ has <u>9</u> cone types

$c(0,0)$ is <u>all</u> geodesic paths

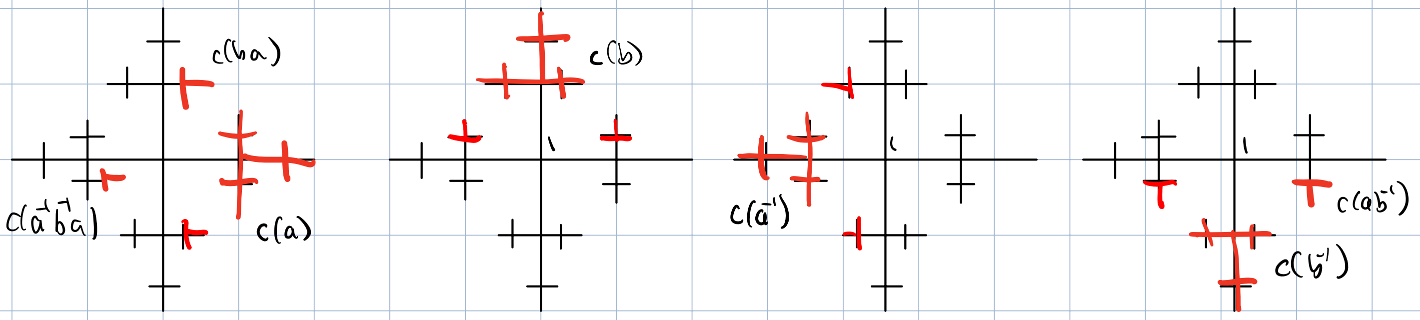
$c(1,0)$ is all geodesic paths from $(0,0)$ to (m,n) with $m \geq 0$



$c(1,1)$ is all geodesic paths in the first quadrant etc.

Notice that isometries don't preserve cone type!

2. F_2 has 5 cone types: $c(1)$ = <u>all</u> geodesics, plus



$c(a)=c(ba)=c(a^{-1}b^{-1}a)$ .... $c(a^{-1})=c(b^{-1}a^{-1})$ $c(ab^{-1})=c(b^{-1})=$ ...

Note $c(w)$ only depends on the last letter of w ...

<u>Proposition</u> A hyperbolic group has only finitely many cone types.

<u>Corollary</u>: If G is an infinite hyperbolic group, it has an element of infinite order.

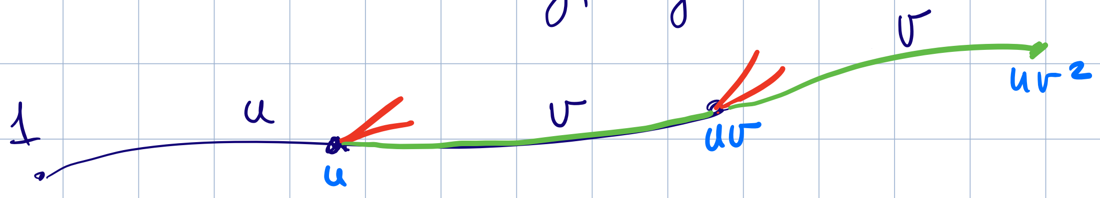
<u>Proof</u> Since G is infinite and balls in $\mathcal{C}(G,S)$ are finite, we can find arbitrarily long geodesics starting at 1.

Choose one that is longer than the number of cone types; it must contain two vertices with the same cone type:

$d(1,uv) = d(1,u) + d(u,uv) = d(1,u) + \ell(v)$

Since γ is a geodesic, v is in the cone type of u

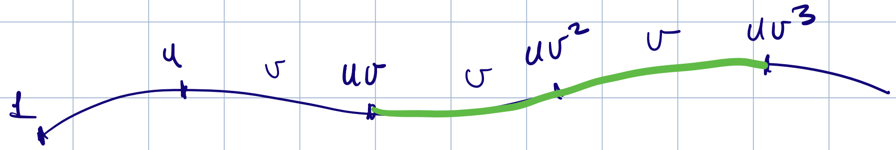
Since u and uv have the same cone type, v is also in the cone type of uv



so $d(1,uv^2) = d(1,u) + d(u,uv^2) = d(1,u) + 2\ell(v)$

so v^2 is in the cone type of u

So v^2 is also in the cone type of uv:



so $d(1,uv^3) = d(1,u) + d(u,uv^3) = d(1,u) + 3\ell(v)$

etc: $d(1,uv^n) = d(1,u) + d(u,uv^n) = d(1,u) + n\ell(v)$

for all n

so $d(1,v^n) = d(u,uv^n) = n\ell(v) \to \infty$, so

v has ∞ order.

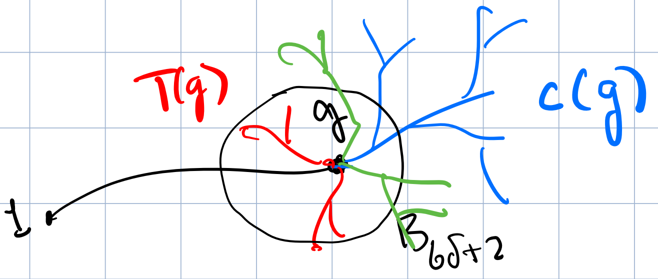
So we just need to show:

Proposition A hyperbolic group has only finitely many cone types.

Proof

We claim that the cone type of g is determined by a short "tail", namely

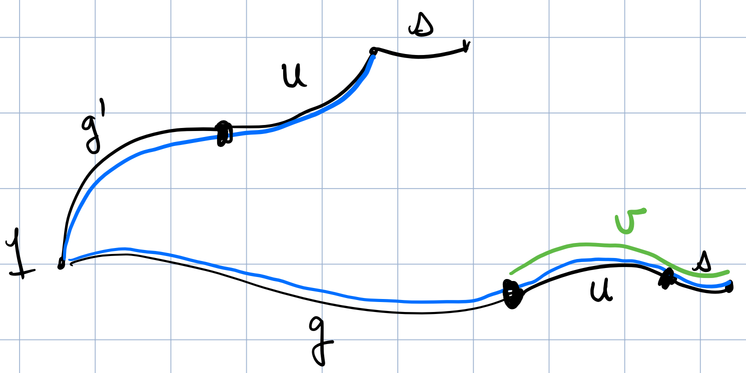
$$T(g) = \{u \in B_{6\delta+2} \mid d(1, gu) < d(1,g)\}$$



Since $B_{6\delta+2}$ is finite, this implies there are only finitely many possibilities for $T(g)$, so for $c(g)$

So suppose $T(g) = T(g')$. To show $c(g) = c(g')$ we will induct on the length $\ell(v)$ for $v \in c(g)$

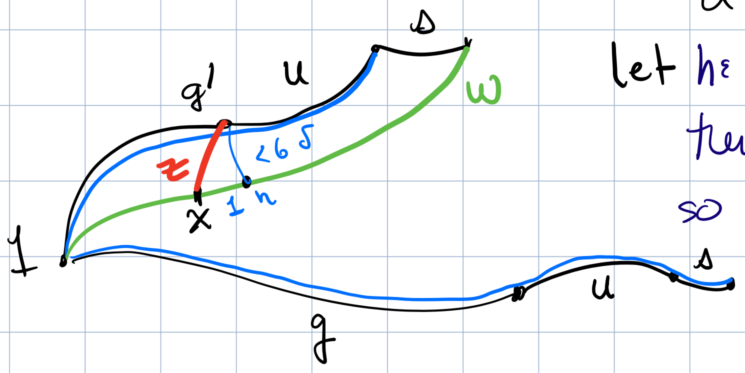
If $\ell(v) = 0$, this is trivial, so assume $\ell(v) \geq 1$,
Suppose $v \in c(g)$. Write $v = us$ with $s \in S$,
$\ell(u) = \ell(v) - 1$, so by induction, $u \in c(g')$



blue paths are geodesic

Suppose $g'us$ is not a geodesic. Draw a geodesic w from 1 to $g'us$, mark $x \in w$ with

$$d(1,x) = d(1,g') - 1$$



let $h \in w$ with $d(1,h) = d(1,g')$,

then $d(h,g') \leq 6\delta$

so $d(g',x) \leq 6\delta + 1 < 6\delta + 2$

So a geodesic z from g' to x is in $T(g')$

$$\ell(w) \leq \ell(g') + \ell(u) + 1$$

$$d(1,x) + d(x,w) < \ell(g') + \ell(u) + 1$$

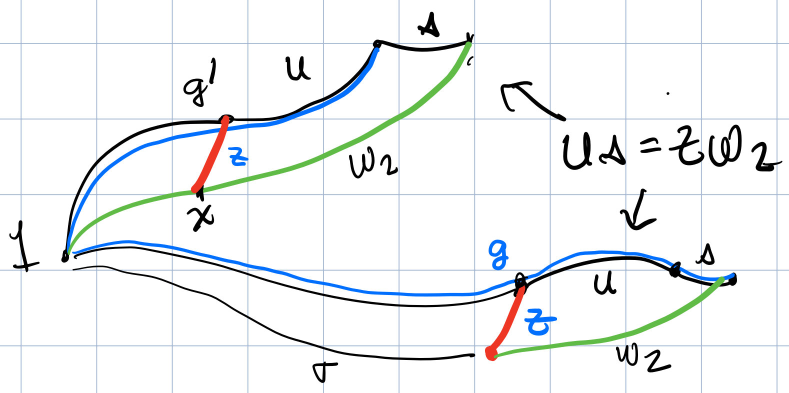
$$\cancel{\ell(g')} - 1 + d(x,w) < \cancel{\ell(g')} + \ell(u) + 1$$

$$d(x,w) < \ell(u) + 2$$

$$d(x,w) \leq \ell(u) + 1$$

Let w_2 be the section of w between x and w.

(We just showed $\ell(w_2) \leq \ell(u) + 1$)



$$T(g') = T(g) \Rightarrow z \in T(g) \Rightarrow d(1,gz) < d(1,g)$$

Let σ be a geodesic from 1 to gz

Then the path σw_2 has length

$$< d(1,g) + \ell(u) + 1 = d(1,gus)$$

contradicting the fact that the path gus is a geodesic ✓

Hyperbolic groups don't contain $\mathbb{Z}^2$

Can they contain F_2?

$\mathbb{Z}$ is hyperbolic and doesn't contain F_2.

Neither does any finite extension $\mathbb{Z}$:

$$1 \to \mathbb{Z} \to G \to K \to 1, \quad K \text{ finite}$$

(like $G = D_\infty$)

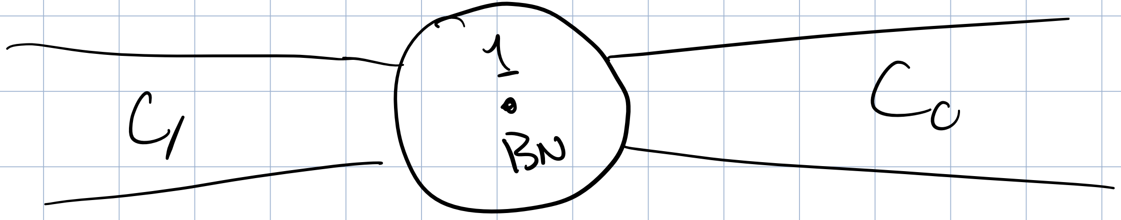
(we say G is "virtually $\mathbb{Z}$")

Exercise: If G finitely generated and has 2 ends, it *is* virtually $\mathbb{Z}$.

Proof

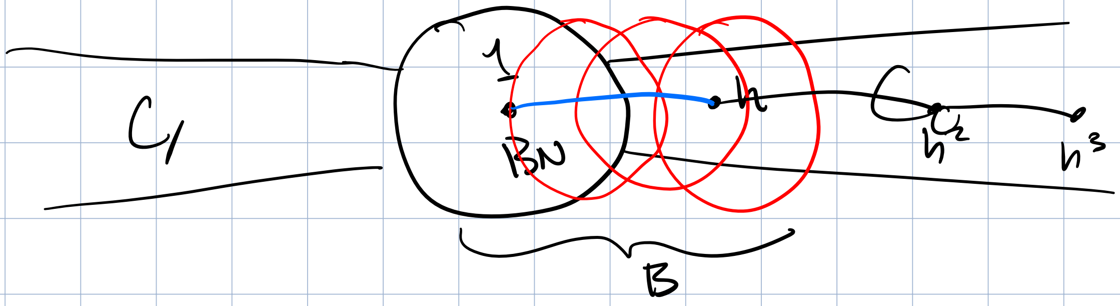
Take N big enough so that $B_N = B_N(1)$ cuts $\mathcal{G}$ into two infinite components

$$\mathcal{G} \setminus B_N = C_0 \cup C_1$$



(Then for any g, $B_N(g)$ cuts $\mathcal{G}$ in two)

Take h outside B_N that preserves the ends (G acts on ends, stab $\varepsilon = \ker G \to \mathbb{Z}/2$)



σ a geodesic 1 to h

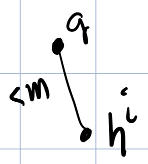
Cover σ with B_N's, let $B = \bigcup_{v \in \sigma} B_N(v)$

and $m = \max_{g \in B} d(g,1)$

$\mathcal{G} \setminus B$ has two components, one $\subset C_2$

translates $h^i B$, $i>0$ cover C_2, so every $g \in C_2 \cup B_N$ is $\leq m$ from some h^i, $i > 0$

translates $h^{-i} B$ cover C_1. So every $g \in \mathcal{G}$ is $\leq m$ from <u>some</u> h^i



$d(g^{-1}, h^i) < k$

$\Rightarrow d(1, g h^i) < k$

$\Rightarrow$ all cosets $g\langle h \rangle$ enter B

But B finite $\Rightarrow$ only room for fin. many cosets.

$\Rightarrow \langle h \rangle$ has finite index in G.

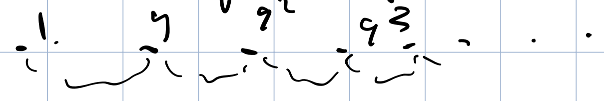
Next theorem: If a hyperbolic group G has infinitely many ends, it contains a copy of F_2.

Proof: Play ping-pong on $\mathcal{C} = \mathcal{C}(G,S)$

$\mathcal{C}$ has ∞ many ends $\Rightarrow$ G is infinite

$\Rightarrow$ G has an element g of infinite order.

$\Rightarrow$ $\{g^i\}_{i \geq 0}$ is a quasi-geodesic ray



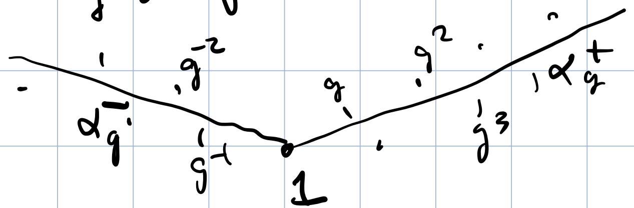
(you can think of a continuous quasi-geodesic ray if you want)

Prop 1 There is a geodesic ray α_g^+ and $K>0$

w/ $\beta_g^+ \subset N_K(\alpha_g^+)$ and $\alpha_g^+ \subset N_K(\alpha_g^+)$

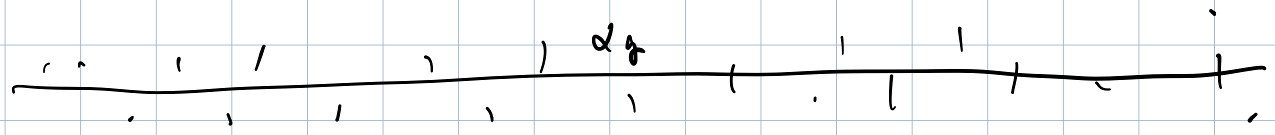
Since a geodesic that leaves $B_N(1)$ can never return, α_g^+ goes out one end of $\mathcal{C}$.

Prop 2 α_g^- $(=\alpha_{g^{-1}}^+)$ goes out a different end.



Prop 3 There is a bi-infinite geodesic α_g and K' with $\{g^i\}_{i\in\mathbb{Z}} \subset N_{K'}(\alpha_g)$

$$\alpha_g \subset N_{K'}(\{g^i\}_{i\in\mathbb{Z}})$$



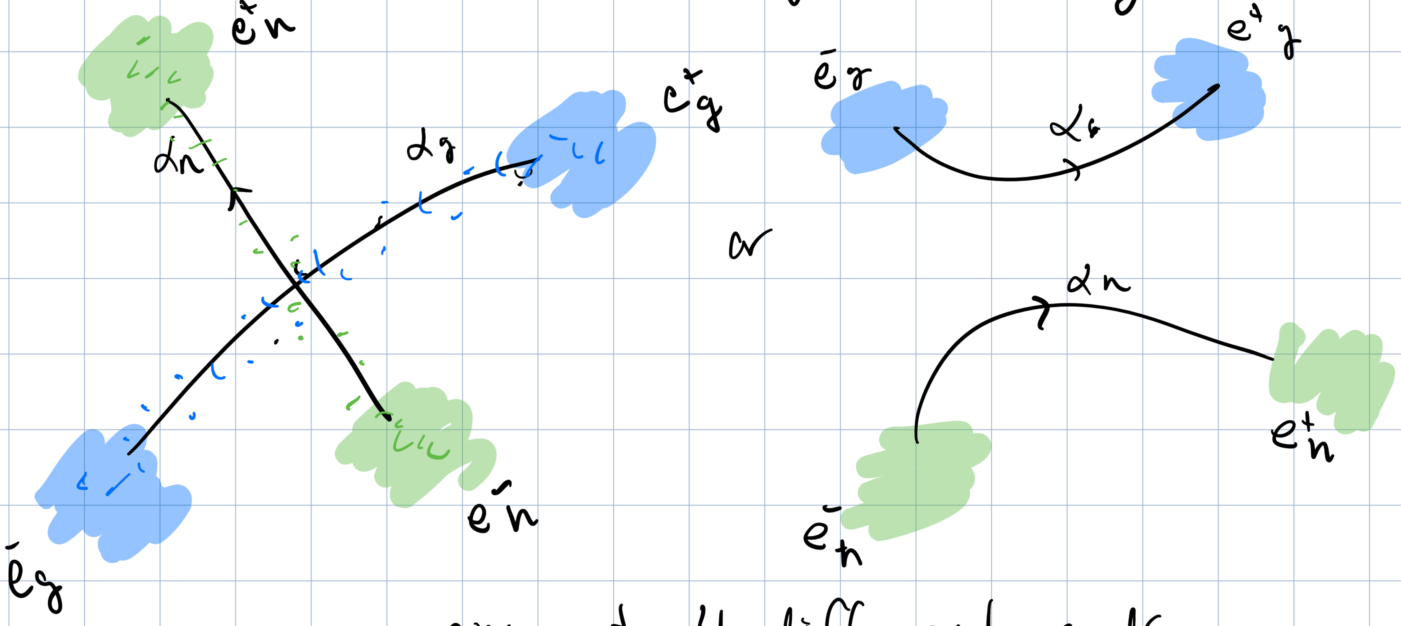
So α_g goes out the same ends as α_g^+, α_g^-

Prop 4 There is h of infinite order going out two different ends

(Proof uses the fact that there are ∞ many ends left over)

Proofs of props 1-4 are not difficult.

We now have two ∞ geodesics α_g, α_h



going out 4 different ends

We want to play Ping-pong with

$A =$ "points in a nbd of e_g^+ or e_g^-"

and $B =$ "points in a nbd of e_h^+ or e_h^-"

We need to find neighborhoods such that

$$A \cap B = \emptyset$$

and

We need to show $\exists N$ s.t. $g^N(\mathcal{B} \setminus A) \subset A$

(in particular $g^N B \subset A$)

and $h^N(\mathcal{B} \setminus B) \subset B$

(in particular $h^N A \subset B$.)

Then ping-pong tells us g^N and h^N generate a free subgroup of G.

To find these neighborhoods, Gromov used the product $(x,y)_w$ you studied in the exercises:

$$(x,y)_w = \frac{1}{2}\left(d(w,x) + d(w,y) - d(x,y)\right)$$

Instead of thinking about ends, he used equivalence classes of sequences in $\mathcal{G}$ to define "points at infinity" of $\mathcal{G}$ and neighborhoods of these points

$\{x_i\}$ a sequence, $w \in \mathcal{G}$

$\{x_i\}_{i \in \mathbb{N}} \longrightarrow \infty$ if $\forall R, \exists N$ s.t.

$$i, j > N \Rightarrow (x_i, x_j)_w > R$$

$$\{x_i\} \sim \{y_i\} \text{ if } (x_i, y_i)_w \longrightarrow \infty \text{ as } i \to \infty$$

Df: if $\{x_i\} \to \infty$, then $x_\infty =$ equiv class of $\{x_i\}$.

Then show $\{g^i\} \to \infty$, define $g^+_\infty = [\{g^i\}]$

and $g^-_\infty = [\{g^{-i}\}]$

prove $g^+_\infty \neq g^-_\infty$

Neighborhoods: $x \in X$,

$\pi_g x =$ nearest point on $\{g^i\}$ to x

$\pi_h x$ = nearest point on $\{h^i\}$ to x

Lemma For some $M > 0$, either $d(1, \pi_g x) < M$ or $d(1, \pi_h x) < M$

Then

$$A = X_g = \{x \mid \pi_g x = g^n, \ d(1, g^n) > M\}$$

$$B = X_h = \{x \mid \pi_h x = h^n, \ d(1, h^n) > M\}$$

are neighborhoods of $g_\infty \cup g^{-1}_\infty$

and $h_\infty \cup h^{-1}_\infty$

You then need to prove $\exists N$ st $g^N(\mathcal{C} \setminus A) \subset A$ and $h^N(\mathcal{C} \setminus B) \subset B$. to finish the argument.

What is known about Hyperbolic groups G?

We showed: If G is hyperbolic then

- G has a finite presentation
- The centralizer of an infinite order element g is a finite extension of $\langle g \rangle$
  (which implies G doesn't contain $\mathbb{Z}^2$)
- If G is infinite it has an element of infinite order
- G has only finitely many conjugacy classes of finite elements (in the Exercises)

We gave an indication of how to prove

- If G has ∞ many ends, it contains a copy of F_2

Other things people have proved:

- If G has more than 2 ends, it contains an infinite normal subgroup H with infinite quotient G/H
  (G is <u>really</u> not simple!)

- G has solvable word, conjugacy and isomorphism problems (The isomorphism problem is particularly difficult to solve - was first proved by Sela for torsion-free hyperbolic groups)

- The homology $H_i(G)$ is finite-dimensional for all i (this generalizes the fact that G is finitely presented, since finitely presented G have finite-dimensional $H_2(G)$.)

- If G has more than 2 ends, then the number of elements of length n (in any finite generating set) grows exponentially with n

- A random presentation gives a hyperbolic group, for a suitable notion of "random."

Dehn functions. Given a finite presentation $\langle S \mid R \rangle$ for G, any word w in $S \cup S^{-1}$ that is 1 in G is a product

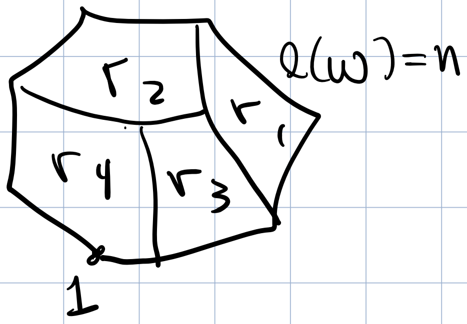
$$w = \prod_{i=1}^{n} g_i r_i g_i^{-1} \text{ for some } r_i \in R,\ g_i \in G.$$

Let $d(w)$ = minimum length of such an expression for w

and, for $n \in \mathbb{N}$, $\delta(n) = \max\limits_{l(w)=n} d(w)$

The function $\delta : \mathbb{N} \to \mathbb{N}$ is called the Dehn function of $\langle S | R \rangle$.

It measures how many relators you need to fill in a loop of length n in the Cayley graph:



So is sometimes called an isoperimetric function

- The Dehn function of any finite presentation for G is linear
- If G has a subquadratic Dehn function then G is hyperbolic

More information about hyperbolic groups can be found in Gromov's original article or in Bridson-Haefliger's book.